Cairo University Discret Structures (M sc521)

FGSSR Lecture 6

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Some important additional connectives

(1) "Nand" (Not and) "↑".

The truth table of this operator is defined as follows:

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

(2) " (Not or)"

The truth table is defined as follows:

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \downarrow Q}$
$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$
$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$
$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$
$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$

Example 6.1: Show that **Demorgan's laws** are satisfied.

Solution: We are to show that**:**

(1) $\neg(\boldsymbol{P} \downarrow \boldsymbol{Q}) \Leftrightarrow (\neg\boldsymbol{P} \uparrow \neg\boldsymbol{Q})$ **.**

(2) $\neg(\boldsymbol{P} \uparrow \boldsymbol{Q}) \Leftrightarrow (\neg\boldsymbol{P} \downarrow \neg\boldsymbol{Q})$**.**

Proof:

(1) By definition and direct application of Demorgan's laws one gets:
$\neg(P \uparrow Q) \Leftrightarrow \neg\neg(P \wedge Q) \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow$
$(\neg P \downarrow \neg Q)$.

(2) Similarly we easily have:

$$\neg(P \downarrow Q) \Leftrightarrow \neg\neg(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow (\neg P \uparrow \neg Q).$$

Example 6.2 : Use the operator "↑" only to represent the following statements:

(a) $\neg P$. **(b)** $P \vee Q$. **(c)** $P \wedge Q$.

(d) $P \to Q$. **(e)** $P \leftrightarrow Q$.

Solution:

(a) Using "idempotent property of "∧",and the definition of "↑", we get:

$$\neg(P) \Leftrightarrow \neg(P \wedge P) \Leftrightarrow (P \uparrow P).$$

(b) Using the rule of double negation, Demorgan's laws, and the definition of "↑" one gets:

$$(P \vee Q) \Leftrightarrow \neg\neg(P \vee Q) \Leftrightarrow \neg(\neg P \ \wedge \neg Q) \Leftrightarrow$$
$$(\neg P \uparrow \neg Q) \Leftrightarrow [(P \uparrow P) \uparrow (Q \uparrow Q)].$$

(c) Using the rule of double negation, the definition of "↑" , and the result of part (a), one gets:

$$(P \wedge Q) \Leftrightarrow \neg\neg(P \wedge Q) \Leftrightarrow \neg(P \uparrow Q)$$
$$\Leftrightarrow [(P \uparrow Q) \uparrow (P \uparrow Q)].$$

(d) Using Demorgan's law, the definition of "↑", and the result of part (a) one gets:

$$(\boldsymbol{P} \longrightarrow \boldsymbol{Q}) \Leftrightarrow (\neg\boldsymbol{P} \vee \boldsymbol{Q}) \Leftrightarrow \neg(\boldsymbol{P} \wedge \neg\boldsymbol{Q}) \Leftrightarrow$$
$$(\boldsymbol{P} \uparrow \neg\boldsymbol{Q}) \Leftrightarrow [\boldsymbol{P} \uparrow (\boldsymbol{Q} \uparrow \boldsymbol{Q})].$$

(e) using part (d) and part c , one easily gets:

(e) $(\boldsymbol{P} \leftrightarrow \boldsymbol{Q}) \Leftrightarrow [(\boldsymbol{P} \longrightarrow \boldsymbol{Q}) \wedge (\boldsymbol{Q} \to \boldsymbol{P})] \Leftrightarrow$

$$\Leftrightarrow \big[[\boldsymbol{P} \uparrow (\boldsymbol{Q} \uparrow \boldsymbol{Q})] \wedge [\boldsymbol{Q} \uparrow (\boldsymbol{P} \uparrow \boldsymbol{P})]\big]$$

$$\Big[\big[[\boldsymbol{P} \uparrow (\boldsymbol{Q} \uparrow \boldsymbol{Q})] \uparrow [\boldsymbol{Q} \uparrow (\boldsymbol{P} \uparrow \boldsymbol{P})]\big] \uparrow \big[[\boldsymbol{P} \uparrow (\boldsymbol{Q} \uparrow \boldsymbol{Q})] \uparrow [\boldsymbol{Q} \uparrow (\boldsymbol{P} \uparrow \boldsymbol{P})]\big]\Big].$$

Example 6.3: Use the operator "↓"only to express the following statements:

(a) $\neg \boldsymbol{P}$. **(b)** $\boldsymbol{P} \vee \boldsymbol{Q}$. **(c)** $\mathbf{P} \wedge \mathbf{Q}$.

(d) $\boldsymbol{P} \rightarrow \boldsymbol{Q}$. **(e)** $P \leftrightarrow Q$.

Solution:

(a) Using the idempotent property of " ∨"**,** and the definition of **"↓"** one gets**:**

$$\neg \boldsymbol{P} \Leftrightarrow \neg(\boldsymbol{P} \vee \boldsymbol{P}) \Leftrightarrow (\boldsymbol{P} \downarrow \boldsymbol{P}).$$

(b) Using the double negation rule, the substitution rule, and the definition of "↓", one gets:

$$(\boldsymbol{P} \vee \boldsymbol{Q}) \Leftrightarrow \neg\neg(\boldsymbol{P} \vee \boldsymbol{Q}) \Leftrightarrow \neg(\boldsymbol{P} \downarrow \boldsymbol{Q})$$
$$\Leftrightarrow [(\boldsymbol{P} \downarrow \boldsymbol{Q}) \downarrow (\boldsymbol{P} \downarrow \boldsymbol{Q})].$$

