Cairo University **Discrete Structures (Msc521)**

FGSSR **Lecture 7**

Department of Mathematical Statistics **Prof. Ali A. A-Rahman**

Valid Argument

Consider the implication $\boldsymbol{p_1} \wedge \boldsymbol{p_2} \wedge \cdots \boldsymbol{p_n} \rightarrow$ **q,** where n is a positive integer, the statements $p_1, p_2, \cdots p_n$ $are\ called\ the\ premises$ of the argument, and the statement q is the conclusion for the argument. The preceding argument is called valid if whenever each of premises $p_1, p_2, p_{3,} \cdots p_n$ is true, the then conclusion is likewise true.

Remark7.1: If any one of the premises $p_i, i \in \{1,2, \cdots, n\}$ is false, then the hypothesis $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ false and the implication $p_1 \wedge p_2 \wedge \cdots \wedge p_n \rightarrow q$ is automatically true, regardless of truth value of q. Therefore, one way to establish the validity of the given argument is to show that the statement $p_1 \wedge p_2 \wedge \cdots p_n \rightarrow q$ is a tautology.

Example7.1: Let p, q, r denote the primitive statements given as:

P: Roger studies.

q: Ruger plays tennis.

r: Roger passes discrete mathematics.

Consider the following premises:

p_1: if Ruger studies, then he will pass discrete math.

p_2: if Ruger does not play tennis, then he will study.

p_3 : Ruger failed discrete mathemsntics.

Determine whether the argument: $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is valid.

Solution: to prove the validity of this argument we must show that

$[p_1 \wedge p_2 \wedge p_3] \to q$ is a tautology. This can be shown using truth table method or using some mathematical operations as follows:

$$[(p_1 \wedge p_2 \wedge p_3) \to q] \Leftrightarrow [[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q]$$

$$\Leftrightarrow [\neg[(\neg p \vee r) \wedge (\neg\neg q \vee p) \wedge \neg r] \vee q]$$

$$\Leftrightarrow [\neg[(\neg p \vee r) \wedge (q \vee p) \wedge \neg r] \vee q]$$

$$\Leftrightarrow [\neg[(\neg p \vee r) \wedge \neg r \wedge (q \vee p)] \vee q]$$

$$\Leftrightarrow [\neg[((\neg p \wedge \neg r) \vee (r \wedge \neg r)) \wedge (q \vee p)] \vee q]$$

$$\Leftrightarrow \left[\neg\left[\left((\neg p \wedge \neg r) \vee F\right) \wedge (q \vee p)\right] \vee q\right]$$

$$\Leftrightarrow [\neg[(\neg p \wedge \neg r) \wedge (q \vee p)] \vee q]$$

$$\Leftrightarrow \left[\left((p \vee r) \vee \neg(q \vee p)\right) \vee q\right]$$

$$\Leftrightarrow \left[[(p \vee r) \vee (\neg q \wedge \neg p)] \vee q\right]$$

$$\Leftrightarrow \left[(p \vee r) \vee \left((\neg q \vee q) \wedge (\neg p \vee q)\right)\right]$$

$$\Leftrightarrow \left[(p \vee r) \vee \left(T \wedge (\neg p \vee q)\right)\right] \Leftrightarrow [(p \vee r) \vee (\neg p \vee q)]$$

$$\Leftrightarrow [p \vee \neg p \vee r \vee q] \Leftrightarrow [T \vee r \vee q] \Leftrightarrow T.$$

Another solution: If we perform the truth table, we note that in order to establish the validity of the argument, we need to consider only those rows of the table where each of the three premises $p \to r$, $\neg q \to p$, and $\neg r$ is true. If $\neg r$ is true then r will be false. But as $p \to r$ is true, it follows that p must be false. Again $\neg q \to p$ is true implies that $\neg q$ is false. Hence q must be always true. Therefore the implication is tautology and the argument is valid.

Remark 7.2: It is clear that to prove the validity of an argument, the truth table method is easier than using mathematical rules. But probably are two disadvantages for using the truth table method:

(1) The number of rows needed to write the truth table is 2^n, where $n =$ the number primitive statements included in the propositions, therefore, as the number of premises gets larger, the number of rows increase rapidly so this method loses its appeal.

(2) It is impractical to use truth table method since in any problem we have premises and we want to get a conclusion (which may be unknown).

Therefore, it is necessary to find another way to prove the validity of the argument.

Some Rules of Inference

The rules of inference are fundamental in the development of a step by step validation of how the conclusion q logically follows from the premises $p_i, i \in \{1,2,3, \dots n\}$ in implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots p_n) \rightarrow \text{q}.$$

Such a development will establish the validity of the given argument, it will how the truth of the conclusion can be deduced from the truth of the premises. Each rule of inference arises from a logical implication.Many rules of inferences arise in the study of logic. We concentrate on those that help us validate the arguments that arise in our study of logic. These rules will also help us to prove some mathematical theorems

(1) Modus ponens (Rule of Detachment)rule:

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

Proof:

$$\big[[p \wedge (p \rightarrow q)] \rightarrow q\big] \Leftrightarrow \big[[p \wedge (\neg p \vee q)] \rightarrow q\big]$$

$$\Leftrightarrow [\neg[p \wedge (\neg p \vee q)] \vee q] \Leftrightarrow \big[[\neg p \vee \neg(\neg p \vee q)] \vee q\big]$$

$$\Leftrightarrow [(\neg p \vee q) \vee \neg(\neg p \vee q)] \Leftrightarrow [s \vee \neg s] \Leftrightarrow T.$$

Summary of steps used in the previous proof

(1) We (in the first and second steps) make use of the fact : $(p \to q) \Leftrightarrow (\neg p \vee q)$.
(2) In the third step we make use of D'Morgan's rule.
(3) In forth step we make use of the associativity property of the operation "$\vee$".
(4) In the last step we make use of the substitution rule.

