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Transmuted Generalized Inverted Exponential Distribution

Abstract: This paper introduces a transmuted generalized inverted exponential distribution. We generalize the two parameter generalized inverted exponential distribution using the quadratic rank transmutation map proposed by Shaw et al. (2007) to develop a transmuted generalized inverted exponential distribution. The properties of the transmuted generalized inverted exponential distribution are discussed. We derive the moments and examine the order statistics. Moreover, the maximum likelihood estimators for the parameters is briefly investigated and the information matrix is derived.

Keywords: Order Statistics, Transmutation Map, Maximum Likelihood Estimation, Reliability Function.

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1 Introduction

The exponential distribution is the most widely used lifetime model in reliability theory, because of its simplicity and mathematical feasibility. If a random variable $X$ has an exponential distribution, then $Y = \frac{1}{X}$ has an inverted exponential distribution (IED). IED has been discussed as a lifetime model by Lin et al. [9] in detail. They obtained maximum likelihood estimates (MLEs), confidence limits and uniformly minimum variance unbiased estimators for the parameter and reliability function of IED with complete samples. Later IED has been considered by Killer and Kamath [8] and among many others. The distribution function and density function of IED are given below:

\[
F_Y(x) = 1 - e^{-\theta x}, x \geq 0, \theta > 0, \tag{1}
\]

and

\[
f_Y(x) = \theta x e^{-\theta x}. \tag{2}
\]

The exponential distribution was generalized, by introducing a shape parameter, and studied extensively by Gupta and Kundu [5, 6]. Raqab and Madi [11] studied generalized exponential distribution (GED) from a Bayesian point of view. Hare Krishna and Kapil Kumar [7] introduced reliability estimation in generalized inverted exponential distribution with progressively type II censored samples.

On the same lines, Abouammoh and Alshingiti [1] introduced a shape parameter in the IED to obtain generalized inverted exponential distribution (GIED). They derived many distributional properties and reliability characteristics of GIED. Assuming it to be a good lifetime model they obtained maximum likelihood estimators, least square estimators and confidence intervals of the two parameters involved. A life random variable $X$ is said to have a generalized inverted exponential distribution denoted by GIED $(\theta, \alpha)$, if its distribution function has the form:

\[
F(x) = 1 - \left(1 - e^{-\frac{x}{\theta}}\right)^\alpha, \quad x \geq 0, \alpha, \theta > 0, \tag{3}
\]
and density function given by:

\[ f(x) = \frac{\alpha \theta}{x^2} e^{-\frac{x}{\theta}} \left( 1 - e^{-\frac{x}{\theta}} \right)^{\alpha - 1} \]

(4)

In this article we present a new generalization of generalized inverted exponential distribution called the transmuted generalized inverted exponential distribution. We derive the subject distribution using the quadratic rank transmutation map studied by Shaw and Buckley [12]. Let \( X \) be a random variable with distribution function \( F_X \) and density function \( f_X \). The random variable \( X|\lambda \) is said to have transmuted distribution with base distribution given by \( F_X \) if its distribution function \( F_{X|\lambda}(x) \) is given by:

\[ F_{X|\lambda}(x) = (1 + \lambda)F_X(x) - \lambda F_X(x)^2, |\lambda| \leq 1, \]

(5)

Differentiation yields the density function of \( X|\lambda \):

\[ f_{X|\lambda}(x) = f_X(x) \left( (1 + \lambda) - 2\lambda F_X(x) \right) \]

(6)

An extensive study of the quadratic rank transmutation map, i.e. a functional composition of the distribution function of one distribution with the quantile function of another, is given by Shaw and Buckley [12]. Observe that for \( \lambda = 0 \), the distribution of the base random variable is obtained.

