Bulletin of the **S**ection of **L**ogic
Volume 51/3 (2022), pp. 411–453
https://doi.org/10.18778/0138-0680.2022.17

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COMPLETE REPRESENTATIONS AND NEAT EMBEDDINGS

Abstract

Let $2 < n < \omega$. Then CA_n denotes the class of cylindric algebras of dimension n, RCA_n denotes the class of representable CA_ns, CRCA_n denotes the class of completely representable CA_ns, and $\mathsf{Nr}_n\mathsf{CA}_\omega(\subseteq \mathsf{CA}_n)$ denotes the class of n-neat reducts of CA_ωs. The elementary closure of the class CRCA_ns ($\mathbf{K_n}$) and the non-elementary class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ are characterized using two-player zero-sum games, where At is the operator of forming atom structures. It is shown that $\mathbf{K_n}$ is not finitely axiomatizable and that it coincides with the class of atomic algebras in the elementary closure of $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ where $\mathbf{S_c}$ is the operation of forming complete subalgebras. For any class $\mathbf{L}$ such that $\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{L} \subseteq \mathsf{At}\mathbf{K_n}$, it is proved that $\mathbf{SP}\mathfrak{Cm}\mathbf{L} = \mathsf{RCA}_n$, where $\mathfrak{Cm}$ is the dual operator to At; that of forming complex algebras. It is also shown that any class $\mathbf{K}$ between $\mathsf{CRCA}_n \cap \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not first order definable, where $\mathbf{S_d}$ is the operation of forming dense subalgebras, and that for any $2 < n < m$, any $l \geq n+3$ any any class $\mathbf{K}$ such that $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m \cap \mathsf{CRCA}_n) \subseteq \mathbf{K} \subseteq \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_l$, $\mathbf{K}$ is not not first order definable either.

Keywords: Algebraic logic, cylindric algebras, relation algebras, atom-canonicity, combinatorial game theory.

2010 Mathematical Subject Classification: 03B45, 03G15.

We follow the notation of [1] which is in conformity with the notation in the monograph [3]. In particular, for any pair of ordinal $\alpha < \beta$, CA_α stands for the class of cylindric algebras of dimension α, RCA_α denotes the class

Presented by: Andrzej Indrzejczak
Received: June 20, 2021
Published online: September 9, 2022

of representable CA_αs and $\mathsf{Nr}_\alpha\mathsf{CA}_\beta(\subseteq \mathsf{CA}_\alpha)$ denotes the class of α-neat reducts of CA_βs.

Definition 0.1. Assume that $\alpha < \beta$ are ordinals and that $\mathfrak{B} \in \mathsf{CA}_\beta$. Then the *$\alpha$-neat reduct* of $\mathfrak{B}$, in symbols $\mathfrak{Nr}_\alpha\mathfrak{B}$, is the algebra obtained from $\mathfrak{B}$, by discarding cylindrifiers and diagonal elements whose indices are in $\beta \setminus \alpha$, and restricting the universe to the set

$$Nr_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : \mathsf{c}_i x \neq x\} \subseteq \alpha\}.$$

It is straightforward to check that $\mathfrak{Nr}_\alpha\mathfrak{B} \in \mathsf{CA}_\alpha$. Let $\alpha < \beta$ be ordinals. If $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathsf{CA}_\beta$, then we say that $\mathfrak{A}$ *neatly embeds* in $\mathfrak{B}$, and that $\mathfrak{B}$ is a *β-dilation of* $\mathfrak{A}$, or simply a *dilation* of $\mathfrak{A}$ if β is clear from context. For $\mathbf{K} \subseteq \mathsf{CA}_\beta$, we write $\mathsf{Nr}_\alpha\mathbf{K}$ for the class $\{\mathfrak{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in \mathbf{K}\}$. Following [3], Cs_n denotes the class of *cylindric set algebras of dimension* n, and Gs_n denotes the class of *generalized cylindric set algebra of dimension* n; $\mathfrak{C} \in \mathsf{Gs}_n$, if $\mathfrak{C}$ has top element V a disjoint union of cartesian squares, that is $V = \bigcup_{i\in I} {}^nU_i$, I is a non-empty indexing set, $U_i \neq \emptyset$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$. The operations of $\mathfrak{C}$ are defined like in cylindric set algebras of dimension n relativized to V.

Definition 0.2. An algebra $\mathfrak{A} \in \mathsf{CA}_n$ is *completely representable* $\iff$ there exists $\mathfrak{C} \in \mathsf{Gs}_n$, and an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that for all $X \subseteq \mathfrak{A}$, $f(\sum X) = \bigcup_{x\in X} f(x)$, whenever $\sum X$ exists in $\mathfrak{A}$. If $\sum X$ exists in $\mathfrak{A}$, we denote this supremum by $\sum^{\mathfrak{A}} X$. In this case, we say that $\mathfrak{A}$ is *completely representable via* f.

It is known that $\mathfrak{A}$ is completely representable via $f : \mathfrak{A} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_n$ has top element V say $\iff$ $\mathfrak{A}$ is atomic and f is *atomic* in the sense that $f(\sum \mathsf{At}\mathfrak{A}) = \bigcup_{x\in \mathsf{At}\mathfrak{A}} f(x) = V$ [5] where $\mathsf{At}\mathfrak{A}$ denotes the set of atoms of $\mathfrak{A}$. We denote the class of completely representable CA_ns by CRCA_n.

For an atomic Boolean algebra with operators $\mathfrak{A}$ say, we may write $\mathsf{At}\mathfrak{A}$ to denote its atom structures, i.e. the set of atoms *expanded with the accessiblity relations corresponding to the non- Boolean operations*- which is a first order structure. In modal logic terminology, this atom structure is nothing more than a *Kripke frame*. It will be clear from context what we mean by $\mathsf{At}\mathfrak{A}$ (either the atom structure of $\mathfrak{A}$ or the set of atoms of $\mathfrak{A}$). No confusion is likely to ensue. We write $\mathfrak{A} \subseteq_d \mathfrak{B}$ if $\mathfrak{A}$ is dense subalgebra of $\mathfrak{B}$. Recall that $\mathfrak{A} \subseteq_d \mathfrak{B}$ if $\mathfrak{A}$ is a subalgebra of B, in symbols $\mathfrak{A} \subseteq \mathfrak{B}$, and for all non-zero $b \in \mathfrak{B}$, there exists a non-zero $a \in \mathfrak{A}$ such that $a \leq b$. Let $\mathbf{S_d}$ denote the class of forming dense subalgebas; that is to say, for a class K of Boolean algebras with operators $\mathbf{S_d}\mathsf{K} = \{\mathfrak{A} : (\exists \mathfrak{B} \in \mathsf{K})(\mathfrak{A} \subseteq_d \mathfrak{B})\}$. Given two Boolean algebras with operators $\mathfrak{A}, \mathfrak{B}$ having the same signature, we write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$ in the sense that for all $X \subseteq A$, if $\sum^{\mathfrak{A}} X = 1$ then $\sum^{\mathfrak{B}} X = 1$.[1] We write $\mathbf{S_c}$ for the operation of forming subalgebras, that is to say for a class K of Boolean algebras with operators, $\mathbf{S_c}\mathsf{K} = \{\mathfrak{A} : (\exists \mathfrak{B} \in \mathsf{K})(\mathfrak{A} \subseteq_c \mathfrak{B})\}$. It is known that the class CRCA_n coincides with the class of atomic algebras in $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ as long as the number of atoms is countable [14, Theorem 5.3.6]. However, unlike ordinary reprsentations, this charactrization using complete neat embeddings does not generalize to the uncountable case. This will be proved below in Theorem 1.16, where an atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ having uncountably many atoms but $\mathfrak{A}$ has no complete representation, is constructed.

Define the class LCA_n as follows: $\mathfrak{A} \in \mathsf{LCA}_n \iff \mathfrak{A}$ is atomic and $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{A})$ for all $k < \omega$, where G_k is the k rounded game defined on atomic networks in [7, Definition 3.3.2] truncated to k rounds. Then this class is elementary, because a winning strategy for $\exists$ in G_k can be coded by a first order sentence; call it ρ_k. Hirsch and Hodkinson study the class of atom structures of this class denoted by LCAS_n on [7, p. 73] that they call atom structures satisfying the 'Lyndon conditions' [7]. In our context, working now on the algebra level, the Lyndon conditions that Hirsch and Hodkinson use can be lifted to the algebra level as first order formulas that are just the ρ_ks.

Layout: Fix $2 < n < \omega$. In the following Section 1, the class $\mathbf{El}\mathsf{CRCA}_n$ is characterized using neat embeddings. It is shown that $\mathbf{El}\mathsf{CRCA}_n$ coincides with the elementary class LCA_n defined by the Lyndon conditions and that $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n = \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) = (\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega) \cap \mathbf{At}$, cf. Theorem 1.4. In particular, $\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{LCA}_n$. We show that LCA_n is not finitely axiomatizable, and we prove that RCA_n is generated by $\mathsf{At}(\mathsf{LCA}_n)$ in the following strong sense $\mathsf{RCA}_n = \mathbf{S}\mathfrak{Cm}\mathsf{At}(\mathsf{LCA}_n)$ and by $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ in the weaker sense $\mathsf{RCA}_n = \mathbf{SP}\mathfrak{Cm}\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, cf. Theorem 1.17. We also show that for any $2 < n < l < m$, there exists an atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that its Dedekind–MacNeille completion[2], namely, the complex alge-

[1] This is different from that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is complete.

[2] Sometimes referred to as minimal or Monk completion.

bra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, is outside RCA_n, cf. Theorem 1.12. In Section 2 we continue study atom-canonicity for varieties of cylindric algebras and introduce a new notion of 'degrees of representability' cf. Theorems 2.2, which enables one to measure in a precise sense the degree of representability of a given $\mathfrak{A} \in \mathsf{RCA}_n$; some algebras are more representable than others: Given an atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n$ and $n < m \leq \omega$, then $\mathfrak{A}$ is representable up to m if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. In the final Section 4, using certain atomic games, we characterize the non-elementary class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ and it is shown, using such games, that any class $\mathbf{K}$ such that $\mathsf{CRCA}_n \cap \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary, cf. Theorem 3.1.

1. Complete representations and the Lyndon conditions

Fix a finite ordinal $n > 2$. For a class $\mathbf{K}$, $\mathsf{El}\mathbf{K}$ denotes its elementary closure. By the Keisler-Shelah Ultrapower Theorem, $\mathsf{El}\mathsf{K} = \mathbf{UpUr}\mathsf{K}$ where $\mathbf{Up}(\mathbf{Ur})$ denotes the operation of forming ultraproducts (ultraroots). For a Boolean algebra $\mathfrak{A}$ and $a \in \mathfrak{A}$, $\mathfrak{Rl}_a\mathfrak{A}$ is the Boolean with universe $\{x \in \mathfrak{A} : x \leq a\}$ and Boolean operations those of $\mathfrak{A}$ relativized to the universe. For a Boolean algebra $\mathfrak{A}$, we write $\mathfrak{A}^+$ to denote its canonical extension.

Definition 1.1. [3, Definition 3.1.2] Let α be an ordinal. A *weak space of dimension* α is a set V of the form $\{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ where U is a non-empty set and $p \in {}^\alpha U$. We denote V by ${}^\alpha U^{(p)}$. Following [3], Ws_α denotes the class of weak set algebra of dimension α. The top elements of Ws_αs are weak spaces of dimension α and the operations are defined like in cylindric set algebras of dimension α relativized to the top element.

Observe that when $\alpha < \omega$, $\mathsf{Ws}_\alpha = \mathsf{Cs}_\alpha$. To define certain deterministic games to be used in the sequel, we recall the notions of *atomic networks*, and *atomic games* [6, 7]. Let $i < n$. For n-ary sequences $\bar{x}$ and $\bar{y}$ $\iff$ $\bar{y}(j) = \bar{x}(j)$ for all $j \neq i$.

Definition 1.2. Fix finite $n > 2$ and assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic.

(1) An *n-dimensional atomic network* on $\mathfrak{A}$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non-empty set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$.

For n-dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

(2) Assume that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$*, or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds [7, Definition 3.3.2], where $\forall$ is offered only one move, namely, *a cylindrifier move*: At round zero $\forall$ picks an atom $a \in A$. Then $\exists$ has to respond with a network N and a tuple $\bar{y}$ such that $N(\bar{y}) = a$. Suppose that we are at round $t > 0$. Then $\forall$ picks the played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $x \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

(3) The ω-rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that $\forall$ has the bonus to reuse the m nodes in play.[3]

LEMMA 1.3. *Let* $2 < n < m < \omega$ *and assume that* $\mathfrak{A} \in \mathsf{CA}_n$ *is atomic. If* $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$*, then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$*.*

PROOF: [15, Lemma 4.3]. □

For a class K of BAOs, recall that $\mathsf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in K. Let $\mathsf{Fs}_n = \{\mathfrak{A} \in \mathsf{Cs}_n : A = \wp({}^nU) \text{ some non-empty set } U\}$.

THEOREM 1.4. *For* $2 < n < \omega$ *the following hold:*

1. $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$,
2. *If* $\mathfrak{A} \in \mathsf{CRCA}_n$*, then* $\exists$ *has a winning strategy in* $G_\omega(\mathsf{At}\mathfrak{A})$ *and* $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$,

[3] The games G^m and $\mathbf{G}^m$ are based on a private Ehrenfeucht–Fraïssé deterministic games on two relational structures A and B between two players $\exists$ lloise and $\forall$ belard. Each player chooses a pebble from a particular pebble pair outside the board of the game and places it on one of the structures, A say. The other responds with the other pebble in this pair putting it on the other structure B. The aim of $\exists$ is to show that A and B are alike while the 'spoiler' $\forall$ wants to show that they are different—the 'likeness' here may be measured by existence of isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$, or partial isomorphisms or elementary equivalence, ... etc. In G^m once $\forall$ has chosen a pebble in his private game Ehrenfeucht–Fraïssé game, he cannot use it again. However, in $\mathbf{G}^m$ the pebbles chosen by $\forall$ always remain outside the board of the play, so that $\forall$ has the option to re-use them in every round of the game. This of course makes it harder for $\exists$ to win.

3. *All reverse inclusions and implications in the previous two items hold, if algebras considered have countably many atoms,*

4. *Non of the classes in the first item is elementary,*

5. $\mathsf{CRCA}_n = \mathbf{S_c P}\mathsf{Fs}_n$,

6. $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \nsubseteq \mathsf{CRCA}_n$, $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ *and* $\mathsf{CRCA}_n \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$.

7. *Neither of the classes* CRCA_n *and* $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$ *are contained in each other. In particular,* $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \subsetneq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$.

Proof: 1. Let $\mathfrak{A} \in \mathsf{CRCA}_n$. Assume that M is the base of a complete representation of $\mathfrak{A}$, whose unit is a generalized cartesian space, that is, $1^{\mathsf{M}} = \bigcup {}^nU_i$, where ${}^nU_i \cap {}^nU_j = \emptyset$ for distinct i and j, in some index set I, that is, we have an isomorphism $t : \mathfrak{B} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathsf{Gs}_n$ has unit 1^{M}, and t preserves arbitrary meets carrying them to set-theoretic intersections. For $i \in I$, let $E_i = {}^nU_i$ and pick an arbitrary $f_i \in {}^\omega U_i$ and let W_i be the ω-dimensional weak space $\{f \in {}^\omega U_i^{(f_i)} : |\{k \in \omega : f(k) \neq f_i(k)\}| < \omega\}$. Identifying set algebras with their domain let $\mathfrak{C}_i = \wp(W_i)$. Then $\mathfrak{C}_i \in \mathsf{Ws}_\omega$ and is atomic; indeed the atoms are the singletons sets $\{f\}$ for $f \in W_i$. Note, for $f, g \in W_i$ ad $i < \omega$ if $f \upharpoonright \omega \sim \{i\} = g \upharpoonright \omega \sim \{i\}$, then $\{f\} \leq C_i\{g\}$.