(c) Using the double negation rule, Demorgan's rule, and the definition of "↓", one gets:

$$(P \wedge Q) \Leftrightarrow \neg\neg(P \wedge Q) \Leftrightarrow \neg(\neg P \ \vee \neg Q)$$

$$\Leftrightarrow (\neg P \downarrow \neg Q) \ \Leftrightarrow [(P \downarrow P) \downarrow (Q \downarrow Q)].$$

(d) Writing the if statement in terms of the or statement and making use of part (b), one finds:

$$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q) \Leftrightarrow [(\neg P \downarrow Q) \downarrow (\neg P \downarrow Q)]$$
$$\Leftrightarrow \left[\left((\boldsymbol{P} \downarrow \boldsymbol{P}) \downarrow \boldsymbol{Q}\right) \downarrow \left((\boldsymbol{P} \downarrow \boldsymbol{P}) \downarrow \boldsymbol{Q}\right)\right].$$

(e) using part (c), (d), and substitution rule, we get:

$$(P \leftrightarrow Q) \Leftrightarrow [(P \longrightarrow Q) \wedge (Q \rightarrow P)] \ \Leftrightarrow$$

$$\left[\left[\left((\boldsymbol{P} \downarrow \boldsymbol{P}) \downarrow \boldsymbol{Q}\right) \downarrow \left((\boldsymbol{P} \downarrow \boldsymbol{P}) \downarrow \boldsymbol{Q}\right)\right] \wedge \left[\left((\boldsymbol{Q} \downarrow \boldsymbol{Q}) \downarrow \boldsymbol{P}\right) \downarrow (\boldsymbol{Q} \downarrow \boldsymbol{Q}) \downarrow \boldsymbol{P}\right]\right]$$

$$\Leftrightarrow \left((\boldsymbol{r} \downarrow \boldsymbol{r}) \downarrow (\boldsymbol{s} \downarrow \boldsymbol{s})\right).$$

Where r: $\left[\left((P \downarrow P) \downarrow Q\right) \downarrow \left((P \downarrow P) \downarrow Q\right)\right]$,

$s: [((Q \downarrow Q) \downarrow P) \downarrow ((Q \downarrow Q) \downarrow P)].$

Example 6.4: Show that:

$$(p \underline{\vee} q) \Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg p)] \Leftrightarrow \neg(p \leftrightarrow q) .$$

Solution:

$$\neg(\boldsymbol{P} \leftrightarrow \boldsymbol{Q}) \Leftrightarrow \neg[(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{q} \to \boldsymbol{p})]$$

$$\Leftrightarrow \neg[(\neg \boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{q} \vee \boldsymbol{p})]$$

$$\Leftrightarrow [\neg(\neg \boldsymbol{p} \vee \boldsymbol{q}) \vee \neg(\neg \boldsymbol{q} \vee \boldsymbol{p})]$$

$$\Leftrightarrow [(\boldsymbol{p} \wedge \neg \boldsymbol{q}) \vee (\boldsymbol{q} \wedge \neg \boldsymbol{p})] . \qquad \textbf{(1)}$$

Now write r: $p \wedge \neg q$, $s: q \wedge \neg p$,

$x: p \underline{\vee} q$, and $z: r \vee s$

We shall use the truth table technique to show that the equivalence of the first two parts of the problem

p	q	x	r	s	z
t	t	f	f	f	f
t	f	t	t	f	t
f	t	t	f	t	t
f	f	f	f	f	f

Therefore, we have:

$(p \underline{\vee} q) \Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg p)]$ **(2)**

From (1) and (2) we have the result.

Definition 6.4: if p, q are arbitrary statements such that p$\to q$ is a tautology, then we say that p logically implies q and we write p$\Longrightarrow q$ to denote this situation.

Example 6.5: without using truth table method prove the following logical implications:

(a) $(p \wedge q) \Longrightarrow p.$ **(b)** $p \Longrightarrow (p \vee q).$

(c) $[(p \vee q) \wedge \neg p] \Longrightarrow q$.

Solution :

(a) We are **to** show that $(p \wedge q) \rightarrow \boldsymbol{p}$ is tautology.

$[(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \boldsymbol{p}] \Leftrightarrow [\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \vee \boldsymbol{p}]$

$(\neg \boldsymbol{p} \vee \neg \boldsymbol{q} \vee \boldsymbol{p}) \Leftrightarrow [(\neg \boldsymbol{p} \vee \boldsymbol{p}) \vee \boldsymbol{q}] \Leftrightarrow (\boldsymbol{T} \vee \boldsymbol{q}) \Leftrightarrow \boldsymbol{T}.$

(b) $[\boldsymbol{p} \rightarrow (\boldsymbol{p} \vee \boldsymbol{q})] \Leftrightarrow [\neg \boldsymbol{p} \ \vee (\boldsymbol{p} \vee \boldsymbol{q})]$

$\Leftrightarrow [(\neg p \vee p) \vee q] \Leftrightarrow (T \vee q) \Leftrightarrow T.$

(c) $\left[\left((p \vee q) \wedge \neg p\right) \rightarrow q\right] \Leftrightarrow [\neg([p \vee q] \wedge \neg p) \vee q]$

$\Leftrightarrow [((\neg p \wedge \neg q) \vee p) \vee q] \Leftrightarrow [[(\neg p \vee p) \wedge (\neg q \vee p)] \vee q]$

$\Leftrightarrow [[T \wedge (\neg q \vee p)] \vee q] \Leftrightarrow [(\neg q \vee p) \vee q]$

$\Leftrightarrow [p \vee (\neg q \vee q)] \Leftrightarrow [p \vee T] \Leftrightarrow T$.