Many authors investigated the generalization of some well-known distributions. Aryal and Tsokos [3] defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution, since it has been observed that the transmuted Gumbel can be used to model climate data. Moreover, Aryal and Tsokos [2] presented a generalization of the Weibull distribution called the transmuted Weibull distribution. Recently, Aryal [4] proposed and studied the various structural properties of the transmuted Log-Logistic distribution, and Khan and King [10] introduced the transmuted modified Weibull distribution which extended the developments on transmuted Weibull distribution by Aryal and Tsokos [2]. They studied the mathematical properties and maximum likelihood estimation of the parameters. In the present paper we provide mathematical formulation of the transmuted generalized inverted exponential distribution and some of its properties. We also provide possible areas of applications.

The rest of the paper is organized as follows. In Section 2 we demonstrate the concept of transmuted probability distributions. In Section 3, we find the reliability functions of the subject model. The statistical properties including quantile functions, moments and moment generating functions are derived in Section 4. The minimum, maximum and median order statistics are discussed in Section 5. Finally, in Section 6 we demonstrate the maximum likelihood estimates and the asymptotic confidence intervals of the unknown parameters.

### 2 Transmuted Generalized Inverted Exponential Distribution

In this section we study the transmuted generalized inverted exponential (TGIE) distribution and the sub-models of this distribution. Using (5) and (6) we obtain the distribution function of the transmuted generalized inverted exponential distribution:

\[ F_{TGIE}(x) = \left( 1 - \left( 1 - e^{-\frac{x}{\theta}} \right)^{\alpha} \right) \left( 1 + \lambda \left( 1 - e^{-\frac{x}{\theta}} \right)^{\alpha} \right) \]

where \( \theta \) and \( \alpha \) are the scale and shape parameters representing the different patterns of the transmuted generalized inverted exponential distribution and \( \lambda \) is the transmuted parameter.

The restrictions in Equation (7) on the values of \( \theta, \alpha \) and \( \lambda \) are always the same. The density function of the transmuted generalized inverted exponential distribution is given by:

\[ f_{TGIE}(x) = \frac{\alpha \theta}{x^2} e^{-\frac{x}{\theta}} \left( 1 - e^{-\frac{x}{\theta}} \right)^{\alpha - 1} \left( 1 - \lambda + 2\lambda \left( 1 - e^{-\frac{x}{\theta}} \right)^{\alpha} \right). \]

(8)
The transmuted generalized inverted exponential distribution is a very flexible model that approaches many different distributions when its parameters are changed.

- If \( \alpha = 1 \) we get the transmuted inverted exponential \( TIE(\theta, \lambda) \).
- If \( \lambda = 0 \) we get the generalized inverted exponential \( GIE(\theta, \alpha) \).
- If \( \alpha = 1 \) and \( \lambda = 0 \) we get the inverted exponential \( IE(\theta) \).

3 Reliability Analysis

The transmuted generalized inverted exponential distribution can be used for describing a random lifetime in reliability analysis. The reliability function \( R \) of the transmuted generalized inverted exponential distribution is denoted by \( R_{TGIE} \) which is also known as the survival function. It is given by:

\[
R_{TGIE}(x) = 1 - F_{TGIE}(x) = 1 - \left(1 - \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right) \left(1 + \lambda \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right).
\]  

(9)

Note that \( R_{TGIE}(x) + F_{TGIE}(x) = 1 \). Another characteristic in reliability analysis is the hazard rate function \( h \) defined for the transmuted generalized inverted exponential distribution by:

\[
h_{TGIE}(x) = \frac{f_{TGIE}(x)}{1 - F_{TGIE}(x)} = \frac{\alpha \theta e^{-\frac{x}{\theta}} \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha-1} \left(1 - \lambda + 2\lambda \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right)}{1 - \left(1 - \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right) \left(1 + \lambda \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right)}.
\]  

(10)

It is important to note that the units for \( h_{TGIE}(x) \) is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted generalized inverted exponential distribution is denoted by \( H_{TGIE}(x) \) and is defined as:

\[
H_{TGIE}(x) = -\log \left|\left(1 - \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right) \left(1 + \lambda \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right)\right|.
\]  

(11)

It is important to note that the units for \( H_{TGIE}(x) \) is the cumulative probability of failure per unit of time, distance or cycles. For all choices of the distribution has decreasing cumulative instantaneous failure rate.