Let $x \in \mathfrak{Nr}_n\mathfrak{C}_i$, that is $\mathsf{c}_i x = x$ for all $n \leq i < \omega$. Now if $f \in x$ and $g \in W_i$ satisfy $g(k) = f(k)$ for all $k < n$, then $g \in x$ because $|\{n \leq i < \omega : f(i) \neq g(i)\}| < \omega$. Hence $\mathfrak{Nr}_n\mathfrak{C}_i$ is atomic; its atoms are $\{\{g \in W_i : \{g(i) = d : i < n\}, d \in U_i\}$. Define $h_i : \mathfrak{A} \to \mathfrak{Nr}_n\mathfrak{C}_i$ by $h_i(a) = \{f \in W_i : \exists a' \in \mathsf{At}\mathfrak{A}, a' \leq a; (f(i) : i < n) \in t(a')\}$. Let $\mathfrak{D} = \mathbf{P}_i\mathfrak{C}_i$. Let $\pi_i : \mathfrak{D} \to \mathfrak{C}_i$ be the ith projection map. Now clearly $\mathfrak{D}$ is atomic, because it is a product of atomic algebras, and its atoms are $(\pi_i(\beta) : \beta \in \mathsf{At}(\mathfrak{C}_i))$. Now $\mathfrak{A}$ embeds into $\mathfrak{Nr}_n\mathfrak{D}$ via $J : a \mapsto (\pi_i(a) : i \in I)$. If $x \in \mathfrak{Nr}_n\mathfrak{D}$, then for each i, we have $\pi_i(x) \in \mathfrak{Nr}_n\mathfrak{C}_i$, and if x is non-zero, then $\pi_i(x) \neq 0$. By atomicity of $\mathfrak{C}_i$, there is an n-ary tuple y, such that $\{g \in W_i : g(k) = y_k\} \subseteq \pi_i(x)$. It follows that there is an atom of $b \in \mathfrak{A}$, such that $y \in t(b)$. Hence $\{g \in U_i : g(i) = y_i\} \subseteq \pi_i(< x \cdot J(b) >$, so $x \cdot J(b) \neq 0$, and so the embedding is atomic, hence complete. We have shown that $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, and since $\mathfrak{A}$ is atomic because $\mathfrak{A} \in \mathsf{CRCA}_n$ we are done with the first inclusion. The second inclusion is straightforward since $\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{CA}_\omega$.

2. [7, Theorem 3.3.3]. Follows too from the first item taken together with lemma 1.3.

3. Follows by observing that the class $\mathsf{CRCA_n}$ coincides with the class $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ on atomic algebras having countably many atoms, cf. [14, Theorem 5.3.6], taken together with [7, Theorem 3.3.3]. Strictly speaking, in [14] it is shown that the two classes CRCA_n and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ coincide on countable atomic algebras. One can show that they coincide on the larger class of atomic agebras having countably many atoms by observing that if $\mathfrak{A}$ is an atomic algebra having countably many atoms, then $\mathfrak{TmAt}\mathfrak{A}$ is countable and $\mathfrak{TmAt}\mathfrak{A} \in \mathsf{CRCA}_n \iff \mathfrak{A} \in \mathsf{CRCA}_n$.

4. To show that non of the classes in the first item is elementary, let $\mathfrak{D}$ be an atomic RCA_n with countably many atoms that is not completely representable, but is elementary equivalent to some $\mathfrak{B} \in \mathsf{CRCA}_n$. Such algebras exist; see e.g. [5]. Another such algebra is the algebra $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ used in theorem 3.1 below. Then $\mathfrak{D}$ is not in any of the aforementiond classes because it has countably many atoms, and by the first item $\mathfrak{B}$ is in all three classes, proving the required.

5. The inclusion $\subseteq$ is straightforward. Conversely, assume that $\mathfrak{A} \subseteq_c \mathbf{P}_{i\in I}\wp({}^nU_i)$. Then $\mathfrak{B} = \mathbf{P}_{i\in I}\wp({}^nU_i) \cong \wp(V)$, where V is the disjoint union of the ${}^nU_i$, is clearly completely representable. Then since $\mathfrak{A} \subseteq_c \mathfrak{B}$, and so $\mathfrak{A}$ is completely representable, too.

6. First $\nsubseteq$ follows from the construction in [12], cf. corollary 1.16 for more details. Second $\subsetneq$ follows from item (3) of Theorem 2.2. Last $\subsetneq$ follows from the first two parts in this item together with the inclusions in the first item.

7. That $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \nsubseteq \mathsf{CRCA}_n$ follows from the first part of item (6) of theorem 1.4, cf. also corollary 1.16. To show that, conversely $\mathsf{CRCA}_n \nsubseteq \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, we slighty modify the construction in [14, Lemma 5.1.3, Theorem 5.1.4] lifted to any finite $n > 2$. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in *op. cit.* satisfy that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but it can be fixed that they are atomic, giving the same result, by interpreting the uncountably many n-ary relations in the signature of $\mathfrak{M}$ defined in [14, Lemma 5.1.3] for $n = 3$, which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be *disjoint* in $\mathfrak{M}$, not just distinct. In fact the construction is presented in this way in [11]. Let us explain why. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in *op. cit.* lifts verbatim to any such n. Let $u \in {}^nn$. Write $\mathbf{1}_u$ for $\chi_u^{\mathfrak{M}}$ (denoted by 1_u (for $n = 3$) in [14, Theorem 5.1.4].) We denote by $\mathfrak{A}_u$ the Boolean

algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathbf{1}_u\}$ and similarly for $\mathfrak{B}$, writing $\mathfrak{B}_u$ short hand for the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathbf{1}_u\}$. Using that $\mathfrak{M}$ has quantifier elimination we get, using the same argument in *op. cit.* that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. The property that $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ is also still maintained. To see why consider the substitution operator ${}_n\mathsf{s}(0,1)$ (using one spare dimension) as defined in the proof of [14, Theorem 5.1.4].

Assume for contradiction that $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$, with $\mathfrak{C} \in \mathsf{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots, n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term ${}_n\mathsf{s}(0,1)$ acts like a substitution operator corresponding to the transposition $[0, 1]$; it 'swaps' the first two coordinates. Now one can show that ${}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u \subseteq \mathfrak{B}_{[0,1]\circ u} = \mathfrak{B}_{Id}$, so $|{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{Id}$ was forced by construction to be countable. But ${}_n\mathsf{s}(0,1)$ is a Boolean automorpism with inverse ${}_n\mathsf{s}(1,0)$, so that $|\mathfrak{B}_{Id}| = |{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u| > \omega$, contradiction. One proves that $\mathfrak{A} \equiv \mathfrak{B}$ exactly like in [14]. Take the cardinality κ specifying the signature of $\mathfrak{M}$ to be 2^{2^ω} and assume for contradiction that $\mathfrak{B} \in \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Then $\mathfrak{B} \subseteq_d \mathfrak{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \mathsf{CA}_\omega$ and $\mathfrak{Nr}_n\mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathfrak{Nr}_n\mathfrak{D}$. Then $\mathfrak{B}_{Id} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$; the last algebra is the Boolean algebra with universe $\{x \in \mathfrak{C} : x \leq Id\}$. Since $\mathfrak{C}$ is atomic, then $\mathfrak{Rl}_{Id}\mathfrak{C}$ is also atomic.

Using the same reasoning as above, we get that $|\mathfrak{Rl}_{Id}\mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} \in \mathsf{Nr}_n\mathsf{CA}_\omega$). By the choice of κ, we get that $|\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. By $\mathfrak{B} \subseteq_d \mathfrak{C}$, we get that $\mathfrak{B}_{Id} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$, and that $\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C} \subseteq \mathsf{At}\mathfrak{B}_{Id}$, so $|\mathsf{At}\mathfrak{B}_{Id}| \geq |\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{B}_{Id}| = |\mathsf{At}\mathfrak{B}_{Id}| = \omega$, which is a contradiction and we are done. The algebra $\mathfrak{B}$ so constructed is atomic and is outside $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega$. Furthermore, $\mathfrak{B} \in \mathsf{CRCA}_n$ because $\mathfrak{B} \in \mathsf{Gs}_n$ and $\bigcup \mathsf{At}\mathfrak{B} = \bigcup_{u \in {}^n n} \bigcup \mathsf{At}\mathfrak{B}_u = \bigcup_{u \in {}^n n} \mathbf{1}_u = 1^{\mathfrak{B}}$. Thus the identity may establishes a complete representation of $\mathfrak{B}$. $\square$

Here we review and elaborate on the construction in [2] as our first instance of a so-called *blow up and blur construction* in the sense of [16]. This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive Boolean algebras with opertors (BAOs). One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), *blows it up*, by *splitting*[4] *one or more of its atoms each to infinitely many subatoms*, obtaining an

[4]The idea of splitting one or more atoms in an algebra to get a (bigger) superalgebra tailored to a certain purpose seems to originate with Henkin [3, p. 378, footnote 1] to be reinvented by Hajnal Andréka as a nutcracker for proving non-finite axiomatizability results for varieties of RCA_n.

(infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind–MacNeille completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$. Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom-canonical, for $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom-canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ in [6, SS 17.7] and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathfrak{Ra}\mathsf{CA}_k$ for all $k \geq 3$ in [2]; the construction in [2] will be generalized below, and will applied below to $\mathbf{K} = \mathsf{SNr}_n\mathsf{CA}_{n(n+1)/2}$ and $\mathbf{L} = \mathsf{RCA}_n$, where $\mathfrak{Ra}$ denotes the operator of forming relation algebra reducts (applied to classes) of CAs, respectively, cf. [3, Definition 5.2.7].

Definition 1.5. Let $\mathfrak{R}$ be an atomic relation algebra. An n-dimensional basic matrix, or simply a matrix on $\mathfrak{R}$, is a map $f : {}^2n \to \mathsf{At}\mathfrak{R}$ satsfying the following two consistency conditions $f(x, x) \leq \mathsf{Id}$ and $f(x, y) \leq f(x, z); f(z, y)$ for all $x, y, z < n$. For any f, g basic matrices and $x, y < m$ we write $f \equiv_{xy} g$ if for all $w, z \in m \setminus \{x, y\}$ we have $f(w, z) = g(w, z)$. We may write $f \equiv_x g$ instead of $f \equiv_{xx} g$.

Definition 1.6. An *n-dimensional cylindric basis* for an atomic relaton algebra $\mathfrak{R}$ is a set $\mathsf{CA}lM$ of n-dimensional matrices on $\mathfrak{R}$ with the following properties:

- If $a, b, c \in \mathsf{At}\mathfrak{R}$ and $a \leq b; c$, then there is an $f \in \mathsf{CA}lM$ with $f(0, 1) = a$, $f(0, 2) = b$ and $f(2, 1) = c$
- For all $f, g \in \mathsf{CA}lM$ and $x, y < n$, with $f \equiv_{xy} g$, there is $h \in \mathsf{CA}lM$ such that $f \equiv_x h \equiv_y g$.

For the next lemma, we refer the reader to [6, Definition 12.11] for the definition of of hyperbasis for relation algebras as well as to [6, Chapter 13, Definitions 13.4, 13.6] for the notions of n-flat and n-square representations for relation algebras $(n > 2)$ For a relation algebra $\mathfrak{R}$, recall that $\mathfrak{R}^+$ denotes its canonical extension.

Lemma 1.7. *Let $\mathfrak{R}$ be a relation algebra and $3 < n < \omega$. Then the following hold:*

1. *$\mathfrak{R}^+$ has an n-dimensional infinite basis $\iff$ $\mathfrak{R}$ has an infinite n-square representation.*

2. $\mathfrak{R}^+$ *has an* n*-dimensional infinite hyperbasis* $\iff$ $\mathfrak{R}$ *has an infinite* n*-flat representation.*

Proof: [6, Theorem 13.46, the equivalence (1) $\iff$ (5) for basis, and the equivalence (7) $\iff$ (11) for hyperbasis]. □

One can construct a CA_n in a natural way from an n-dimensional cylindric basis which can be viewed as an atom structure of a CA_n (like in [6, Definition 12.17] addressing hyperbasis). For an atomic relation algebra $\mathfrak{R}$ and $l > 3$, we denote by $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ the set of all n-dimensional basic matrices on $\mathfrak{R}$. $\mathsf{Mat}_n(\mathsf{At}\mathfrak{R})$ is not always an n-dimensional cylindric basis, but sometimes it is, as will be the case described next. On the other hand, $\mathsf{Mat}_3(\mathsf{At}\mathfrak{R})$ is always a 3-dimensional cylindric basis; a result of Maddux's, so that $\mathfrak{Cm}\mathsf{Mat}_3(\mathsf{At}\mathfrak{R}) \in \mathsf{CA}_3$. The following definition to be used in the sequel is taken from [2]:

Definition 1.8. [2, Definition 3.1] Let $\mathfrak{R}$ be a relation algebra, with non-identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$.

1. We say that (J, E) is an n*-blur* for $\mathfrak{R}$, if J is a *complex* n*-blur* defined as follows:

   (a) Each element of J is non-empty,

   (b) $\bigcup J = I$,

   (c) $(\forall P \in I)(\forall W \in J)(I \subseteq P; W)$,

   (d) $(\forall V_1, \dots V_n, W_2, \dots W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$, that is there is for $v \in V_i$, $w \in W_i$ and $t \in T$, we have $v; w \leq t$,

   (e) $(\forall P_2, \dots P_n, Q_2, \dots Q_n \in I)(\forall W \in J) W \cap P_2; Q_n \cap \dots P_n; Q_n \neq \emptyset$.

   and the tenary relation E is an *index blur* defined as in item (ii) of [2, Definition 3.1].

2. We say that (J, E) is a *strong* n*-blur*, if it (J, E) is an n-blur, such that the complex n-blur satisfies:

$$(\forall V_1, \dots V_n, W_2, \dots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T).$$

Definition 1.9. An atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$ is *strongly representable* if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$.

LEMMA 1.10. *Let* $\mathfrak{A} \in \mathsf{CA}_n$ *be completely representable. Then* $\mathfrak{A}$ *is strongly representable.*

PROOF: Since $\mathfrak{A}$ is completely representable, then it is atomic. Let $f : \mathfrak{A} \to \mathfrak{B}$ be a complete representation of $\mathfrak{A}$ via f, with $\mathfrak{B} \in \mathsf{Gs}_n$. Then one extends f to $\hat{f}$ from $\mathfrak{CmAt}\mathfrak{A}$ to $\mathfrak{B}$ by defining $\hat{f}(a) = \sum^{\mathfrak{CmAt}\mathfrak{A}}_{x \in \mathsf{At}\mathfrak{A}, x \leq a} f(x)$. The last suprema is well defined because $\mathfrak{CmAt}\mathfrak{A}$ is complete. It is easy to check that $\hat{f}$ is an isomorphism and so $\mathfrak{CmAt}\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, hence, by definition, $\mathfrak{CmAt}\mathfrak{A}$ is representable. □

DEFINITION 1.11. A completely additive variety V of BAOs is *atom-canonical* if whenever $\mathfrak{A} \in \mathsf{V}$, then its Dedekind–MacNeille completion, which is the complex algebra of its atom structure, namely, $\mathfrak{CmAt}\mathfrak{A}$, is also in V,

Monk prove that CA_n is atom-canonical; this follows from the fact that CA_n can be axiomatized by positive in the wider sense equations, which are are an instance of Sahlqvist equations. However, the variety RCA_n is not atom-canonical; a result of Hodkinson's [10]. We reprove the last result differently based on the construction in [2].

THEOREM 1.12. *For any* $2 < n < l < \omega$, *there is an atomic algebra* $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, *but* $\mathfrak{CmAt}\mathfrak{B} \notin \mathsf{RCA}_n$. *In particular,* $\mathfrak{B}$ *is not completely representable a fortiori* $\mathfrak{B}$ *is not strongly representable, and* RCA_n *is not atom-canonical.*

PROOF: Let $2 < n < m \leq \omega$. First we prove the conditionally the non-atom canonicity of $\mathsf{SNr}_n\mathsf{CA}_m$ depending on the existence of a certain finite relation algebra $\mathfrak{R}$ with strong m blur- satisfying a condition that we highlight as we go along. We use the flexible blow up and blur construction used in [2]. The idea is to use $\mathfrak{R}$ in place of the finite Maddux algebras denoted by $\mathfrak{E}_k(2,3)$ on [2, p. 83]. Here $k(<\omega)$ is the number of non-identity atoms and then take it from there to reach the conditions, we move backwards if you like. The required algebra witnessing non-atom canonicity will be obtained by blowing up and blurring $\mathfrak{R}$ in place of the relation algebra $\mathfrak{E}_k(2,3)$ [2].