Remark 7.3: Tabular form of Modus Ponens Rule:

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

Type equation here.

(2) The second rule of inference (the law of Syllogism):

$$[(p \to q) \wedge (q \to r)] \to (p \to r),$$

Where $p, q, and\ r$ are any primitive statements.

In tabular form we have:

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Proof: it easy to use truth table method to show that

$\big[[(p \to q) \wedge (q \to r)] \to (p \to r)\big]$ is a tautology.

(3) Third Rule of The inference(Modus Tollens):

$\left[[(p \to q) \land \neg q] \to \neg p \right]$, for any primitive statements, p and q. In tabular form, we have:

$$p \to q$$

$$\underline{\neg q}$$

$$\therefore \neg p$$

Proof: $\left[[(p \to q) \land \neg q] \to \neg p \right]$

$$\Leftrightarrow [\neg[(\neg p \lor q) \land \neg q] \lor \neg p]$$

$$\Leftrightarrow \left[[\neg(\neg p \lor q) \lor \neg\neg q] \lor \neg p \right]$$

$$\Leftrightarrow [\neg(\neg p \lor q) \lor (q \lor \neg p)] \Leftrightarrow T$$

(4) The Rule of conjunction: for any primitive statements p and q

$$p$$

$$\underline{q}$$

$$\therefore p \land q$$

(5) Rule of Disjunctive Syllogism:

$\left[[(\boldsymbol{p} \lor \boldsymbol{q}) \land \neg \boldsymbol{p}] \to \boldsymbol{q} \right]$**,**

for any primitive statements p and q.

$$\boldsymbol{p} \lor \mathbf{q}$$

$$\underline{\neg p}$$

$$\therefore q$$

Proof: $\left[[(\boldsymbol{p} \lor \boldsymbol{q}) \land \neg \boldsymbol{p}] \to \boldsymbol{q} \right]$

$$\Leftrightarrow [\neg[(p \lor q) \land \neg p] \lor q]$$

$$\Leftrightarrow [[\neg(p \vee q) \vee \neg\neg p] \vee q]$$
$$\Leftrightarrow [\neg(p \vee q) \vee (p \vee q)]$$
$$\Leftrightarrow T.$$

(6) Rule of contradiction:

$$[\neg \boldsymbol{p} \rightarrow \boldsymbol{F}] \rightarrow \boldsymbol{p},$$

for any primitive statement.

Proof: $[(\neg \boldsymbol{p} \rightarrow \boldsymbol{F}) \rightarrow \boldsymbol{p}] \Leftrightarrow [\neg[(\neg\neg p \vee F) \vee p]]$

$\Leftrightarrow [\neg[(\boldsymbol{p} \vee \boldsymbol{F})] \vee \boldsymbol{p}] \Leftrightarrow (\neg \boldsymbol{p} \vee \boldsymbol{p}) \Leftrightarrow \boldsymbol{T}.$

Example 7.4:

Show the validity of this argument:

$$\begin{array}{l} p \\ p \rightarrow \neg q \\ \underline{\neg q \rightarrow \neg r} \\ \therefore \neg r \end{array}$$

Solution:

Steps	Reasons
(1) $\boldsymbol{p}$	**premise**
(2) $p \rightarrow \neg q$	**premise**
(3) $\neg q$	**from(1),(2)by modus ponens.**
(4) $\neg \boldsymbol{q} \rightarrow \neg \mathbf{r}$	**premise.**
(5) $\therefore \neg \boldsymbol{r}$	**from (3),(4) by modus ponens.**

Example 7.5:

Show that the following argument is valid

$$\begin{array}{l} p \rightarrow r \\ \boldsymbol{r} \rightarrow \boldsymbol{s} \\ \boldsymbol{t} \vee \neg \boldsymbol{s} \\ \neg \boldsymbol{t} \vee \boldsymbol{u} \\ \underline{\neg u \quad\quad} \\ \neg p \end{array}$$

Solution:

Steps	Reasons
(1) $p \to r$	premise
(2) $r \to s$	premise
(3) $p \to s$	from (1),(2) syllogism
(4) $t \vee \neg s$	premise
(5) $\neg s \vee t$	from (1),(2), commutativity of $\vee$.
(6) $s \to t$	logically equivalence to (5).
(7) $p \to t$	from (3),(6) by syllogism.
(8) $\neg t \vee u$	premise.
(9) $t \to u$	logically equivalence to 8.
(10) $p \to u$	from (7),(9), syllogism.
(11) $\neg u$	premise.
(12) $\neg p$	from (10),(11) by modus Tollens.

Example 7.6:

Show that the following argument is valid

$$p \to q$$

$$q \to (r \wedge s)$$

$$\neg r \vee (\neg t \vee u)$$

$$\underline{p \wedge t \quad\quad}$$

$$\therefore u$$

Solution:

Steps	Reasons
(1) $p \to q$	premise.

(2) $q \to (r \wedge s)$ premise.

(3) $p \to (r \wedge s)$ from (1),(2) syllogism.

(4) $p \wedge t$ premise.

(5) p from (4),conjunctive simplification

(6) t from (4),conjunctive simplification.

(7) $r \wedge s$ from (3),(5),modus ponens.

(8) r from 7,conjunctive simplification

(9) $\neg r \vee (\neg t \vee u)$ premise

(10) $\neg t \vee u$ from 8,9,disjunctive syllogism.

(11) u from 6,10,disjunctive syllogism.