**Theorem 1.** The hazard rate function \( h_{TGIE} \) of the transmuted generalized inverted exponential distribution has the following properties

- If \( \alpha = 1 \) the failure rate is same as that of \( TIE(\theta, \lambda) \).
- If \( \lambda = 0 \) the failure rate is same as that of \( GIE(\theta, \alpha) \).
- If \( \alpha = 1 \) and \( \lambda = 0 \) the failure rate is same as that of \( IE(\theta) \).

**Proof.** The hazard function \( h_{TGIE} \) of the transmuted generalized inverted exponential distribution is given by (10). It has the following special cases:

- For \( \alpha = 1 \) the failure rate is same as that of \( TIE(\theta, \lambda) \), because:

\[
h_{TIE}(x) = \frac{\theta e^{-\frac{x}{\theta}} \left(1 + \lambda - 2\lambda e^{-\frac{x}{\theta}}\right)}{1 - \left(1 - (1 - e^{-\frac{x}{\theta}})\right) \left(1 + \lambda \left(1 - e^{-\frac{x}{\theta}}\right)\right)}.
\]  

(12)

- If \( \lambda = 0 \) the failure rate is same as that of \( GIE(\theta, \alpha) \), because:

\[
h_{GIE}(x) = \frac{\alpha \theta e^{-\frac{x}{\theta}}}{x^2 (1 - e^{-\frac{x}{\theta}})}.
\]  

(13)
If $\alpha = 1$ and $\lambda = 0$ the failure rate is same as that of $IE(\theta, x)$, because:

$$h_{IE}(x) = \frac{\theta e^{-\theta x}}{x^2 \left(1 - e^{-\theta x}\right)}$$

(14)

4 Statistical Properties

In this section we discuss the statistical properties of the transmuted generalized inverted exponential distribution.

4.1 Quantile and median

By solving $F_{TGIE}(x_q) = q$ for given value $q$, the quantile $x_q$ is obtained:

$$x_q = \frac{\theta}{\log \left(\frac{(\lambda-1)+\sqrt{(\lambda+1)^2-4\lambda q}}{2\lambda} \right)^{\frac{1}{\alpha}} - 1}$$

(15)

Setting $q = 0.5$, yields the median of $TGIE(\theta, \alpha, \lambda)$.

4.2 Random number generation

Let $U \sim U(0, 1)$ be a uniformly distributed random variable on the unit interval. Then the random number generation for $TGIE(\theta, \alpha, \lambda)$ is defined by:

$$U = \left(1 - \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right) \left(1 + \lambda \left(1 - e^{-\frac{x}{\theta}}\right)^{\alpha}\right)$$

(16)

thus for given realization $u$ we obtain the random number $x$ by:

$$x = \frac{\theta}{\log \left(\frac{(\lambda-1)+\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda} \right)^{\frac{1}{\alpha}} - 1}.$$ 

(17)

4.3 Moments

The following theorem gives the $r_{th}$ moment $\mu_r$ of $TGIE(\theta, \alpha, \lambda)$.

Theorem 2. If $X$ has the $TGIE(\theta, \alpha, \lambda)$ with $|\lambda| \leq 1$, then the $r_{th}$ moment of $X$ is given by:

$$\mu_r = \sum_{j=0}^{\infty} (-1)^j (j+1)^{r-1} \Gamma(1-r) \left(1 + \lambda \left(\begin{array}{c} \alpha-1 \\ j \end{array}\right) + 2\lambda \left(\begin{array}{c} 2\alpha-1 \\ j \end{array}\right)\right),$$

(18)

and for $\alpha = 1$, we obtain:

$$\mu_r = \theta^r \Gamma(1-r)(1 + \lambda - \lambda^2^r)$$

(19)
Proof. Starting with:

\[ \mu_r = \int_0^\infty x^r f_{TGIE}(x)\,dx \]

\[ = \int_0^\infty x^r \frac{\alpha \theta}{x^2} e^{-\frac{\theta}{x}} \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha-1} \left(1 + 2\lambda\left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha}\right)\,dx. \]

\[ = (1 - \lambda) \int_0^\infty \alpha \theta x^{r-2} e^{-\frac{\theta}{x}} \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha-1} \,dx \]

\[ + 2\lambda \int_0^\infty \alpha \theta x^{r-2} e^{-\frac{\theta}{x}} \left(1 - e^{-\frac{\theta}{x}}\right)^{2\alpha-1} \,dx. \quad (20) \]