Our exposition addresses an (abstract) finite relation algebra $\mathfrak{R}$ having an l-blur in the sense of definition [2, Definition 3.1], with $3 \leq l \leq k < \omega$ and k depending on l. Occasionally we use the concrete Maddux algebra $\mathfrak{E}_k(2,3)$ to make certain concepts more tangible. We use the notation in [2]. Let $2 < n \leq l < \omega$. One starts with a finite relation algebra $\mathfrak{R}$ that has only representations, if any, on finite sets (bases), having an l-blur (J, E) as in [2, Definition 3.1] recalled in definition 1.8. After *blowing up and bluring* $\mathfrak{R}$, by splitting each of its atoms into infinitely many, one gets an infinite atomic representable relation algebra $\mathsf{Bb}(\mathfrak{R}, J, E)$ [2, p. 73], whose atom structure $\mathbf{At}$ is weakly but not strongly representable. The atom structure $\mathbf{At}$ is not strongly representable, because $\mathfrak{R}$ is *not blurred* in $\mathsf{Cm}\mathbf{At}$. The finite relation algebra $\mathfrak{R}$ embeds into $\mathfrak{Cm}\mathbf{At}$, so that a representation of $\mathfrak{Cm}\mathbf{At}$, necessarily on an infinite base, induces one of $\mathfrak{R}$ on the same base, which is impossible. The representability of $\mathsf{Bb}(\mathfrak{R}, J, E)$ depend on the properties of the l-blur, which *blurs* $\mathfrak{R}$ *in* $\mathsf{Bb}(\mathfrak{R}, J, E)$. The set of blurs here, namely, J is finite. In the case of $\mathfrak{E}_k(2, 3)$ used in [2], the set of blurs is the set of all subsets of non-identity atoms having the same size $l < \omega$, where $k = f(l) \geq l$ for some recursive function f from $\omega \to \omega$, so that k depends recursively on l.

One (but not the only) way to define the *index blur* $E \subseteq {}^3\omega$ is as follows [13, Theorem 3.1.1]: $E(i, j, k) \iff (\exists p, q, r)(\{p, q, r\} = \{i, j, k\}$ and $r - q = q - p$. This is a concrete instance of an index blur as defined in [2, Definition 3.1(iii)] (recalled in definition 1.8 above), but defined uniformly, it does not depends on the blurs. The underlying set of $\mathbf{At}$, the atom structure of $\mathsf{Bb}(\mathfrak{R}, J, E)$ is the following set consisting of triplets: $At = \{(i, P, W) : i \in \omega, P \in \mathsf{At}\mathfrak{R} \sim \{\mathsf{Id}\}, W \in J\} \cup \{\mathsf{Id}\}$. When $\mathfrak{R} = \mathfrak{E}_k(2, 3)$ (some finite $k > 0$), composition is defined by singling out the following (together with their Peircian transforms), as the consistent triples: (a, b, c) is consistent $\iff$ one of a, b, c is Id and the other two are equal, or if $a = (i, P, S), b = (j, Q, Z), c = (k, R, W)$

$$S \cap Z \cap W \neq \emptyset \implies E(i, j, k) \& |\{P, Q, R\}| \neq 1.$$

(We are avoiding mononchromatic triangles). That is if for $W \in J$, $E^W = \{(i, P, W) : i \in \omega, P \in W\}$, then

$$(i, P, S); (j, Q, Z) = \bigcup \{E^W : S \cap Z \cap W = \emptyset\}$$
$$\bigcup \{(k, R, W) : E(i, j, k), |\{P, Q, R\}| \neq 1\}.$$

More generally, for the $\mathfrak{R}$ as postulated in the hypothesis, composition in $\mathbf{At}$ is defined as follow. First the index blur E can be taken to be like above. Now the triple $((i, P, S), (j, Q, Z), (k, R, W))$ in which no two entries are equal, is consistent if either S, Z, W are *safe*, briefly $\mathsf{safe}(S, Z, W)$, witness item (4) in definition 1.8 (which vacuously hold oif $S \cap Z \cap W = \emptyset$), or $E(i, j, k)$ and $P ; Q \leq R$ in $\mathfrak{R}$. This generalizes the above definition of composition, because in $\mathfrak{E}_k(2, 3)$, the triple of non-identity atoms (P, Q, R) is consistent $\iff$ they do not have the same colour $\iff |\{P, Q, R\}| \not= 1$. Having specified its atom structure, its timely to specfiy the relation algebra $\mathsf{Bb}(\mathfrak{R}, J, E) \subseteq \mathfrak{Cm}\mathbf{At}$. The relation algebra $\mathsf{Bb}(\mathfrak{R}, J, E)$ is $\mathfrak{Tm}\mathbf{At}$ (the term algebra). Its universe is the set $\{X \subseteq H \cup \{\mathsf{Id}\} : X \cap E^W \in \mathsf{Cof}(E^W), \text{ for all } W \in J\}$, where $\mathsf{Cof}(E^W)$ denotes the set of co-finite subsets of E^W, that is subsets of E^W whose complement is infinite, with E^W as defined above. The relation algebra operations lifted from $\mathbf{At}$ the usual way. The algebra $\mathsf{Bb}(\mathfrak{R}, J, E)$ is proved to be representable [2].

For brevity, denote $\mathsf{Bb}(\mathfrak{R}, J, E)$ by $\mathsf{CA}lR$, and its domain by R. For $a \in \mathbf{At}$, and $W \in J$, set $U^a = \{X \in R : a \in X\}$ and $U^W = \{X \in R : |X \cap E^W| \geq \omega\}$. Then the principal ultrafilters of $\mathsf{CA}lR$ are exactly U^a, $a \in H$ and U^W are non-principal ultrafilters for $W \in J$ when E^W is infinite. Let $J' = \{W \in J : |E^W| \geq \omega\}$, and let $\mathsf{Uf} = \{U^a : a \in F\} \cup \{U^W : W \in J'\}$. Uf is the set of ultrafilters of $\mathsf{CA}lR$ which is used as colours to represent $\mathsf{CA}lR$, cf. [2, pp. 75–77]. The representation is built from coloured graphs whose edges are labelled by elements in Uf in a fairly standard step-by-step construction. The step-by-step construction builds in the way coloured graphs, which are basically networks whose edges are labelled by ultrafilters, with non-principal ultrafilters allowed. So such coloured graphs are networks that are *not atomic* because not only principal ultrafilters are allowed as labels. Furthermore, we *cannot restrict our attension to only atomic networks* because we do not want $\mathsf{Bb}(\mathfrak{R}, J, E)$ to be strongly representable, least completely representable. The 'limit' of a sequence of atomic networks constructed in a step-by-step manner, or obtained via winning strategystrategy for $\exists$ in an ω-rounded atomic game, will necessarily produce a complete representation of $\mathsf{Bb}(\mathfrak{R}, J, E)$. But the required representation will be extracted from *a complete representation* of the *canonical extension of* $\mathsf{Bb}(\mathfrak{R}, J, E)$. Nothing wrong with that. A relation algebra $\mathsf{CA}lR$ is representable $\iff$ its canonical extension is representable. A complete representation of the canonical extension of $\mathsf{CA}lR$ induces a representation of $\mathsf{CA}lR$, because $\mathsf{CA}lR$ embeds into its a canonical extension, but the converse is not necessarily true. So here we are proving more than the mere representablity of $\mathfrak{Bb}(\mathfrak{R}, J, E)$, because we are constructing a complete representation of its canonical extension, namely, the algebra $\mathfrak{Cm}\mathbf{Uf}$, where $\mathbf{Uf}$ is the atom structure having domain Uf, with Uf as defined above.

Now we show why the Dedekind–MacNeille completion $\mathfrak{Cm}\mathbf{At}$ is *not* representable. For $P \in I$, let $H^P = \{(i, P, W) : i \in \omega, W \in J, P \in W\}$. Let $P_1 = \{H^P : P \in I\}$ and $P_2 = \{E^W : W \in J\}$. These are two partitions of At. The partition P_2 was used to *represent*, $\mathsf{Bb}(\mathfrak{R}, J, E)$, in the sense that the tenary relation corresponding to composition was defined on $\mathbf{At}$, in a such a way so that the singletons generate the partition $(E^W : W \in J)$ up to "finite deviations." The partition P_1 will now be used to show that $\mathfrak{Cm}(\mathsf{Bb}(\mathfrak{R}, J, E)) = \mathfrak{Cm}(\mathbf{At})$ is *not* representable. This follows by observing that omposition restricted to P_1 satisfies: $H^P; H^Q = \bigcup\{H^Z : Z; P \leq Q \text{ in } \mathfrak{R}\}$ which means that $\mathfrak{R}$ embeds into the complex algebra $\mathfrak{Cm}\mathbf{At}$ prohibiting its representability, because $\mathfrak{R}$ allows only representations having a finite base.

The construction lifts to higher dimensions expressed in CA_ns, $2 < n < \omega$. Because (J, E) is an l-blur, then by [2, Theorem 3.2 9(iii)], $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathsf{AtBb}(\mathfrak{R}, J, E))$, the set of l by l basic matrices on $\mathbf{At}$ is an l-dimensional cylindric basis, giving an algebra $\mathfrak{B}_l = \mathsf{Bb}_l(\mathfrak{R}, J, E) \in \mathsf{RCA}_l$. Again $\mathbf{At}_{ca}$ is not strongly representable, for had it been then a representation of $\mathfrak{Cm}\mathbf{At}_{ca}$, induces a representation of $\mathfrak{R}$ on an infinite base, because $\mathfrak{Ra}\mathfrak{Cm}\mathbf{At}_{ca} \supseteq \mathfrak{Cm}\mathbf{At} \supseteq \mathfrak{R}$, and the representability of $\mathfrak{Cm}\mathbf{At}_{ca}$ induces one of $\mathfrak{Ra}\mathfrak{Cm}\mathbf{At}_{ca}$, necessarily having an infinite base. For $2 < n \leq l < \omega$, denote by $\mathfrak{C}_l$ the non-representable Dedekind–MacNeille completion of the algebra $\mathsf{Bb}_l(\mathfrak{R}, J, E) \in \mathsf{RCA}_l$, that is $\mathfrak{C}_l = \mathfrak{Cm}\mathsf{At}(\mathsf{Bb}_l(\mathfrak{R}, J, E)) = \mathfrak{Cm}\mathsf{Mat}_l(\mathbf{At})$. If the l-blur happens to be *strong*, in the sense of definition 1.8 and $n \leq m \leq l$, then we get by [2, item (3), p. 80], that $\mathsf{Bb}_m(\mathfrak{R}, J, E) \cong \mathsf{Nr}_m\mathsf{Bb}_l(\mathfrak{R}, J, E)$. This is proved by defining an embedding $h : \mathfrak{Rd}_m\mathfrak{C}_l \to \mathfrak{C}_m$ via $x \mapsto \{M \restriction m : M \in x\}$ and showing that $h \restriction \mathsf{Nr}_m\mathfrak{C}_l$ is an isomorphism onto $\mathfrak{C}_m$ [2, p. 80]. Surjectiveness uses the condition $(J5)_l$ formulated in the second item of definition 1.8 of strong l-blurness. Without this condition, that is if the l-blur (J, E) is not strong, then still $\mathfrak{C}_m$ and $\mathfrak{C}_l$ can be defined because by definition (J, E) is an t-blur for all $m \leq t \leq l$, so $\mathsf{Mat}_\mathsf{t}(\mathbf{At})$ is a cylindric basis and for $t < l$ $\mathfrak{C}_t$ embeds into $\mathsf{Nr}_m\mathfrak{C}_l$ using the same above map, but this embedding might not be surjective. So for every l, now replacing $\mathfrak{R}$ by the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, the algebra $\mathfrak{A}_l = \mathsf{Nr}_n\mathsf{Bb}_l(\mathfrak{E}_{f(l)}(2, 3)), J, E)$– with $f(l)$ depending recursively on l, having strong l-blur due to the properties of the Maddux algebra $\mathfrak{E}_{f(l)}(2, 3)$, is as required. In other words, and more concisely, we have $\mathfrak{A}_l \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{A}_l \notin \mathsf{RCA}_n$. □

The following Theorem summarizes the proof of the previous Theorem, generalizes the construction in [2] and says some more new facts. We use the notation $\mathfrak{Bb}(\mathfrak{R}, J, E)$ with atom structure $\mathbf{At}$ obtained by blowing up and blurring $\mathfrak{R}$ with underlying set is denoted by At on [2, p. 73] and is recalled in the previous proof. The algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)(\in \mathsf{CA}_l)$ is defined in [2, top of p. 78] and also in the immediately previous proof.

A CA_n atom structure $\mathbf{At}$ is *weakly representable* if there is an atomic $\mathfrak{A} \in \mathsf{RCA}_n$ such that $\mathbf{At} = \mathsf{At}\mathfrak{A}$; recall that it is strongly representable if $\mathfrak{Cm}\mathbf{At} \in \mathsf{RCA}_n$. These two notions are distinct as proved in Theorem 1.12.

THEOREM 1.13. *Let $2 < n \leq l < m \leq \omega$.*

1. *Let $\mathfrak{R}$ be a finite relation algebra with an l-blur (J, E) where J is the l-complex blur and E is the index blur.*

   (a) Let $\mathbf{At}$ be the relation algebra atom structure obtained by blowing up and blurring $\mathfrak{R}$ as specified above. Then the set of l by l-dimensional matrices $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$ is an l-dimensional cylindric basis, that is a weakly representable atom structure [2, Theorem 3.2]. The algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$ with atom structure $\mathbf{At}_{ra}$ is in RCA_l. Furthermore, $\mathfrak{R}$ embeds into $\mathfrak{Cm}\mathbf{At}$ which embeds into $\mathfrak{RaCm}(\mathbf{At}_{ca})$.

   (b) If (J, E) is a strong m-blur for $\mathfrak{R}$, then (J, E) is a strong l-blur for $\mathfrak{R}$. Furthermore, $\mathfrak{Bb}_l(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_l\mathfrak{Bb}_m(\mathfrak{R}, J, E)$ and for any $l \leq j \leq m$, $\mathfrak{Bb}(\mathfrak{R}, J, E)$ having atom structure $\mathbf{At}$, is isomorphic to $\mathfrak{Ra}(\mathfrak{Bb}_j(\mathfrak{R}, J, E))$.

2. *For every $n < l$, there is an $\mathfrak{R}$ having a strong l-blur (J, E) but no infinite representations (representations on an infinite base). Hence the atom structures defined in (a) of the previous item (denoted by $\mathbf{At}$ and $\mathbf{At}_{ca}$) for this specific $\mathfrak{R}$ are not strongly representable.*

3. *Let $m < \omega$. If $\mathfrak{R}$ is a finite relation algebra having a strong l-blur, and no m-dimensional hyperbasis, then $l < m$.*

4. *If $n = l < m < \omega$ and $\mathfrak{R}$ is a finite relation algebra with an n blur (J, E) (not necessarily strong) and no infinite m-dimensional hyperbasis, then the algebras $\mathfrak{Cm}\mathsf{At}(\mathfrak{Bb}(\mathfrak{R}, J, E))$ and $\mathfrak{Cm}\mathsf{At}(\mathfrak{Bb}_l(\mathfrak{R}, J, E))$ are outside $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$ and $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, respectively, and the latter two varieties are not atom-canonical.*

Proof: [2, Lemmata 3.2, 4.2, 4.3]. We start by an outline of (a) of item 1. Let $\mathfrak{R}$ be as in the hypothesis. Let $3 < n \leq l$. We blow up and blur $\mathfrak{R}$. $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many defining an (infinite atoms) structure $\mathbf{At}$. $\mathfrak{R}$ is blurred by using a finite set of blurs (or colours) J. The term algebra denoted in [2] by $\mathfrak{Bb}(\mathfrak{R}, J, E)$) over $\mathbf{At}$, is representable using the finite number of blurs. Such blurs are basically non-principal ultrafilters; they are used as colours together with the principal ultrafilters (the atoms) to represent $\mathfrak{Bb}(\mathfrak{R}, J, E)$. This representation is implemented in step-by-step manner, and in fact this step by step construction adopted in [2] *completely represents the canonical extension* of $\mathfrak{Bb}(\mathfrak{R}, J, E)$. Because (J, E) is a complex set of l-blurs, this atom structure has an l-dimensional cylindric basis, namely, $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$. The resulting l-dimensional cylindric term algebra $\mathfrak{TmMat}_l(\mathbf{At})$, and an algebra $\mathfrak{C}$ having atom structure $\mathbf{At}_{ca}$ (denoted in [2] by $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$) such that $\mathfrak{TmMat}_l(\mathbf{At}) \subseteq \mathfrak{C} \subseteq \mathfrak{CmMat}_l(\mathbf{At})$ is shown to be representable.
We prove (b) of item (1): Assume that the m-blur (J, E) is strong, then by definition (J, E) is a strong j blur for all $n \leq j \leq m$. Furthermore, by [2, item (3), p. 80], $\mathfrak{Bb}(\mathfrak{R}, J, E) = \mathfrak{Ra}(\mathfrak{Bb}_j(\mathfrak{R}, J, E))$ and $\mathfrak{Bb}_j(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_j\mathfrak{Bb}_m(\mathfrak{R}, J, E)$.