There are two cases to be considered. The general case refers to \( \alpha, \theta, \gamma > 0 \). Using the expansion of \( (1 - e^{-\frac{\theta}{x}})^{\alpha-1} \)

\[ \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} e^{-\frac{\theta}{x}}, \quad (21) \]

expression (20) takes the following form:

\[ \mu_r = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} (1 - \lambda) \int_0^\infty \alpha \theta x^{r-2} e^{-\frac{\theta}{x}(j+1)} \,dx \]

\[ + 2\lambda \sum_{j=0}^{\infty} (-1)^j \binom{2\alpha-1}{j} \int_0^\infty \alpha \theta x^{r-2} e^{-\frac{\theta}{x}(j+1)} \,dx \]

\[ = \sum_{j=0}^{\infty} (-1)^j (j + 1)^{-1} \Gamma(1 - r) \left(1 + \lambda\right) \binom{\alpha-1}{j} + 2\lambda \binom{2\alpha-1}{j}. \quad (22) \]

The second case refers to \( \alpha = 1 \) and \( \theta > 0 \). In this case (20) reduces to:

\[ \mu_r = \int_0^\infty x^r \frac{\theta}{x^2} e^{-\frac{\theta}{x}} \left(1 + \lambda - 2\lambda e^{-\frac{\theta}{x}}\right) \,dx \]

\[ = \theta(1 + \lambda) \int_0^\infty x^{r-2} e^{-\frac{\theta}{x}} \,dx - 2\lambda \theta \int_0^\infty x^{r-2} e^{-\frac{\theta}{x}} \,dx. \quad (23) \]

Setting \( \frac{\theta}{x} = y \) and integrating yields:

\[ \mu_r = \theta' \Gamma(1 - r)(1 + \lambda - \lambda 2^r). \quad (24) \]

which completes the proof.

Based on Theorem 2 the coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of \( TGIE(\theta, \alpha, \lambda) \) distribution are obtained:

\[ CV_{TMIU} = \sqrt{\frac{\mu_2}{\mu_1^2}} - 1, \quad (25) \]

\[ CS_{TMIU} = \frac{\mu_3 - 3\mu_2 \mu_1 + 2\mu_1^3}{(\mu_2 - \mu_1)^2} \quad (26) \]

\[ CK_{TMIU} = \frac{\mu_4 - 4\mu_3 \mu_1 + 6\mu_2 \mu_1^2}{(\mu_2 - \mu_1)^2}. \quad (27) \]
4.4 Moment Generating Function

Next the moment generating function of $TGIE(\theta, \alpha, \lambda)$ is derived.

**Theorem 3.** If $X$ has $TGIE(\theta, \alpha, \lambda)$ with $|\lambda| \leq 1$, then the moment generating function $M_X$ of $X$ is given as follows:

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!}(-1)^j(j+1)^{-1}\Gamma(1-r)\left(1+\lambda\right)^{\left(\alpha - 1\right)} + 2\lambda\left(\frac{2\alpha - 1}{j}\right),$$

(28)

and for $\alpha = 1$:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t\theta)^r}{r!}\Gamma(1-r)(1 + \lambda - \lambda2^r).$$

(29)

**Proof.**

$$M_X(t) = \int_0^\infty e^{tx} f_{TGIE}(x) dx$$

$$= \int_0^\infty \left(\sum_{r=0}^{\infty} \frac{t^r}{r!}x^r\right) f_{TGIE}(x) dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r$$

(30)

With (23) and (24), we get from (30):

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!}(-1)^j(j+1)^{-1}\Gamma(1-r)\left(1+\lambda\right)^{\left(\alpha - 1\right)} + 2\lambda\left(\frac{2\alpha - 1}{j}\right)$$

(31)

and for $\alpha = 1$:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t\theta)^r}{r!}\Gamma(1-r)(1 + \lambda - \lambda2^r).$$

(32)

which completes the proof.

5 Order Statistics

Order statistics have many applications in reliability and life testing, since order statistics arise in the study of reliability of a system. Let $X_1, X_2, \ldots, X_n$ be a simple random sample from $TGIE(\theta, \alpha, \lambda)$ with distribution function and density function given by (7) and (8), respectively. Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ is used to model the lifetime of an $(n-i+1)$-out-of-$n$ system which consists of $n$ independent and identically distributed components. The density function of $X_{(i:n)}, 1 \leq i \leq n$ is given by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}(F_{TGIE(x)})^{i-1}(1-F_{TGIE(x)})^{n-i}f_{TGIE(x)}$$

(33)

The joint density function of $(X_{(i:n)}, X_{(j:n)})$ for $1 \leq i \leq j \leq n$ is:

$$f_{i:j:n}(x_i, x_j) = C(F_{TGIE(x_j)})^{j-1}(F_{TGIE(x_i)} - F_{TGIE(x_j)})^{j-i-1}(1 - F_{TGIE(x_j)})^{n-j}$$

$$f_{TGIE(x_i)}f_{TGIE(x_j)}$$

(34)
where \( C \) is given by:

\[
C = \frac{\binom{n}{i} \binom{n-i}{j-i} \binom{n-j}{j}}{(i-1)!(j-i-1)!(n-j)!} \tag{35}
\]

The first order statistic is given by \( X_{(1)} = \min(X_1, X_2, \ldots, X_n) \), the last order statistics by \( X_{(n)} = \max(X_1, X_2, \ldots, X_n) \) and the median order statistics by \( X_{m+1} \).

### 5.1 Distribution of Minimum, Maximum and Median

Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables with transmuted generalized inverted exponential distribution. The first, last and median order statistics have the density functions \( f_{1:n}, f_{n:n} \) and \( f_{m+1:n} \) (with \( m = \left\lfloor \frac{n}{2} \right\rfloor \)) given below:

\[
f_{1:n}(x) = n(1 - F_{TGIE}(x))^{n-1} f_{TGIE}(x)
\]

\[
= n \left( 1 - \left( 1 - e^{-\frac{a}{x(1)}} \right)^\alpha \left( 1 + \lambda \left( 1 - e^{-\frac{a}{x(1)}} \right)^\alpha \right) \right)^{n-1} \\
\frac{\alpha \theta}{x(1)^2} e^{-\frac{a}{x(1)}} \left( 1 - e^{-\frac{a}{x(1)}} \right)^{\alpha-1} \left( 1 - \lambda + 2\lambda \left( 1 - e^{-\frac{a}{x(1)}} \right)^\alpha \right) \tag{36}
\]

\[
f_{n:n}(x) = n(F_{TGIE}(x(n)))^{n-1} f_{TGIE}(x(n))
\]

\[
= n \left( \left( 1 - \left( 1 - e^{-\frac{a}{x(n)}} \right)^\alpha \left( 1 + \lambda \left( 1 - e^{-\frac{a}{x(n)}} \right)^\alpha \right) \right)^{n-1} \\
\frac{\alpha \theta}{x(n)^2} e^{-\frac{a}{x(n)}} \left( 1 - e^{-\frac{a}{x(n)}} \right)^{\alpha-1} \left( 1 - \lambda + 2\lambda \left( 1 - e^{-\frac{a}{x(n)}} \right)^\alpha \right) \tag{37}
\]

and

\[
f_{m+1:n}(\tilde{x}) = \frac{(2m+1)!}{m!m!} (F_{TGIE}(\tilde{x}))^m (1 - F_{TGIE}(\tilde{x}))^m f_{TGIE}(\tilde{x})
\]

\[
= \frac{(2m+1)!}{m!m!} \left( \left( 1 - \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^\alpha \left( 1 + \lambda \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^\alpha \right) \right)^{m-1} \\
\left( 1 - \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^\alpha \left( 1 + \lambda \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^\alpha \right) \right) \right)^m \\
\frac{\alpha \theta}{x(m+1)^2} e^{-\frac{a}{x(m+1)}} \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^{\alpha-1} \left( 1 - \lambda + 2\lambda \left( 1 - e^{-\frac{a}{x(m+1)}} \right)^\alpha \right) \tag{38}
\]