2. Like in [2, Lemma 5.1], one takes $l \geq 2n-1$, $k \geq (2n-1)l$, $k \in \omega$. The Maddux integral relation algebra $\mathfrak{E}_k(2,3)$ where k is the number of non-identity atoms is the required $\mathfrak{R}$. In this algebra a triple (a, b, c) of non-identity atoms is consistent $\iff$ $|\{a, b, c\}| \neq 1$, i.e only monochromatic triangles are forbidden.

3. Let (J, E) be the strong l-blur of $\mathfrak{R}$. Assume for contradiction that $m \leq l$. Then we get by [2, item (3), p. 80], that $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}, J, E)$. But the cylindric l-dimensional algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$ is atomic, having atom structure $\mathsf{Mat}_l\mathsf{At}(\mathsf{split}(\mathfrak{R}, J, E))$, so $\mathfrak{A}$ has an atomic l-dilation. So $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{D}$ where $\mathfrak{D} \in \mathsf{CA}_l$ is atomic. But $\mathfrak{R} \subseteq_c \mathfrak{Ra}\mathfrak{Nr}_n\mathfrak{D} \subseteq_c \mathfrak{Ra}\mathfrak{D}$. By [6, Theorem 13.45 (6) $\iff$ (9)], $\mathfrak{R}$ has a complete l-flat representation, thus it has a complete m-flat representation, because $m < l$ and $l \in \omega$. This is a contradiction.

4. Let $\mathfrak{B} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. Then, since (J, E) is an n blur, $\mathfrak{B} \in \mathsf{RCA}_n$. But $\mathfrak{C} = \mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, because $\mathfrak{R} \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$, $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which, in turn, embeds into $\mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{B}$. Similarly, $\mathfrak{Bb}(\mathfrak{R}, J, E) \in \mathsf{RRA}$ and $\mathfrak{Cm}(\mathsf{At}\mathfrak{Bb}(\mathfrak{R}, J, E)) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_m$. Hence the alledged varieties are not atom-canonical. $\square$

THEOREM 1.14. *Let $2 < n < \omega$. Then LCA_n is an elementary class that is not finitely axiomatizable.*

PROOF: For each $2 < n \leq l < \omega$, let $\mathfrak{R}_l$ be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$, as defined on [2, p. 83, S5, in the proof of Theorem 5.1] with l-blur (J_l, E_l) as defined in [2, Definition 3.1] and $f(l) \geq l$ as specified in [2, Lemma 5.1] (denoted by k therein). Let $\mathsf{CA}lR_l = \mathfrak{Bb}(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RRA}$ where $\mathsf{CA}lR_l$ is the relation algebra having atom structure denoted At in [2, p. 73] when the blown up and blurred algebra denoted $\mathfrak{R}_l$ happens to be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathfrak{R}_l, J_l, E_l) \in \mathsf{RCA}_n$ as defined in [2, top of p. 80] (with $\mathfrak{R}_l = \mathfrak{E}_{f(l)}(2,3)$). Then $(\mathsf{AtCA}lR_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. $\square$

We have shown that the three classes in the first item of the theorem 1.4 are not elementary and in the last item of *op. cit.* that at least two are distinct. Now we show that their elementary closure coincide with the class LCA_n.

THEOREM 1.15. *Let $2 < n < \omega$. Then:*

$$\begin{aligned}\mathbf{El}\mathsf{CRCA}_n &= \mathbf{El}[\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}]\\ &= \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}\\ &= \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})\\ &= \mathsf{LCA}_n.\end{aligned}$$

PROOF: We show, as claimed, that all the given classes coincide with LCA_n. Assume that $\mathfrak{A} \in \mathsf{LCA}_n$. Take a countable elementary subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Since LCA_n is elementary, then $\mathfrak{C} \in \mathsf{LCA}_n$, so for $k < \omega$, $\exists$ has a winning strategyρ_k, in $G_k(\mathsf{At}\mathfrak{C})$. Let $\mathfrak{D}$ be a non-principal ultrapower of $\mathfrak{C}$. Then $\exists$ has a winning strategyσ in $G_\omega(\mathsf{At}\mathfrak{D})$ [7, Theorem 3.3.4]. Essentially she uses ρ_k in the k'th component of the ultraproduct so that at each round of $G_\omega(\mathsf{At}\mathfrak{D})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in the following way. One defines $\mathfrak{A}_{i+1}$ to be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that σ selects in a play of $G_\omega(\mathsf{At}\mathfrak{D})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$.

Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, so $\mathfrak{B}$ is completely representable.

Thus $\mathfrak{A} \equiv \mathfrak{C} \equiv \mathfrak{B}$, hence $\mathfrak{A} \in \mathbf{El}\mathsf{CRCA}_n$. We have shown that $\mathsf{LCA}_n \subseteq \mathbf{El}\mathsf{CRCA_n}$. If $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, then by lemma 1.3, $\exists$ has a winning strategy in $\mathbf{G}^\omega(\mathsf{At}\mathfrak{A})$, hence in $G_\omega(\mathsf{At}\mathfrak{A})$, *a fortiori*, in $G_k(\mathsf{At}\mathfrak{A})$ for all $k < \omega$, so $\mathfrak{A} \in \mathsf{LCA}_n$. Since LCA_n is elementary, we get that $\mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$. But $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, hence $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n \subseteq \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathsf{LCA}_n$. Now $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, and the latter class is elementary (if $\mathbf{K}$ is elementary, then $\mathbf{K} \cap \mathbf{At}$ is elementary), so $\mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) \subseteq \mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$.

Conversely, if $\mathfrak{C}$ is in $\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. then $\mathfrak{C}$ is atomic and $\mathfrak{C} \equiv \mathfrak{D}$, for some $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ since $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega$ is closed under ultraproducts. Hence $\mathfrak{D}$ is atomic because atomicity is a first order property, so $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, thus $\mathfrak{C} \in \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$.

We have shown that $\mathbf{El}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} = \mathbf{El}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}) = \mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$. Finally, by lemma 1.3, $\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathsf{LCA}_n$, from which it follows that $\mathbf{El}\mathbf{S_c}[\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}] \subseteq \mathsf{LCA}_n$, since LCA_n is elementary. The other inclusion follows from that, by item (1) of theorem 1.4, $\mathsf{CRCA}_n \subseteq \mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}$, so $\mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n \subseteq \mathbf{El}[\mathbf{S_c}\mathsf{Nr}_n(\mathsf{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At}]$. We have shown that all classes coincide with LCA_n, which is the elementary closure of CRCA_n, and we are done. □

Corollary 1.16. For each $2 < n < \omega$, there is an atomic algebra $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{El}\mathsf{CRCA}_n$, that is not completely representable. In particular, CRCA_n is not elementary [5]. Furthermore, each $\mathfrak{A}_n$ is constructed uniformly from one relation algebra.

Proof: In [12], a relation atomic algebra $\mathfrak{R}$ having uncountably many atoms is constructed such that $\mathfrak{R}$ has an ω-dimensional cylindric basis $\mathsf{CA}lH$ (the latter is defined in *opcit*) and $\mathfrak{R}$ is not completely representable. It is shown in [12] that if one takes $\mathfrak{C} = \mathsf{CA}(\mathsf{CA}lH)$, then $\mathfrak{C} \in \mathsf{CA}_\omega$, $\mathfrak{C}$ is atomless, and $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$. Now fix $2 < n < \omega$. Then the required CA_n is $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$; $\mathfrak{A}_n$ is atomic and has uncountably many atoms. Furthermore, $\mathfrak{B}$ has no complete representation for a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$. Since $\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$, then by theorem 1.15, $\mathfrak{B} \in \mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$.

For the reader's convenience, we give the details of the above proof. We use the following uncountable version of Ramsey's theorem due to Erdos and Rado: If $r \geq 2$ is finite, k an infinite cardinal, then $exp_r(k)^+ \to (k^+)^{r+1}_k$, where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If f is a coloring of the $r+1$ element subsets of a set of cardinality $exp_r(k)^+$ in k many colors, then there is a homogeneous set of cardinality k^+ (a set, all whose $r+1$ element subsets get the same f-value). We will construct the requred $\mathfrak{C} \in \mathsf{CA}_\omega$ from a relation algebra (to be denoted in a while by $\mathfrak{A}$) having an 'ω-dimensional cylindric basis.'

To define the relation algebra, we specify its atoms and forbidden triples. Let κ be the given cardinal in the hypothesis of the Theorem. The atoms are Id, $\mathsf{g}_0^i : i < 2^\kappa$ and $\mathsf{r}_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of (Id, x, y) for $x \neq y$, $(\mathsf{r}_j, \mathsf{r}_j, \mathsf{r}_j)$ for $1 \leq j < \kappa$ and $(\mathsf{g}_0^i, \mathsf{g}_0^{i'}, \mathsf{g}_0^{i^*})$ for $i, i', i^* < 2^\kappa$. Write g_0 for $\{\mathsf{g}_0^i : i < 2^\kappa\}$ and r_+ for $\{\mathsf{r}_j : 1 \leq j < \kappa\}$. Call this atom structure α.

Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff$ $\mathfrak{B}$ is completely representable.

Assume for contradiction that $\mathfrak{A}$ has a complete representation with base M. Let x, y be points in the representation with $\mathsf{M} \models \mathsf{r}_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in \mathsf{M}$ such that $\mathsf{M} \models \mathsf{g}_0^i(x, z_i) \wedge \mathsf{r}_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within Z, each edge is labelled by one of the κ atoms in r_+. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $\mathsf{M} \models \mathsf{r}_j(z^1, z^2) \wedge \mathsf{r}_j(z^2, z^3) \wedge \mathsf{r}_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles).

Let S be the set of all atomic $\mathfrak{A}$-networks N with nodes ω such that $\{\mathsf{r}_i : 1 \leq i < \kappa : \mathsf{r}_i \text{ is the label of an edge in } N\}$ is finite. Then it is straightforward to show S is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$, witness [6, Definition 12.8] for notation. We have S is symmetric, that is, if $N \in S$ and $\theta : \omega \to \omega$ is a finitary function, in the sense that $\{i \in \omega : \theta(i) \neq i\}$ is finite, then $N\theta$

is in S. It follows that the complex algebra $\mathsf{CA}(S) \in \mathsf{QEA}_\omega$. Now let X be the set of finite $\mathfrak{A}$-networks N with nodes $\subseteq \kappa$ such that:

1. each edge of N is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $\mathsf{r}_+ = \{\mathsf{r}_j : 1 \leq j < \kappa\}$ or (c) a cofinite subset of $\mathsf{g}_0 = \{\mathsf{g}_0^i : i < 2^\kappa\}$ and

2. N is 'triangle-closed', i.e. for all $l, m, n \in \mathsf{nodes}(N)$ we have $N(l,n) \leq N(l,m); N(m,n)$. That means if an edge (l,m) is labelled by Id then $N(l,n) = N(m,n)$ and if $N(l,m), N(m,n) \leq \mathsf{g}_0$ then $N(l,n) \cdot \mathsf{g}_0 = 0$ and if $N(l,m) = N(m,n) = \mathsf{r}_j$ (some $1 \leq j < \omega$) then $N(l,n) \cdot \mathsf{r}_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathsf{CA}(S)$ be defined by

$$\{L \in S : L(m,n) \leq N(m,n) \text{ for } m, n \in \mathsf{nodes}(N)\}.$$

For $i \in \omega$, let $N \restriction_{-i}$ be the subgraph of N obtained by deleting the node i. Then if $N \in X$, $i < \omega$ then $\widehat{\mathsf{c}_i N} = \widehat{N \restriction_{-i}}$. The inclusion $\widehat{\mathsf{c}_i N} \subseteq (\widehat{N \restriction_{-i}})$ is clear. Conversely, let $L \in (\widehat{N \restriction_{-i}})$. We seek $M \equiv_i L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{\mathsf{c}_i N}$, as required. Since $L \in S$ the set $T = \{\mathsf{r}_i \notin L\}$ is infinite. Let T be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the ω-network M we must define the labels of all edges involving the node i (other labels are given by $M \equiv_i L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathsf{nodes}(N)$. We must choose $M(i,j) \leq N(i,j)$. If $N(i,j)$ is an atom then of course $M(i,j) = N(i,j)$. Since N is finite, this defines only finitely many labels of M. If $N(i,j)$ is a cofinite subset of g_0 then we let $M(i,j)$ be an arbitrary atom in $N(i,j)$. And if $N(i,j)$ is a cofinite subset of r_+ then let $M(i,j)$ be an element of $N(i,j) \cap Y$ which has not been used as the label of any edge of M which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathsf{nodes}(N)$ then we can let $M(i,j) = \mathsf{r}_k \in Y$ some $1 \leq k < \kappa$ such that no edge of M has already been labelled by r_k. It is not hard to check that each triangle of M is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv_i L$. The labeling avoided all but finitely many elements of Y', so $M \in S$. So $(\widehat{N \restriction_{-i}}) \subseteq \widehat{\mathsf{c}_i N}$.

Now let $\widehat{X} = \{\widehat{N} : N \in X\} \subseteq \mathsf{CA}(S)$. Then we claim that the subalgebra of $\mathsf{CA}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $-\widehat{N} = \bigcup_{m,n \in \mathsf{nodes}(N)} \widehat{N_{mn}}$ where N_{mn} is a network with nodes $\{m, n\}$ and labeling $N_{mn}(m, n) = -N(m, n)$. N_{mn} may not belong to X but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $\mathsf{d}_{ij} \in \mathsf{CA}(S)$ is equal to $\widehat{N}$ where N is a network with nodes $\{i, j\}$ and labeling $N(i, j) = \mathsf{Id}$. Closure under cylindrification is given.

Let $\mathfrak{C}$ be the subalgebra of $\mathsf{CA}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{Ra}\mathfrak{C}$. To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co-finite subset of g_0 and possibly a co-finite subset of r_+. Clearly $\mathfrak{A} \subseteq \mathfrak{Ra}\mathfrak{C}$. Conversely, each element $z \in \mathfrak{Ra}\mathfrak{C}$ is a finite union $\bigcup_{N \in F} \widehat{N}$, for some finite subset F of X, satisfying $\mathsf{c}_i z = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in F. Then, $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} z = \bigcup_{N \in F} \mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} \widehat{N} = \bigcup_{N \in F} \widehat{(N \restriction_{\{0,1\}})} \in \mathfrak{A}$. So $\mathfrak{Ra}\mathfrak{C} \subseteq \mathfrak{A}$. Thus $\mathfrak{A}$ is the relation algebra reduct of $\mathfrak{C} \in \mathsf{CA}_\omega$, but $\mathfrak{A}$ has no complete representation. Let $n > 2$. Let $\mathfrak{B} = \mathfrak{Nr}_n \mathfrak{C}$. Then $\mathfrak{B} \in \mathsf{Nr}_n \mathsf{CA}_\omega$, is atomic, but has no complete representation for plainly a complete representation of $\mathfrak{B}$ induces one of $\mathfrak{A}$.