### 5.2 Joint Distribution of the \( i \)th and \( j \)th Order Statistics

The joint distribution of the the \( i \)th and \( j \)th order statistics from TGIE with \( h_{(k)} = 1 - e^{-\frac{a}{x(k)}} \) for \( k = i, j \) is given by:

\[
f_{i:j:n}(x_i, x_j) = C(F_{TGIE}(x_i))^{i-1} (F_{TGIE}(x_j) - F_{TGIE}(x_i))^{j-i-1} (1 - F_{TGIE}(x_j))^{n-j} f_{TGIE}(x_i) f_{TGIE}(x_j)
\]

\[
= C \left( \left( 1 - h_{(i)}^\alpha \right) \left( 1 + \lambda h_{(i)}^\alpha \right) \right)^{i-1} \\
\left( \left( 1 - h_{(j)}^\alpha \right) \left( 1 + \lambda h_{(j)}^\alpha \right) - \left( 1 - h_{(i)}^\alpha \right) \left( 1 + \lambda h_{(i)}^\alpha \right) \right)^{j-i-1} \\
\left( 1 - \left( 1 - h_{(i)}^\alpha \right) \left( 1 + \lambda h_{(i)}^\alpha \right) \right)^{n-j} \\
\frac{\theta}{x(i)^2} e^{-\frac{a}{x(i)}} h_{(i)}^{\alpha-1} \left( 1 - \lambda + 2\lambda h_{(i)}^\alpha \right)
\]
\[
\frac{\theta}{x_j^{(j)}} e^{-\frac{\theta}{x_j^{(j)}}} h_{(j)}^{\alpha-1} \left(1 - \lambda + 2 \lambda h_{(j)}^{\alpha}\right)
\]  

(39)

For the special case \(i = 1\) and \(j = n\) we get the joint distribution of the minimum and maximum order statistics:

\[
f_{1:n:n}(x_i, x_j) = n(n-1)(F_{TGIE}(x_n) - F_{TGIE}(x_1))^{n-2} f_{TGIE}(x_1) f_{TGIE}(x_n)
\]

\[
= n(n-1) \left(\left(1 - \left(1 - e^{-\frac{\theta}{x_n^{(n)}}}\right)^\alpha\right) \left(1 + \lambda \left(1 - e^{-\frac{\theta}{x_n^{(n)}}}\right)^\alpha\right)\right)^{n-2}
\]

\[
\frac{\alpha \theta}{x_j^{(1)}} e^{-\frac{\theta}{x_j^{(1)}}} \left(1 - e^{-\frac{\theta}{x_j^{(1)}}}\right)^{\alpha-1} \left(1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(1)}}}\right)\right)
\]

\[
\frac{\alpha \theta}{x_n^{(n)}} e^{-\frac{\theta}{x_n^{(n)}}} \left(1 - e^{-\frac{\theta}{x_n^{(n)}}}\right)^{\alpha-1} \left(1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_n^{(n)}}}\right)\right).
\]

(40)

6 Maximum Likelihood Estimators

In this section, maximum likelihood estimators (MLE) and inference for \(T_{\text{GIE}}(\theta, \alpha, \lambda)\) is discussed. Let \(x_1, \ldots, x_n\) be a realization of a random sample of size \(n\) from \(TGIE(\theta, \alpha, \lambda)\) then the likelihood function can be written as:

\[
L(\theta, \alpha, \lambda | x^{(j)}) = \prod_{i=1}^n \frac{\alpha \theta}{x_j^{(i)}} e^{-\frac{\theta}{x_j^{(i)}}} \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)^{\alpha-1} \left(1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)\right)
\]