By Theorem 1.15 $\mathfrak{B}$ is in $\mathbf{El}\mathsf{CRCA}_n = \mathsf{LCA}_n$. It remains to show that the ω-dilation $\mathfrak{C}$ is atomless. For any $N \in X$, we can add an extra node extending N to M such that $\emptyset \subsetneq M' \subsetneq N'$, so that N' cannot be an atom in $\mathfrak{C}$. □

In the next theorem the inclusions in the third item are valid since by Lemma 1.3, $\mathsf{Nr}_n \mathsf{CA}_\omega \cap \mathbf{At} \subseteq \mathsf{LCA}_n$ and the last class is elementary.[5]

THEOREM 1.17. *Let $2 < n < \omega$. Then the following hold:*

1. $\mathbf{S}\mathfrak{Cm}\mathsf{LCAS}_n = \mathsf{RCA}_n$,
2. $\mathbf{SP}\mathfrak{Cm}\mathsf{At}(\mathsf{Nr}_n \mathsf{CA}_\omega) = \mathsf{RCA}_n$,
3. *For any class $\mathbf{L}$ such that $\mathsf{At}(\mathsf{Nr}_n \mathsf{CA}_\omega) \subseteq \mathbf{L} \subseteq \mathsf{LCAS}_n$, $\mathbf{SP}\mathfrak{Cm}\mathbf{L} = \mathsf{RCA}_n$.*

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[5]The last incusion was implicitly prove in Theorem 1.3. To be more explicit, assume that $\mathfrak{A} \in \mathsf{Nr}_n \mathsf{CA}_\omega$ is atomic. Then by lemma 1.3, ∃ has a winning strategy in $\mathbf{G}^\omega$, since there are infinitely many nodes, reusing them is superfluous, so ∃ has a winning strategyactually in (the harder to win game), $G_\omega(\mathsf{At}\mathfrak{A})$, and so ∃ has a winning strategy in all k rounded game $G_k(\mathsf{At}\mathfrak{A})$, so by definition $\mathfrak{A} \in \mathsf{LCA}_n$.

In particular, $\mathbf{SP}\mathfrak{Cm}(\mathbf{El}\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)) = \mathsf{RCA}_n$.

Proof: 1. If $\mathfrak{A} \in \mathsf{RCA}_n$, then $\mathfrak{A}^+$ is completely representable [5], so $\mathsf{At}\mathfrak{A}^+ \in \mathsf{LCAS}_n$. By $\mathfrak{A} \subseteq \mathfrak{A}^+ = \mathfrak{Cm}\mathsf{At}\mathfrak{A}^+$, and $\mathfrak{Cm}\mathsf{At}\mathfrak{A}^+ \in \mathfrak{Cm}\mathsf{LCAS}_n$, we are done.

2. This follows from that $\mathsf{Fs}_n \subseteq \mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$. Indeed, suppose that $\mathfrak{A} \in \mathsf{Fs}_n$, then $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, hence $\mathsf{At}\mathfrak{A} \in \mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathfrak{A} = \mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$. Thus $\mathsf{RCA}_n = \mathbf{SP}\mathsf{Fs}_n \subseteq \mathbf{SP}\mathfrak{Cm}\mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{SP}\mathfrak{Cm}\mathsf{LCAS}_n \subseteq \mathsf{RCA}_n$.

3. Follows immediately from the previous item. □

2. Atom-canonicity and degrees of representability

In this section, unless otherwise indicated, n is a finite ordinal > 2. We study closure properties of the classes $\mathsf{Nr}_n\mathsf{CA}_m$ $(m > n)$ and CRCA_n. We also introduce several new classes defined via the complex algebra operator $\mathfrak{Cm}$ and the neat reduct operator Nr and study their properties. The most general exposition of CA rainbow constructions is given in [7, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [5]. Let G, R be two relational structures. Let $2 < n < \omega$. Then the colours used are:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $(i < j \in n$,
- shades of yellow : $\mathsf{y}_S : S$ a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 2.1. Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M an *i-cone* if $M(x_0, z) = \mathsf{g}_0^i$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green.

$(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone*.

The rainbow algebra depending on G and R from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$(\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^{*}), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) \qquad \text{any } 1 \leq i \leq n-2, \tag{2.1}$$

$$(\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) \qquad \text{any } j, k \in \mathsf{G}, \tag{2.2}$$

$$(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) \qquad \text{unless } |\{(j,k), (j',k'), (j^*,k^*)\}| = 3 \tag{2.3}$$

   and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an i cone with apex δ, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade y_S then $i \in S$.

Let G and R be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [5]; the nodes are possibly different but the graph structure is the same. Let At be the atom structure with underlying set $J \sim$. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth-diagonal element, and ith-cylindrifier, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a] T_i [b]$ iff $a \upharpoonright n \smallsetminus \{i\} = b \upharpoonright n \smallsetminus \{i\}$,

This, as easily checked, defines a CA_n atom structure. The complex CA_n over this atom structure will be denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension of $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$, always finite and > 2, will be clear from context. For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs [5, Lemma 30], so for $2 < n < m \leq \omega$, we use the graph versions of the games G_k^m, $k \leq \omega$, and $\mathbf{G}^m$ played on rainbow atom structures of dimension m [5, pp. 841–842]. The the atomic k rounded game game G^m_k where the number of nodes are limited to n to games on coloured graphs [5, lemma 30]. The game $\mathbf{G}^m$ lifts to a game on coloured graphs, that is like the graph games G^m_ω [5], where the number of nodes of graphs played during the ω rounded game does not exceed m, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match [5, 4.3]. So roughly if $|G|$ is larger that $|R|$ substantially, then $\forall$ can win; otherwise $\exists$ wins for if there is a winning strategy for $\forall$ it must be implemented as just described. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context.

THEOREM 2.2. *Let* $2 < n < \omega$.

1. *There exists* $\mathfrak{A} \in \mathsf{RCA}_n$ *such that* $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, *where* $t(n) = n(n+1)/2$. *Therefore any completely additive variety* V *such that* $\mathsf{RCA}_n \subseteq \mathsf{V} \subseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$ *is not atom-canonical.*
2. *There exists* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ *such that* $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$,
3. *There exists* $\mathfrak{B} \in \mathsf{Cs}_n$, $\mathfrak{B} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$, *but* $\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ *and* $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$

PROOF: 1. The proof of the first item is given in full detail in [16, Theorem 1]; here we give the main ingredients of the proof as another instance of a blow up and blur construction. Take the finite rainbow cylindric algebra $R(\Gamma)$ as defined in [7, Definition 3.6.9], where Γ (the reds) is taken to be the complete irreflexive graph m, and the greens are $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n(n-1)/2\}$ so that G is the complete irreflexive graph $n(n-1)/2$.

Call this finite rainbow n-dimensional cylindric algebra, based on $\mathsf{G} = n(n-1)/2$ and $\mathsf{R} = n$, $\mathsf{CA}_{n(n-1)/2+1,n}$ and denote its finite atom structure by $\mathbf{At}_\mathbf{f}$. One then replaces each red colour used in constructing $\mathsf{CA}_{n(n-1)/2,n}$ by infinitely many with superscripts from ω, getting a weakly representable atom structure $\mathbf{At}$, that is, the term algebra $\mathfrak{Tm}\mathbf{At}$ is representable.

The resulting atom structure (with ω-many reds), call it $\mathbf{At}$, is the rainbow atom structure that is like the atom structure of the (atomic set) algebra denoted by $\mathfrak{A}$ in [10, Definition 4.1] except that we have $n(n-1)/2$ greens and not infinitely many as is the case in [10]. Everything else is the same. In particular, the rainbow signature [7, Definition 3.6.9] now consists of $\mathsf{g}_i : 1 \leq i < n-1$, $\mathsf{g}_0^i : 1 \leq i \leq n+1$, $\mathsf{w}_i : i < n-1$, $\mathsf{r}_{kl}^t : k < l < n$, $t \in \omega$, binary relations, and $n-1$ ary relations y_S, $S \subseteq n(n-1)/2$.

There is a shade of red ρ; the latter is a binary relation that is *outside the rainbow signature*. But ρ is used as a label for coloured graphs built during a 'rainbow game', and in fact, $\exists$ can win the rainbow ω-rounded game and she builds an n-homogeneous (coloured graph) model M as indicated in the above outline by using ρ when she is forced a red [10, Proposition 2.6, Lemma 2.7]. Then, it can be shown exactly as in [10], that $\mathfrak{Tm}\mathsf{At}$ is representable as a set algebra with unit ${}^n\mathsf{M}$.

We next embed $\mathsf{CA}_{n(n-1)/2,n}$ into the complex algebra $\mathfrak{Cm}\mathbf{At}$, the Dedekind–MacNeille completion of $\mathfrak{Tm}\mathbf{At}$. Let CRG_f denote the class of coloured graphs on $\mathbf{At}_\mathbf{f}$ and CRG be the class of coloured graph on $\mathbf{At}$. We can assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CRG}$. Here we identify a with $[a]$; no harm will ensue.

We define the (equivalence) relation $\sim$ on At by $M_b \sim N_a$, $(M, N \in \mathsf{CRG}) \iff$ they are everywhere identical except possibly at red edges:

$$M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k, \text{ for some } l, k \in \omega.$$

We say that M_a is a *copy of* N_b if $M_a \sim N_b$. Now we define a map $\Theta : \mathsf{CA}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_\mathbf{f}$ to $\mathfrak{Cm}\mathsf{At}$, by specifing first its values on At_f, via $M_a \mapsto \sum_j M_a^{(j)}$; where $M_a^{(j)}$ is a copy of M_a; each atom maps to the suprema of its copies. (If M_a has no red edges, then by $\sum_j M_a^{(j)}$, we understand M_a). This map is extended to $\mathsf{CA}_{n+1,n}$ the obvious way. The map Θ is well defined, because $\mathfrak{Cm}\mathsf{At}$ is complete. It is not hard to show that the map Θ is an injective homomorphim.

One next proves that $\forall$ has a winning strategy for $\exists$ in $\mathbf{G}^{t(n)}\mathsf{At}(\mathsf{CA}_{n(n-1)/2,n})$, where $t(n) = n(n+1)/2 + 1$ using the usual rainbow strategy by bombarding $\exists$ with cones having the same base and distinct green tints. He needs $t(n)$ nodes to implement his winning strategy. In fact, he needs $t(n)$ nodes to force a win in the weaker game $G_\omega^{t(n)}(\mathsf{At}\mathfrak{A}_{n(n-1)/2,n})$ without the need to resue the nodes in play. To see why, first it is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1 (\leq n(n-1)/2+1)$ and n rounds

$\mathsf{EF}_{n+1}^{n+1}(n+1, n)$ since $n+1$ is 'longer' than n. $\forall$ lifts his winning strategy from the last private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n(n-1)/2,n})$ see [5, p. 841] forcing a win using $t(n)$ nodes. One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$. Pick any point $x \in C$. Then there are $> n(n-1)/2$ points y in $C \setminus \{x\}$. There are only $n(n-1)/2$ red relations. So there must be distinct $y, z \in C \setminus \{x\}$ such that (x, y) and (x, z) both have the same red label (it will be some $\mathsf{r}_{ij}{}^m$ for $i < j < n$). But (y, z) is also red, and this contradicts the consistency condition of reds. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P = \{(i, j) : i < j < n\}$ forming a red clique, triangle with two edges satisfying the same $r_p{}^m$ for $p \in P$. Calculating, we get $t(n) = n-1+n(n-1)/2+2 = n(n+1)/2+1$. We proved that $\forall$ lifts his winning strategy from the last private game to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathsf{CA}_{n(n-1)/2,n}$ forcing a win using $t(n)$ nodes.

2. This follows from the proof of Theorem 1.12; we give a more streamlined proof. Like before, we use the construction in [2]. Let $\mathfrak{R}$ be a relation algebra, with non-identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. (J, E) is an *n-blur* for $\mathfrak{R}$, if J is a *complex n-blur* and the tenary relation E is an *index blur* defined as in item (ii) of [2, Definition 3.1]. Recall that (J, E) is a *strong n-blur*, if it (J, E) is an n-blur, such that the complex n-blur satisfies: $(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$ (with notation as in [2]). Now let $l \geq 2n-1$, $k \geq (2n-1)l$, $k \in \omega$. One takes the finite integral relation algebra $\mathfrak{R}_l = \mathfrak{E}_k(2,3)$ where k is the number of non-identity atoms in $\mathfrak{R}_l$. Then $\mathfrak{R}_l$ has a strong l-blur, (J, E) and it can only be represented on a finite basis [2]. Then $\mathfrak{Bb}_n(\mathfrak{R}_l, J, E) = \mathsf{Nr}_n\mathfrak{Bl}_l(\mathfrak{R}_l, J, E)$ has no complete representation, so $\mathfrak{CmAt}\mathfrak{Bb}_n(\mathfrak{R}_l, J, E)$ is not representable.

3. Let $V = {}^n\mathbb{Q}$ and let $\mathfrak{A} \in \mathsf{Cs}_n$ has universe $\wp(V)$. Then clearly $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. To see why, let $W = {}^\omega\mathbb{Q}$ and let $\mathfrak{D} \in \mathsf{Cs}_\omega$ have universe $\wp(W)$. Then the map $\theta : \mathfrak{A} \to \wp(\mathfrak{D})$ defined via $a \mapsto \{s \in W : (s \upharpoonright \alpha) \in a\}$, is an injective homomorphism from $\mathfrak{A}$ into $\mathfrak{Rd}_n\mathfrak{D}$ that is onto $\mathfrak{Nr}_n\mathfrak{D}$. Let y denote the following n-ary relation: $y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}$. Let y_s be the singleton containing s, i.e. $y_s = \{s\}$ and $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{A}}\{y, y_s : s \in y\}$. It is shown in [17] that $\{s\} \in \mathfrak{B}$, for all $s \in V$.

Now $\mathfrak{B}$ and $\mathfrak{A}$ having same top element V, share the same atom structure, namely, the singletons, so $\mathfrak{B} \subseteq_d \mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} = \mathfrak{A}$. Furthermore, plainly $\mathfrak{A}, \mathfrak{B} \in \mathsf{CRCA}_n$; the identity maps establishes a complete representation for both, since $\bigcup_{s \in V}\{s\} = V$. Since $\mathfrak{B} \subseteq_d \mathfrak{A}$, then $\mathfrak{B} \subseteq_c \mathfrak{A}$, so $\mathfrak{B} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ is atomic. As proved in [17], $\mathfrak{B} \notin \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}(\supseteq \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$). □

Recall that $\mathbf{S_c}$ denotes the operation of forming complete sublgebras and $\mathbf{S_d}$ denotes the opeartion of forming dense subalgebras. We let $\mathbf{I}$ denote the operation of forming isomorphic images. For any class of BAO, $\mathbf{I}\mathsf{K} \subseteq \mathbf{S_d}\mathsf{K} \subseteq \mathbf{S_c}\mathsf{K}$. (It is not hard to show that for Boolean algebras the inclusion are proper).

Definition 2.3. Let $2 < n \leq l \leq m \leq \omega$. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_d}, \mathbf{S_c}, \mathbf{I}\}$.

1. An algebra $\mathfrak{A} \in \mathsf{CA}_n$ has the $\mathbf{O}$ *neat embedding property up to* m if $\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. If $m = \omega$ and $\mathbf{O} = \mathbf{S}$, we say simply that $\mathfrak{A}$ has *the neat embedding property.* (Observe that the last condition is equivalent to that $\mathfrak{A} \in \mathsf{RCA}_n$).
2. An atomic algebra $\mathfrak{A} \in \mathsf{CA_n}$ has the *complex* $\mathbf{O}$ *neat embedding property up to* m, if $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_m$. The word 'complex' here refers to the *involvement of the complex algebra* in the definition.
3. An atomic algebra $\mathfrak{A} \in \mathsf{RCA}_n$ is *strongly representable up to* l *and* m if $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$. If $l = n$ and $m = \omega$, we say that $\mathfrak{A}$ is *strongly representable.*
4. Let $\mathbf{L} \subseteq \mathbf{K}$ be subclasses of CA_n. We say that $\mathbf{L}$ is *not atom-canonical relative* $\mathbf{K}$ if there exists an atomic algebra $\mathfrak{A} \in \mathbf{L}$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{K}$. Observe that if $\mathbf{L}$ is not atom-canonical relative to itself, then $\mathbf{L}$ is not atom-canonical.

Example 2.4.

1. The algebra $\mathfrak{A}$ constructed in the third item of theorem 2.2 has the neat embedding property, but not the complex $\mathbf{S}$ neat embedding propery up to m for any $m \geq n(n+1)/2$. In particular, $\mathfrak{A}$ is not strongly representable and $\mathfrak{A}$ lacks a complete representation. Furthermore, the algebra $\mathfrak{A}$ witnesses that RCA_n is not atom-canonical relative to $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$ for any $k \geq n(n+1)/2$.

2. For every $2 < n < l < \omega$, the algebra $\mathfrak{B} = \mathfrak{Bb}_n(\mathfrak{E}_k(2,3), J, E)$ used in the second item of Theorem 2.2 based on Theorem 1.12, where k depends on l and (J, E) is the strong l-blur of the Maddux algebra $\mathfrak{E}_k(2,3)$ as specified in *op. cit.*, is in $\mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$, but is not strongly representable up to l and ω. In particular, $\mathfrak{B}$, like $\mathfrak{A}$ in the first item, is also not strongly representable and lacks a complete representation. The algebra $\mathfrak{B}$ witnesses that $\mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$ is not atom-canonical relative to RCA_n.

3. The algebra $\mathfrak{B}$ used in the last item of theorem 2.2 has the complex $\mathbf{I}$ neat embedding property up to m for any $m \geq n$ but does not have the $\mathbf{I}$ neat embedding property up to $n+1$, *a fortiori* up to any $m \geq n+1$, cf. the second item of the forthcoming theorem 2.5.

Let $2 < n \leq l \leq m \leq \omega$. Let $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_d}, \mathbf{S_c}, \mathbf{I}\}$. Denote the class of CA_ns having the complex $\mathbf{O}$ neat embedding property up to m by $\mathsf{CNPCA}^{\mathbf{O}}_{n,m}$, and let $\mathsf{RCA}^{\mathbf{O}}_{n,m} := \mathsf{CNPCA}^{\mathbf{O}}_{n,m} \cap \mathsf{RCA}_n$. Denote the class of strongly representable CA_ns up to l and m by $\mathsf{RCA}_n^{l,m}$. Call an algebra $\mathfrak{A} \in \mathsf{CA}_n$ *strongly representable* if $\mathfrak{A}$ is atomic and $\mathsf{At}\mathfrak{A}$ is strongly representable; that is $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \in \mathsf{RCA}_n$. Observe that $\mathsf{RCA}_n^{n,m} = \mathsf{RCA}^{\mathbf{S}}_{n,m}$ and that when $m = \omega$ both classes coincide with the class of strongly representable CA_ns. For a class $\mathbf{K}$ of BAOs, $\mathbf{K} \cap \mathsf{Count}$ denotes the class of countable algebras in $\mathbf{K}$, and recall that $\mathbf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in $\mathbf{K}$.

THEOREM 2.5. *Let $2 < n \leq l < m \leq \omega$ and $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}, \mathbf{I}\}$. Then the following hold:*

1. $\mathsf{RCA}^{\mathbf{O}}_{n,m} \subseteq \mathsf{RCA}^{\mathbf{O}}_{n,l}$ *and* $\mathsf{RCA}^{\mathbf{I}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S_d}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S_c}}_{n,l} \subseteq \mathsf{RCA}^{\mathbf{S}}_{n,l}$. *The last inclusion is proper for $l \geq n(n+1)/2$,*

2. *For $\mathbf{O} \in \{\mathbf{S}, \mathbf{S_c}, \mathbf{S_d}\}$, $\mathsf{CNPCA}^{\mathbf{O}}_{n,l} \subseteq \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ (that is the complex $\mathbf{O}$ neat embedding property is stronger than the $\mathbf{O}$ neat embedding property), and for $\mathbf{O} = \mathbf{S}$, the inclusion is proper for $l \geq n+3$. But for $\mathbf{O} = \mathbf{I}$, $\mathsf{CNPCA}^{\mathbf{I}}_{n,l} \nsubseteq \mathsf{Nr}_n\mathsf{CA}_l$ (so the complex $\mathbf{I}$ neat embedding property does not imply the $\mathbf{I}$ neat embedding property),*

3. *If $\mathfrak{A}$ is finite, then $\mathfrak{A} \in \mathsf{CNPCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$ and $\mathfrak{A} \in \mathsf{RCA}^{\mathbf{O}}_{n,l} \iff \mathfrak{A} \in \mathsf{RCA}_n \cap \mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. Furthermore, for any positive k, $\mathsf{CNPCA}^{\mathbf{O}}_{n,n+k+1} \subsetneq \mathsf{CNPCA}^{\mathbf{O}}_{n,n+k}$, and finally $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega} \subsetneq \mathsf{RCA}_n$,*

4. $(\exists\mathfrak{A}\in\mathsf{RCA}_n\cap\mathbf{At}\sim\mathsf{CNPCA}^{\mathbf{S}}_{n,l})\implies\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom-canonical for all* $k\geq l$. *In particular,* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k$ *is not atom-canonical for all* $k\geq n+3$,

5. *If* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ *is atom-canonical, then* $\mathsf{RCA}^{\mathbf{S}}_{n,l}$ *is first order definable. There exists a finite* $k>n+1$, *such that* $\mathsf{RCA}^{\mathbf{S}}_{n,k}$ *is not first order definable.*

6. *Let* $2<n<l\leq\omega$. *Then* $\mathsf{RCA}^{l,\omega}_n\cap\mathsf{Count}\neq\emptyset\iff l<\omega$.

Proof: 1. The inclusions follow from the definition and the strictness of the last inclusion in this item is witnessed by the algebra $\mathfrak{C}=\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ used in Theorem 3.1, since $\mathfrak{C}$ satisfies $\mathfrak{C}=\mathfrak{CmAt}\mathfrak{C}\in\mathsf{RCA}_n$ but $\mathfrak{C}\notin\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_l$ for $l\geq n+3$.

2. Let $\mathbf{O}\in\{\mathbf{S},\mathbf{S_c},\mathbf{S_d}\}$. If $\mathfrak{CmAt}\mathfrak{A}\in\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$, then $\mathfrak{A}\subseteq_d\mathfrak{CmAt}\mathfrak{A}$, so $\mathfrak{A}\in\mathbf{S_d}\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l\subseteq\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_l$. This proves the first part. The strictness of the last inclusion follows from the first part of Theorem 2.2 since the atomic countable algebra $\mathfrak{A}$ constructed in *op. cit.* is in RCA_n, but $\mathfrak{CmAt}\mathfrak{A}\notin\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ for any $l\geq n(n+1)/2$.

For the last non-inclusion in item (2), we use the set algebras $\mathfrak{A}$ and $\mathfrak{B}$ in item (3) of Theorem 2.2. Now $\mathfrak{B}\subseteq_d\mathfrak{A}$, $\mathfrak{A}\in\mathsf{Cs}_n$, and clearly $\mathfrak{CmAt}\mathfrak{B}=\mathfrak{A}(\in\mathsf{Nr}_n\mathsf{CA}_\omega)$. As proved in [17], $\mathfrak{B}\notin\mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$, so $\mathfrak{B}\notin\mathsf{Nr}_n\mathsf{CA}_{n+1}(\supseteq\mathsf{Nr}_n\mathsf{CA}_l)$. But $\mathfrak{CmAt}\mathfrak{B}\in\mathsf{Nr}_n\mathsf{CA}_\omega$, hence $\mathfrak{B}\in\mathsf{RCA}^{\mathbf{I}}_{n,l}$. We have shown that $\mathfrak{B}\in\mathsf{RCA}^{\mathbf{I}}_{n,l}\sim\mathsf{Nr}_n\mathsf{CA}_l$, and we are through with the last required in item (2). Here we basically use that $\mathsf{Nr}_n\mathsf{CA}_m$ is not closed under $\mathbf{S_d}$, *a fortiori* under $\mathbf{S_c}$, while, conversely, CRCA_n is closed under $\mathbf{S_c}$ since $\mathbf{S_c}$ is an idempotent operator ($\mathbf{S_c}\mathbf{S_c}=\mathbf{S_c}$), *a fortiori* CRCA_n is closed under $\mathbf{S_d}$.

3. Follows by definition observing that if $\mathfrak{A}$ is finite then $\mathfrak{A}=\mathfrak{CmAt}\mathfrak{A}$. The strictness of the first inclusion follows from the construction in [9] where it shown that for any positive k, there is a *finite algebra* $\mathfrak{A}$ in $\mathsf{Nr}_n\mathsf{CA}_{n+k}\sim\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k+1}$ (witness the appendix for a simplified version of the construction in [9]). The inclusion $\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega}\subseteq\mathsf{RCA}_n$ holds because if $\mathfrak{B}\in\mathsf{CNPCA}^{\mathbf{O}}_{n,\omega}$, then $\mathfrak{B}\subseteq\mathfrak{CmAt}\mathfrak{B}\in\mathbf{O}\mathsf{Nr}_n\mathsf{CA}_\omega\subseteq\mathsf{RCA}_n$. The $\mathfrak{A}$ used in the last item of theorem 2.2 witnesses the strictness of the last inclusion proving the last required in this item.

4. Follows from the definition and the construction used in item (3) of theorem 2.2.

5. Follows from that $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$ is canonical. So if it is atom-canonical too, then $\mathsf{At}(\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l)=\{\mathfrak{F}:\mathfrak{Cm}\mathfrak{F}\in\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l\}$, the former class is ele-

mentary [6, Theorem 2.84], and the last class is elementray $\iff \mathsf{RCA}^{\mathbf{S}}_{n,l}$ is elementary. Non-elementarity follows from [7, Corollary 3.7.2] where it is proved that $\mathsf{RCA}^{\mathbf{S}}_{n,\omega}$ is not elementary, together with the fact that $\bigcap_{n<k<\omega}\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_k = \mathsf{RCA}_n$. In more detail, let $\mathfrak{A}_i$ be the sequence of strongly representable CA_ns with $\mathfrak{CmAt}\mathfrak{A}_i = \mathfrak{A}_i$ and $\mathfrak{A} = \Pi_{i/U}\mathfrak{A}_i$ is not strongly representable. Hence $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_\omega = \bigcap_{i\in\omega}\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+i}$, so $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{K}_l$ for all $l > k$, for some $k \in \omega$, $k > n$. But for each such l, $\mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l(\supseteq \mathsf{RCA}_n)$, so $\mathfrak{A}_i$ is a sequence of algebras such that $\mathfrak{CmAt}\mathfrak{A}_i = \mathfrak{A}_i \in \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, but $\mathfrak{Cm}(\mathsf{At}(\Pi_{i/U}\mathfrak{A}_i)) = \mathfrak{CmAt}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_l$, for all $l \geq k$. That k has to be strictly greater than $n+1$, follows because $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ is atom-canonical.

6. $\Longleftarrow$: Let $l < \omega$. Then the required follows from theorem 1.12, and item (2) in Theorem 2.2 that there exists a countable $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathsf{RCA}_n$. Now we prove $\Longrightarrow$: Assume for contradiction that there is an $\mathfrak{A} \in \mathsf{RCA}_n^{\omega,\omega} \cap \mathsf{Count}$. Then by definition $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, so by [14, Theorem 5.3.6], we have $\mathfrak{A} \in \mathsf{CRCA}_n$. But this complete representation, induces a(n ordinary) representation of $\mathfrak{CmAt}\mathfrak{A}$ which is a contradiction. Indeed by Lemma 1.10, if $f : \mathfrak{A} \to \mathfrak{B}$ is a complete representation of $\mathfrak{A}$ via f then one extends f to $\hat{f}$ from $\mathfrak{CmAt}\mathfrak{A}$ to $\mathfrak{B}$ by defining $\hat{f}(a) = \sum^{\mathfrak{CmAt}\mathfrak{A}}_{x\in\mathsf{At}\mathfrak{A}, x\leq a} f(x)$. □

3. Non-elementary classes

Still $\mathbf{S_d}$ stands for the operation of forming dense subalgebras and for K a class of BAOs, $\mathbf{S_c}\mathsf{K} = \{\mathfrak{B} : (\exists \mathfrak{A} \in \mathsf{K})(\sum^{\mathfrak{A}} X = 1 \implies \sum^{\mathfrak{B}} X = 1\}$.

THEOREM 3.1. *Let $2 < n < \omega$. Any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not first order definable. Furthermore any class between $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$ and $\mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$ is not first order definable.*

PROOF: The proof is long and is divided into four parts:

(a) We define an ω-rounded (atomic) game $\mathbf{H}(\alpha)$ played on so-called atomic λ-neat hypernetworks-λ a 'label'.

(b) If α is a countable atom structure, and $\exists$ has a winning strategy in $\mathbf{H}(\alpha)$, then any algebra $\mathfrak{F}$ having atom structure α is completely representable, $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\alpha \in \mathsf{AtNr}_\alpha\mathsf{CA}_\omega$. In fact, there will exist a complete $\mathfrak{D} \in \mathsf{CA}_\omega$ such that $\mathfrak{Cm}\alpha \cong \mathsf{Nr}_n\mathfrak{D}$ and $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$,

(c) Then the game $\mathbf{H}$ will be applied to the atom structure of a rainbow-like CA_n denoted below by $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$. From a winning strategyof $\exists$ in $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ (where $\mathbf{H}_k$ is $\mathbf{H}$ truncated to k rounds) for all $k \leq \omega$–so that $\mathbf{H}_\omega = \mathbf{H}$– it will follow that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$ for some completely representable atom structure $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, for which $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. On the other hand, we prove that $\forall$ has a winning strategy in $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$, so by lemma 1.3 $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$.

(d) The term algebra $\mathfrak{Tm}\alpha$ will be used to show that any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not elementary.

(a) Defining the game $\mathbf{H}_k(k \leq \omega)$ which is $\mathbf{H}$ restricted to k rounds
This new game $\mathbf{H}_k$ is stronger than G_k. In $\mathbf{H}_k$ not only the moves are more (which they are), but now the board of the play is different.

Fix $k \leq \omega$. The new game $\mathbf{H}_k$ is played on so-called λ-*neat hypernetworks*, λ a 'hyperlabel' and it has k rounds. These are similar to $m(< n)$-dimensional hypernetworks as defined in item(3) of definition 1.2; they are roughly networks endowed with labelled hyperedges, whose length gets arbitrarily long, but is still finite. Unlike m-dimensional hypernetworks here the lengths of hyperedges are not uniformly bounded. So a hypernetwork of an atomic $\mathfrak{A} \in \mathsf{CA}_n$ has two parts (N^a, N^h) where N^a is network whose n-hyperdges are labelled by atoms of $\mathfrak{A}$ and $N^h :^{<\omega} \mathsf{nodes}(N) \to \Lambda$, where hyperedges get their hyperlabels from a non-empty set (of hyperlabels) Λ.

There is a compatibility condition between N^a and N^h which is a CA analogue of condition (3) in [6, Definition 12.1] formulated for hypernetworks of relation algebras. This condition for hypernetworks as defined in [4], is given in [4, Definition 28]. The form for CAs needed is entirely analogous to the condition in item (3) of definition 1.2. In any such hypernetwork $N = (N^a, N^h)$, there are so-called *short hyperedges* and *long hyperedges* in N^h. The hypernetworks whose short hyperedges are constantly labelled by a hyperlabel $\lambda \in \Lambda$ are called λ-neat hypernetworks. The game $\mathbf{H}$ offers $\forall$ three moves delivered by $\forall$ during the play. There is a *cylindrifier move* analagous to the cylindrifier move in G adapted the obvious way to λ-neat hypernetworks and two more *amalgamation moves.*

First amalgamation move: $\forall$ can play a *transformation move* by picking a previously played λ-neat hypernetwork N and a partial, finite surjection

$\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N, θ). $\exists$'s response is mandatory. She must respond with $N\theta$.