(41)

By taking logarithm, the log-likelihood function of \(\log L(\theta, \alpha, \lambda | x^{(j)})\) can be written as:

\[
\log L = n \log \alpha + n \log \theta + \sum_{i=1}^n \log \left(\frac{1}{x_j^{(i)}}\right)
\]

\[
- \theta \sum_{i=1}^n \frac{1}{x_j^{(i)}} + (\alpha - 1) \sum_{i=1}^n \log \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)
\]

\[
+ \sum_{i=1}^n \log \left(1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)\right)
\]

(42)

Differentiating (42) with respect to \(\theta, \alpha, \) and \(\lambda\) and equating the derivatives to zero results in the normal equations. The derivatives are given below:

\[
\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n \frac{1}{x_j^{(i)}} + (\alpha - 1) \sum_{i=1}^n \frac{1}{x_j^{(i)}} e^{-\frac{\theta}{x_j^{(i)}}}
\]

\[
+ \sum_{i=1}^n \frac{2 \lambda \alpha}{1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)} \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)^{\alpha-1} \frac{1}{x_j^{(i)}} e^{-\frac{\theta}{x_j^{(i)}}}
\]

(43)

\[
\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)
\]

\[
+ \sum_{i=1}^n \frac{2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right) \log \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)}{1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)^\alpha}
\]

(44)

\[
\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^n \frac{2 \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)^{\alpha-1}}{1 - \lambda + 2 \lambda \left(1 - e^{-\frac{\theta}{x_j^{(i)}}}\right)^\alpha}.
\]

(45)
The estimates are obtained by solving the normal equations simultaneously. The solutions represent the ML estimates \( \hat{\alpha}, \hat{\theta}, \) and \( \hat{\lambda} \).

For the three parameters of \( TGIE(\theta, \alpha, \lambda) \) all the second order derivatives of the log-likelihood function exist. Thus, the inverse dispersion matrix is given by:

\[
\begin{pmatrix}
\hat{\alpha} \\
\hat{\theta} \\
\hat{\lambda}
\end{pmatrix} \sim N
\begin{pmatrix}
\alpha \\
\theta \\
\lambda
\end{pmatrix},
\begin{pmatrix}
V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\lambda} \\
V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\lambda} \\
V_{\lambda\alpha} & V_{\lambda\theta} & V_{\lambda\lambda}
\end{pmatrix}
\]

\( (46) \)

\[
V^{-1} = -E \begin{pmatrix}
V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\lambda} \\
V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\lambda} \\
V_{\lambda\alpha} & V_{\lambda\theta} & V_{\lambda\lambda}
\end{pmatrix}
\]

\( (47) \)

where

\[
\begin{align*}
V_{\alpha\alpha} &= \frac{\partial^2 L}{\partial \alpha^2}, \\
V_{\theta\theta} &= \frac{\partial^2 L}{\partial \theta^2}, \\
V_{\lambda\lambda} &= \frac{\partial^2 L}{\partial \lambda^2}, \\
V_{\alpha\theta} &= \frac{\partial^2 L}{\partial \alpha \partial \theta}, \\
V_{\alpha\lambda} &= \frac{\partial^2 L}{\partial \alpha \partial \lambda},
\end{align*}
\]

\( (48) \)

By determining the inverse dispersion matrix, the asymptotic variances and covariances of the ML estimators for \( \alpha, \theta, \) and \( \lambda \) may be obtained. Using \( (46) \), approximate 100(1 – \( \gamma \))% confidence intervals for \( \alpha, \theta, \) and \( \lambda \) are determined respectively as:

\[
\hat{\alpha} \pm z_{\gamma/2} \sqrt{V_{\alpha\alpha}}, \hat{\theta} \pm z_{\gamma/2} \sqrt{V_{\theta\theta}}, \text{and} \hat{\lambda} \pm z_{\gamma/2} \sqrt{V_{\lambda\lambda}}
\]

where \( z_{\gamma} \) is the upper 100\( \gamma \)th quantile of the standard normal distribution.

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References