Second amalgmation move: $\forall$ can play an *amalgamation move* by picking previously played λ-neat hypernetworks M, N such that $M \restriction_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} = N \restriction_{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)}$, and $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$. This move is denoted (M, N). To make a legal response, $\exists$ must play a λ-neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.

(b) Forming the required ω-dilation $\mathfrak{D}$ Fix some $a \in \alpha$. The game $\mathbf{H}_\omega$ is designed so that using $\exists$ s winning strategy in the game $\mathbf{H}_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of λ-neat hypernetworks where M_0 is $\exists$'s response to the initial $\forall$-move a, such that: If M_r is in the sequence and $M_r(\bar{x}) \leq \mathsf{c}_i a$ for an atom a and some $i < n$, then there is $s \geq r$ and $d \in \mathsf{nodes}(M_s)$ such that $M_s(\bar{y}) = a$, $\bar{y}_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if M_r is in the sequence and θ is any partial isomorphism of M_r, then there is $s \geq r$ and a partial isomorphism θ^+ of M_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(M_r)$ (This can be done using $\exists$'s responses to amalgamation moves). Now let $\mathfrak{M}_a$ be the limit of this sequence, that is $\mathfrak{M}_a = \bigcup M_i$, the labelling of $n-1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the M_is are nested. Let L be the signature with one n-ary relation for each $b \in \alpha$, and one k-ary predicate symbol for each k-ary hyperlabel λ. *Now we work in* $L_{\infty,\omega}$. For fixed $f_a \in {}^\omega\mathsf{nodes}(\mathfrak{M}_a)$, let $\mathfrak{U}_a = \{f \in {}^\omega\mathsf{nodes}(\mathfrak{M}_a) : \{i < \omega : g(i) \neq f_a(i)\}$ is finite$\}$. We make $\mathfrak{U}_a$ into the base of an L relativized structure $\mathsf{CA}lM_a$ like in [4, Theorem 29] except that we allow a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0 \ldots, i_{k-1} < \omega$, k-ary hyperlabels λ, and all L-formulas ϕ, ϕ_i, ψ, and $f \in U_a$:

$$\begin{aligned}
\mathsf{CA}lM_a, f \models b(x_{l_0} \ldots, x_{l_{n-1}}) &\iff \mathsf{CA}lM_a(f(l_0), \ldots, f(l_{n-1})) = b,\\
\mathsf{CA}lM_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff \mathsf{CA}lM_a(f(i_0), \ldots, f(i_{k-1})) = \lambda,\\
\mathsf{CA}lM_a, f \models \neg\phi &\iff \mathsf{CA}lM_a, f \not\models \phi,\\
\mathsf{CA}lM_a, f \models (\bigvee_{i\in I} \phi_i) &\iff (\exists i \in I)(\mathsf{CA}lM_a, f \models \phi_i),\\
\mathsf{CA}lM_a, f \models \exists x_i\phi &\iff \mathsf{CA}lM_a, f[i/m] \models \phi,\\
&\qquad \text{some } m \in \mathsf{nodes}(\mathsf{CA}lM_a).
\end{aligned}$$

For any such L-formula ϕ, write $\phi^{\mathsf{CA}lM_a}$ for $\{f \in \mathfrak{U}_a : \mathsf{CA}lM_a, f \models \phi\}$. Let $D_a = \{\phi^{\mathsf{CA}lM_a} : \phi \text{ is an } L\text{-formula}\}$ and $\mathfrak{D}_a$ be the weak set algebra with universe D_a. Let $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$. Then $\mathfrak{D}$ is a generalized *complete* weak set algebra [3, Definition 3.1.2 (iv)]. Now we show that $\alpha \cong \mathsf{At}\mathfrak{Nr}_n\mathfrak{D}$ and $\mathfrak{Cm}\alpha \cong \mathfrak{Nr}_n\mathfrak{D}$. Let $x \in \mathfrak{D}$. Then $x = (x_a : a \in \alpha)$, where $x_a \in \mathfrak{D}_a$. For $b \in \alpha$ let $\pi_b : \mathfrak{D} \to \mathfrak{D}_b$ be the projection map defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathfrak{D}_a \to \mathfrak{D}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Suppose $x \in \mathfrak{Nr}_n\mathfrak{D}\setminus\{0\}$. Since $x \neq 0$, then it has a non-zero component $\pi_a(x) \in \mathfrak{D}_a$, for some $a \in \alpha$. Assume that $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} = \pi_a(x)$, for some L-formula $\phi(x_{i_0}, \ldots, x_{i_{k-1}})$. We have $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ and assume that $\mathsf{CA}lM_a, f \models b(x_0, \ldots x_{n-1})$ for some $b \in \alpha$. We show that $b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$. Take any $g \in b(x_0, x_1 \ldots, x_{n-1})^{\mathfrak{D}_a}$, so that $\mathsf{CA}lM_a, g \models b(x_0, \ldots x_{n-1})$. The map $\{(f(i), g(i)) : i < n\}$ is a partial isomorphism of $\mathsf{CA}lM_a$. Here that short hyperedges are constantly labelled by λ is used. This map extends to a finite partial isomorphism θ of M_a whose domain includes $f(i_0), \ldots, f(i_{k-1})$. Let $g' \in \mathsf{CA}lM_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathsf{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

We have $\mathsf{CA}lM_a, g' \models \phi(x_{i_0}, \ldots, x_{i_{k-1}})$. But $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so g is identical to g' over n and it differs from g' on only a finite set. Since $\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a} \in \mathfrak{Nr}_n\mathfrak{D}_a$, we get that $\mathsf{CA}lM_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}$ (this can be proved by induction on quantifier depth of formulas). This proves that

$$b(x_0, x_1 \ldots x_{n-1})^{\mathfrak{D}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathfrak{D}_a} = \pi_a(x),$$

and so

$$\iota_a(b(x_0, x_1, \ldots x_{n-1})^{\mathfrak{D}_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{\mathfrak{D}_a}) \leq x \in \mathfrak{D}_a \setminus \{0\}.$$

Now every non-zero element x of $\mathfrak{Nr}_n\mathfrak{D}_a$ is above a non-zero element of the following form $\iota_a(b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a})$ (some $a, b \in \alpha$) and these are the atoms of $\mathfrak{Nr}_n\mathfrak{D}_a$. The map defined via $b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{\mathfrak{D}_a} : a \in \alpha)$ is an isomorphism of atom structures, so that $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$. Let $X \subseteq \mathfrak{Nr}_n\mathfrak{D}$. Then by completeness of $\mathfrak{D}$, we get that $d = \sum^{\mathfrak{D}} X$ exists. Assume that $i \notin n$, then $\mathsf{c}_i d = \mathsf{c}_i \sum X = \sum_{x\in X} \mathsf{c}_i x = \sum X = d$, because the c_is are completely additive and $\mathsf{c}_i x = x$, for all $i \notin n$, since $x \in \mathfrak{Nr}_n\mathfrak{D}$. We conclude that $d \in \mathfrak{Nr}_n\mathfrak{D}$, hence d is an upper bound of X in $\mathfrak{Nr}_n\mathfrak{D}$. Since $d = \sum_{x\in X}^{\mathfrak{D}} X$ there can be no $b \in \mathfrak{Nr}_n\mathfrak{D}$ $(\subseteq \mathfrak{D})$ with $b < d$ such that b is an upper bound of X for else it will be an upper bound of X in $\mathfrak{D}$. Thus $\sum_{x\in X}^{\mathfrak{Nr}_n\mathfrak{D}} X = d$ We have shown that $\mathfrak{Nr}_n\mathfrak{D}$ is complete. Making the legitimate identification $\mathfrak{Nr}_n\mathfrak{D} \subseteq_d \mathfrak{Cm}\alpha$ by density, we get that $\mathfrak{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ (since $\mathfrak{Nr}_n\mathfrak{D}$ is complete), hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$.

Finally, to show that any atomic algebra having atom structure α is completely representable one can reason in one of the two following ways:

One: The game $\mathbf{H}$ is stronger than G and a winning strategyof $\exists$ in $G(\alpha)$ implies that the atom structure α is completely representable, hence any atomic algebra having the atom structure α will be completely representable.

Two: The complex algebra $\mathfrak{Cm}\alpha$ has countably many atoms and is in $\mathsf{Nr}_n\mathsf{CA}_\omega$, so by the third item of theorem 1.4 it is completely representable. Thus, any atomic algebra $\mathfrak{F}$ sharing the atom structure α is also completely representable.

(c) Applying H to a rainbow-like atom structure; excluding first order definability of classes between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ We apply the new game $\mathbf{H}$ to the rainbow algebra $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ based on the ordered structures $\mathbb{Z}$ and $\mathfrak{N}$. The reds R are the set $\{\mathsf{r}_{ij} : i < j < \omega(= \mathfrak{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathfrak{N}$, but we add a forbidden triple in coloured graphs. The triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is *forbidden* if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathfrak{N}$. In [15], it is shown that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{B}$ for some countable $\mathfrak{B} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. This is proved by showing that $\exists$ has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ for all $k \in \omega$, hence using ultrapowers followed by an elementary chain argument (like the argument used in the proof of theorem 1.15), we get that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{B}$, and $\exists$ has a winning strategy in $G_\omega(\mathsf{At}\mathfrak{B})$, hence by [7, Theorem 3.3.3] $\mathfrak{B} \in \mathsf{CRCA}_n \subseteq \mathbf{S_c}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At})$; the last inclusion follows from the first item of theorem 1.4. With some significantly more effort one can prove more: It can be shown that that $\exists$ can win the game $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ which is the game $\mathbf{H}$ truncated to k rounds (on the same $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ based on $\mathbb{Z}$ and $\mathfrak{N}$) for all $k < \omega$. Recall that $\mathbf{H}$ is stronger than G hence $\mathbf{H}_k$ is stronger than G_k. Using ultrapowers followed by an elementary chain argument, it follows $\exists$

has a winning strategy in $\mathbf{H}(\alpha)$ for a countable atom structure α, such that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$. We show that $\forall$ has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}})$ played on coloured graphs [5]. The rough idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathfrak{N}$. In ω many rounds $\forall$ forces a win, so by the first item of lemma 1.3, $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. More rigorously, $\forall$ plays as follows: In the initial round $\forall$ plays a graph M with nodes $0, 1, \ldots, n-1$ such that $M(i,j) = \mathsf{w}_0$ for $i < j < n-1$ and $M(i, n-1) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, $M(0, n-1) = \mathsf{g}_0^0$ and $M(0, 1, \ldots, n-2) = \mathsf{y}_{\mathbb{Z}}$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0, \ldots, n-2)$ and demands a node n with $M_2(i,n) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, and $M_2(0,n) = \mathsf{g}_0^{-1}$. $\exists$ must choose a label for the edge $(n+1, n)$ of M_2. It must be a red atom r_{mk}, $m, k \in \mathfrak{N}$. Since $-1 < 0$, then by the 'order preserving' condition we have $m < k$. In the next move $\forall$ plays the face $(0, \ldots, n-2)$ and demands a node $n+1$, with $M_3(i,n) = \mathsf{g}_i$ $(i = 1, \ldots, n-2)$, such that $M_3(0, n+2) = \mathsf{g}_0^{-2}$. Then $M_3(n+1, n)$ and $M_3(n+1, n-1)$ both being red, the indices must match. $M_3(n+1, n) = r_{lk}$ and $M_3(n+1, r-1) = r_{km}$ with $l < m \in \mathfrak{N}$. In the next round $\forall$ plays $(0, 1, \ldots n-2)$ and re-uses the node 2 such that $M_4(0,2) = \mathsf{g}_0^{-3}$. This time we have $M_4(n, n-1) = \mathsf{r}_{jl}$ for some $j < l < m \in \mathfrak{N}$. Continuing in this manner leads to a decreasing sequence in $\mathfrak{N}$. We have proved the required.

(d): Putting (a), (b), (c) together We get that $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$, where α is a countable atom structure, such that $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, any atomic $\mathfrak{F} \in \mathsf{CA}_n$ having atom structure α is completely representable, and $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. So $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{Tm}\alpha \in \mathsf{CRCA}_n$ and $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Let K be any class between $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$. Then $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3} \supseteq \mathsf{K}$. But $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \equiv \mathfrak{Tm}\alpha$, and $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$, and $\mathfrak{Tm}\alpha \in \mathsf{CRCA}_n$, so $\mathfrak{Tm}\alpha \in \mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathsf{K}$. We have shown that $\mathfrak{Tm}\alpha \equiv \mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$, $\mathfrak{Tm}\alpha \in \mathsf{K}$ but $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{K}$, and we are done.[6]

[6]Let $m > n$. It is easy to show that if $\mathfrak{D} \in \mathsf{CA}_n$ and $\mathsf{At}\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$, then $\mathfrak{D} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$. Since $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$, by the (contrapositive of the) above obser-

We have also proved that any K between $\mathsf{AtNr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not elementary, because $\alpha \equiv \mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$, $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$ but $\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$ lest $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3}$.[7] □

Remark 3.2. In forming the required ω-dilation $\mathfrak{D}$ we made use of the 'stronger part' of the game $\mathbf{H}$, involving the amalgamation moves on λ-neat hypernetworks, where λ is the constant hyperlabel kept on short hypernetworks to build the ω-dilation $\mathfrak{D}$ which is a *generalized weak set algebra* of dimension ω, that is a set algebra, whose top element is a disjoint union of *weak spaces of dimension* ω; any such weak space is a set of sequences that agree co-finitely with sequences in ${}^\omega U$ (some non-empty set U). This ω-dilation $\mathfrak{D}$ can be (and was) described in a model theoretic framework. Using $\exists$'s winning strategy in $\mathbf{H}$, one builds an ω-dilation $\mathfrak{D}_a$ of $\mathfrak{Tm}\alpha$ for every $a \in \alpha$, based on a structure M_a in some signature specified above. Strictly speaking, M_a is a *weak model* [13, Definition 3.2.1], where assignments are required to agree co-finitely with a fixed sequence in ${}^\omega\mathsf{M}_a$. Thus $\mathfrak{D}_a$ is a weak set algebra of dimension n with base M_a This weak model M_a was taken in a signature L consisting of one n-ary relation for each $b \in \alpha$ and a k-ary relation symbol for each hyperedge of length k labelled by λ.

For $a \in \alpha$, the weak model M_a is the limit of the play $\mathbf{H}_\omega$; in the sense that M_a is the union of the λ-neat hypernetworks on α played during the game $\mathbf{H}_\omega$, with starting point the initial atom a that $\forall$ chose in the first move. Labels for the edges and hyperedges in M_a were defined the obvious way, inherited from the λ-neat hypernetworks played during the game; these are nested so this labelling is well defined, giving an interpretation of only the atomic formulas of L in M_a.

However, there is some freedom here in 'completing' the interpretation. One can use *any extension* $\mathfrak{L}$*, not necessarily a proper one, of* $L_{\omega,\omega}$ as a vehicle for constructing $\mathfrak{D}_a$. The algebra $\mathfrak{D}_a$ constructed above was a weak

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vation, $\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}} \notin \mathsf{At}(\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3})$, and $\alpha \equiv \mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$ because an atom structure of an atomic algebra is interpretable in the algebra, then we have already proved the required. However, if $\mathsf{At}\mathfrak{D} \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m)$ for some $\mathfrak{D} \in \mathsf{CA}_m$ and some $m > n$ does not imply that $\mathfrak{D} \in \mathsf{Nr}_n\mathsf{CA}_m$, even if the Dedekind–MacNeille completion of $\mathfrak{D}$ is in $\mathsf{Nr}_n\mathsf{CA}_m$, cf. the last item of Theorem 2.2.

[7] There is subtle distinction between $\mathsf{Nr}_n\mathsf{CA}_m$ and the larger $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$ for $1 < n < m \leq \omega$ that we should point out and that is the following: While if $\mathsf{At}\mathfrak{A} \in \mathsf{AtNr}_n\mathsf{K}_m$ *this does not imply that* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_m$; but on the contrary if $\mathsf{At}\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$, then $\mathfrak{A} \in \mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_m$.

set algebra of dimension ω consisting of $\mathfrak{L}$-formulas taken in the signature L. The base of $\mathfrak{D}_a$ is M_a, and the set-theoretic operations of $\mathfrak{D}_a$ are read off the semantics of the connectives avialable in $\mathfrak{L}$. In all cases, as long as $\mathfrak{L}$ contains $L_{\omega,\omega}$ as a fragment, we get that $\mathfrak{Tm}\alpha \subseteq \mathsf{Nr}_n\mathfrak{D}$, where $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$. There are three possibilites measuring 'how close' $\mathfrak{Tm}\alpha$ is to $\mathsf{Nr}_n\mathfrak{D}$. We go from the closest to the less close. Either (a) $\mathfrak{Tm}\alpha = \mathsf{Nr}_n\mathfrak{D}$ or (b) $\mathfrak{Tm}\alpha \subseteq_d \mathsf{Nr}_n\mathfrak{D}$ or (c) $\mathfrak{Tm}\alpha \subseteq_c \mathsf{Nr}_n\mathfrak{D}$. It is reasonable to expect that the stronger (the logic) $\mathfrak{L}$ is, the 'more control' α has over the hitherto obtained ω-dilation $\mathfrak{D}$; the closer $\mathfrak{Tm}\alpha$ is to the neat n-reduct of $\mathfrak{D}$ based on $\mathfrak{L}$-formulas.

Suppose we take $\mathfrak{L} = L_{\omega,\omega}$. Then using the fact that in the λ-neat hypernetworks played during the game $\mathbf{H}$ short hyperedges are constantly labelled by λ, one shows that $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$; the isomorphism defined via $b \mapsto (b^{\mathfrak{D}_a}(x_0, \ldots x_{n-1}) : a \in \alpha)$. But using $\mathfrak{L} = L_{\infty,\omega}$ in the same signature, the resulting algebra $\mathfrak{D}$ which is isomorphic to a generalized ω-dimensionl weak set algebra in the sense of [3, Definition 3.1.2 (iv)] (with top element the disjoint union of top elements of the $\mathfrak{D}_a$) based on the (now) $L_{\infty,\omega}$ weak models M_a taken in the same signature L, $a \in \alpha$, will be *complete*. This is so, because the $\mathfrak{D}_a$s are complete; $\sum_{i\in I}^{\mathfrak{D}_a} \phi_i^{\mathfrak{D}_a} = (\bigvee_{i\in I} \phi_i)^{\mathfrak{D}_a}$. Here $\phi^{\mathfrak{D}_a}$ is the set of all sequences s agreeing co-finitely with a fixed sequence in ${}^{\omega}\mathsf{M}_a$ such that $\mathsf{M}_a, s \models \phi$. So both $\mathfrak{D} = \mathbf{P}_{a\in\alpha}\mathfrak{D}_a$ and its n–neat reduct $\mathsf{Nr}_n\mathfrak{D}$ will be complete. Accordingly, one makes the identification $\mathsf{Nr}_n\mathfrak{D} \subseteq_d \mathfrak{Cm}\alpha$. By density, we get that $\mathsf{Nr}_n\mathfrak{D} = \mathfrak{Cm}\alpha$ (since $\mathsf{Nr}_n\mathfrak{D}$ is complete), hence $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and so we get (b) (and (c)) since $\mathfrak{Tm}\alpha \subseteq_d \mathfrak{Cm}\alpha$. Also the property that $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$ is plainly maintained when we passed from $L_{\omega,\omega}$ to $L_{\infty,\omega}$.

For a class $\mathbf{K}$ of algebras, we denote by $\mathbf{K} \cap \mathsf{Count}$ the class of countable algebras in $\mathbf{K}$. Observe that the game $\mathbf{H}_\omega$ 'captures' the class $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega) \cap \mathsf{Count}$ in the sense that if α is a countable atom structure and $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$. Conversely, it can be proved that if $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{Count})$, then $\exists$ has winning strategy in a game with the same moves as $\mathbf{H}$ but played on networks not λ-neat hypernetworks. However, $\mathbf{H}_\omega$ does not characterize the class $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At} \cap \mathsf{Count}$ for it can be shown that $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$ where $\mathfrak{B}$ is the atomic algebra used in item (3) of Theorem 2.2, but $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1} (\supseteq \mathsf{Nr}_n\mathsf{CA}_\omega)$; though (recall that) $\mathsf{At}\mathfrak{B} \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega)$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. On the other hand, the usual

ω-rounded atomic game G characterizes both the class $\mathsf{CRCA}_n \cap \mathsf{Count}$ and the class $\mathsf{At}(\mathsf{CRCA}_n \cap \mathsf{Count})$ (the class of countable completely representable atom structures), and [7, Theorem 3.3.3].

Corollary 3.3. For any $2 < n < m$, any class K such that

$$\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_m \cap \mathsf{CRCA}_n) \subseteq \mathsf{K} \subseteq \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3},$$

K is not elementary

Proof: . Let β be he atom structure of $\mathfrak{C}_{\mathbb{Z},\mathfrak{N}}$. Then $\beta \equiv \alpha$ where α is an atom structure such that $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ and $\alpha \in \mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$. So if K is as in the hypothesis, then $\alpha \in \mathsf{K}$, $\beta \equiv \alpha$, but $\beta \notin \mathsf{At}\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+3} \supseteq \mathsf{K}$. □

Corollary 3.4. Let $2 < n < \omega$ and $k \geq 3$. Then the following classes, together with the intersection of any two of them, the last four taken at the same k, are not elementary: CRCA_n [5], $\mathsf{Nr}_n\mathsf{CA}_{n+k}$ [14, Theorem 5.4.1], $\mathbf{S_d}\mathsf{Nr}_n\mathsf{CA}_{n+k}$, $\mathbf{S_c}\mathsf{Nr}_n\mathsf{CA}_{n+k}$.

4. Appendix

Theorem 4.1. *Let $2 < m < n < \omega$. For any $k \geq 0$, the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k}$ and RCA_m is not finitely axiomatizable over $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+l}$ for any $0 < l < \omega$.*

Proof: Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathsf{CA}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$-wide m-dimensional wide ω hypernetworks [6, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [6, Definition 15.2]. Furthermore, for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_n$, $\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{\mathsf{n}+1}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$ by [6, Corollaries 15.7, 5.10, Exercise 2, p. 484, Remark 15.13] □

Theorem 4.2. *For $3 \leq m \leq n$ and $r < \omega$ there exists finite algebras $\mathfrak{D}(m,n,r) \in \mathsf{CA}_m$.*

1. $\mathfrak{D}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_{\mathsf{n}}$,
2. $\mathfrak{D}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1}$,
3. *$\Pi_{r/U}\mathfrak{D}(m,n,r)$ is elementarily equivalent to a $\mathfrak{C} \in \mathsf{Nr}_m\mathsf{CA}_{\mathsf{n}+1}$.*

We define the algebras $\mathfrak{D}(m,n,r)$ for $3 \le m \le n < \omega$ and r and then give a sketch of (II) given in detail in [9, pp. 211–215]. We start with.

DEFINITION 4.3. Define a function $\kappa : \omega \times \omega \to \omega$ by $\kappa(x,0) = 0$ (all $x < \omega$) and $\kappa(x, y+1) = 1 + x \times \kappa(x,y))$ (all $x, y < \omega$). For $n, r < \omega$ let

$$\psi(n,r) = \kappa((n-1)r, (n-1)r) + 1.$$

This is to ensure that $\psi(n,r)$ is sufficiently big compared to n, r for the proof of non-embeddability to work. The second parameter $r < \omega$ may be considered as a finite linear order of length r. For any $n < \omega$ and any linear order r, let

$$\mathfrak{B}(n,r) = \{\mathsf{Id}\} \cup \{a^k(i,j) : i < n-1; j \in r,\ k < \psi(n,r)\}$$

where $\mathsf{Id}, a^k(i,j)$ are distinct objects indexed by k, i, j. (So here every atom $a(i,j)$ is split into $\psi(n,r)$ subatoms). The *forbidden* triples) are:

$$\{(\mathsf{Id}, b, c) : b \ne c \in \mathfrak{B}(n,r)\}$$
$$\cup$$
$$\{(a^k(i,j), a^{k'}(i,j), a^{k^*}(i,j')) : k, k', k^* < \psi(n,r),\ i < n-1,\ j' \le j \in r\}.$$

Let $3 \le m \le n < \omega$. The set of m-basic matrices on $\mathfrak{R}$ is is a QEA_m atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. $\mathfrak{D}(m,n,r)$ is defined to be the complex algebra of the m-dimensional atom structure $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$, that is, $\mathfrak{D}(m,n,r) = \mathfrak{Cm}\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$. Unlike the algebras $\mathfrak{C}(m,n,r)$ used to prove theorem 4.1, the algebras $\mathfrak{D}(m,n,r)$ are now finite. It is not hard to see that $3 \le m$, $2 \le n$ and $r < \omega$ the algebra $\mathfrak{D}(m,n,r)$ satisfies all of the axioms defining CA_m except, perhaps, the commutativity of cylindrifiers which it satisfies because $\mathsf{Mat}_m(\mathsf{At}\mathfrak{R})$ is a (symmetric) cylindric basis, so that overlapping matrices amalgamate. Furthermore, if $3 \le m \le m'$, then $\mathfrak{D}(m,n,r) \cong \mathsf{Nr}_m\mathfrak{D}(m',n,r)$ via $X \mapsto \{f \in \mathsf{Mat}_{m'}(\mathsf{At}\mathfrak{R}) : f \restriction_{m\times m} \in X\}$.

We give a sketch of proof of 4.2(II), which is the heart and soul of the proof. Assume hoping for a contradiction that $\mathfrak{D}(m,n,r) \subseteq \mathsf{Nr}_m\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_{n+1}$, some finite m, n, r. Then for $1 \le t \le n+1$, it can be shown inductively that there must be a 'large set' S_t of distinct elements of $\mathfrak{C}$, satisfying certain inductive assumptions, which we outline next. Here largness depends on t and weakens as t increases; for example S_n has only two elements. For each $s \in S_t$ and $i, j < n+2$ there is an element $\alpha(s,i,j) \in \mathfrak{B}(n,r)$ obtained from s by cylindrifying all dimensions in $(n+1)\setminus\{i,j\}$, then using substitutions to replace i,j by $0,1$. It can be shown that the triple $(\alpha(s,i,j),\alpha(s,j,k),\alpha(s,i,k))$ is consistent (not forbidden). The induction hypothesis says chiefly that $\mathsf{c}_n s$ is constant, for $s\in S_t$, and for $l<n$ there are fixed $i<n-1$, $j<r$ such that for all $s\in S_t$, $\alpha(s,l,n)\leq a(i,j)$. This defines, like in the proof of theorem 15.8 in [7, p. 471] , two functions $I:n\to(n-1)$, $J:n\to r$ such that $\alpha(s,l,n)\leq a(I(l),J(l))$ for all $s\in S_t$. The *rank* $\mathsf{rk}(I,J)$ of (I,J) (as defined in [7, Definition 15.9]) is the sum (over $i<n-1$) of the maximum j with $I(l)=i$, $J(l)=j$ (some $l<n$) or -1 if there is no such j. From S_t one constructs a set S_{t+1} with index functions (I',J'), still relatively large (large in terms of the number of times we need to repeat the induction step) where the same induction hypotheses hold but where $\mathsf{rk}(I',J')>\mathsf{rk}(I,J)$. By repeating this enough times (more than nr times) we obtain a non-empty set T with index functions of rank strictly greater than $(n-1)\times(r-1)$, an impossibility. We sketch the induction step. Since I cannot be injective there must be distinct $l_1,l_2<n$ such that $I(l_1)=I(l_2)$ and $J(l_1)\leq J(l_2)$. We may use l_1 as a "spare dimension" (changing the index functions on l will not reduce the rank). Since $\mathsf{c}_n s$ is constant, we may fix $s_0\in S_{t-1}$ and choose a new element s' below $\mathsf{c}_l s_0\cdot\mathsf{s}_l^n\mathsf{c}_l s$, with certain properties. Let $S_{t+1}=\{s':s\in S_t\setminus\{s_0\}\}$. Re-establishing many of the induction hypotheses for S_{t+1} is not too hard. Also, it can be shown that $J'(l)\geq J(l)$ for all $l<n$. Since $(\alpha(s,i,j),\alpha(s,j,k),\alpha(s,i,k))$ is consistent and by the definition of the forbiden triples either $\mathsf{rng}(I')$ properly extends $\mathsf{rng}(I)$ or there is $l<n$ such that $J'(l)>J(l)$, hence $\mathsf{rk}(I',J')>\mathsf{rk}(I,J)$. The idea of constructing S_{t+1} from S_t is given pictorially on [8, Figure 2, p. 8] in the context of CAs. The essence of the ideas used in [8, 9] is the same. Suppose we are at stage t. Then every $x\in S_t$ gives a set of colours (atoms) denoted in [8] by $x(i,t)$ $(i<t)$. One gets S_{t+1} from S_t by first 'glueing together' any two elements x,z of S_t, using $t+1$ as a spare dimension, first moving the tth co-ordinate of x to $t+1$ forming $\mathsf{s}^t_{t+1}x$. By fixing z and varying x one gets a huge number of different elements. Their $(t,t+1)$th colours cannot be controlled yet; they may not be the same. To get over this hurdle, one uses the pigeon-hole principal to pick the *still large set* S_{t+1} in which the $(t,t+1)$th colour is fixed to be the same. 'Largness' enables one to do so.

We summarize next the essence of the idea used in the solution of [3, Problem 2.12]:

In Figure 2 in [8] there is a top element that is connected by coloured edges to the intermediate elements that are all connected to a bottom element. The number of elements (in this figure) is the number of colours plus one. So one gets the same control as rainbow algebras provided by (the second independent parameter) G*. The key idea here is that the proof of Ramsey in this context does not require an uncontrollable Ramsey number of 'spare dimensions', which were the versions used by Monk and Maddux before proving non finite axiomatizability but only one more than the number of colours used.*

For the above non-representable Monk-style algebras denoted by $\mathfrak{A}(n,r)$, $3 \leq m < n < \omega$ and $r \in \omega$, it is easy to see that $\exists$ cannot win the usual infinite atomic game. But this time one can use 'a hyperbasis game' denoted by $G_r^{m,n+1}$ in [6] with r denoting the number of rounds, to pin point the leask $k > n$ for which $\mathfrak{A}(n,r)$ 'stops to be representable' getting the sharper result we want. The game $G_r^{m,n+1}$ is stronger than G_ω, involving additional amalgamation moves played on $n+1$-dimensional m-wide hypernetworks. One can show that $\forall$ has a winning strategy in $G_r^{m,n+1}(\mathsf{At}\mathfrak{A}(n,r))$, using exactly $n+1$ nodes (for any $r < \omega$), getting the same control we get from rainbows using the parameter G, and in fact the best possible. This is the approach adopted in [7]. Here $\mathfrak{A}(n,r)$ has an n-dimensional cylindric basis, but no $n+1$*-dimensional hypebasis*. Worthy of note, is that the last condition is strictly stronger than 'not having an $n+1$-dimensional cylindric basis'. Relation algebras having n-dimensional cylindric basis but no $n+1$-dimensional cylindric basis were constructed by Maddux. We refer to [8] for more. In the proof of theorem 4.1, one uses that $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$. As stated in the last item of theorem 4.2, we do not guarantee that the ultraproduct on r of the $\mathfrak{D}(m,n,r)$s $(2 < m < n < \omega)$ is representable. A standard Lös argument shows that $\Pi_{r/U}\mathfrak{C}(m,n,r) \cong \mathfrak{C}(m,n,\Pi_{r/U}r)$ and $\Pi_{r/U}r$ contains an infinite ascending sequence. Here one extends the definition of ψ by letting $\psi(n,r) = \omega$, for any infinite linear order r. The infinite algebra $\mathfrak{D}(m,n,J) \in \mathbf{El}\mathsf{Nr}_n\mathsf{CA}_{n+1}$ when J is the infinite linear order as above. Since $\Pi_{r/U}r$ is such, then we get $\Pi_{r/U}\mathfrak{D}(m,n,r) \in \mathbf{El}\mathsf{Nr}_m\mathsf{CA}_{n+1}(\subseteq \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1})$, cf. [9, pp. 216–217]. This suffices to show that for any positive k, the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k+1}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{m+k}$.

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