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What is This?
Thermal buckling and nonlinear flutter behavior of shape memory alloy hybrid composite plates

Hesham Hamed Ibrahim¹, Mohammad Tawfik² and Hani Mohammed Negm²

Abstract
A new nonlinear finite element model is provided for the nonlinear flutter response of shape memory alloy (SMA) hybrid composite plates under the combined effect of thermal and aerodynamic loads. The nonlinear governing equations for moderately thick rectangular plates are obtained using first-order shear-deformable plate theory, von Karman strain-displacement relations and the principle of virtual work. To account for the temperature dependence of material properties, the thermal strain is stated as an integral quantity of thermal expansion coefficient with respect to temperature. The aerodynamic pressure is modeled using the quasi-steady first-order piston theory. Newton-Raphson iteration method is employed to obtain the thermal post-buckling deflection, while the linearized updated mode method is implemented in predicting the limit-cycle oscillation at elevated temperatures. Numerical results are presented to show the thermal buckling and flutter characteristics of SMA hybrid composite plates, illustrating the effect of the SMA volume fraction and pre-strain value on the aero-thermo-mechanical response of such plates.

Keywords
Nonlinear FEM, panel flutter, shape memory alloys, shear deformable plates, thermal buckling

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I. Introduction

I.1. Thermal buckling
The external skin of high speed flight vehicles experiences high temperature rise due to aerodynamic heating which can induce thermal buckling and dynamic instability. In general, thermal buckling does not indicate structural failure. However, the thermal large deflection of the skin panels can change its aerodynamic shape causing reduction in the flight performance.

A comprehensive literature review on thermally-induced flexure, buckling, and vibration of plates and shells was presented by Tauchart (1991) and Thornton (1993). Gray and Mei (1991) investigated the thermal post-buckling behavior and free vibrations of thermally buckled composite plates using the finite element method. Shi and Mei (1999) solved the problem of thermal post-buckling of composite plates with initial imperfections using the finite element method. Jones and Mazumdar (1980) investigated the linear and nonlinear dynamic behavior of plates at elevated temperatures. They presented analytical solutions for the thermal buckling and post-buckling behavior of a plate strip. Shi et al. (1999) investigated the thermal post-buckling behavior of symmetrically laminated and anti-symmetric angle-ply, and deflection of asymmetrically laminated composite plate under mechanical and thermal loads. Ibrahim et al. (2006) investigated the aero-thermal buckling of thin functionally-graded plates. An incremental formulation was adopted to capture the effect of the history of variation of the thermal expansion coefficient with temperature on the panel response.

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1.2. Panel Flutter

Supersonic panel flutter is a self-excited oscillation of panels exposed to aerodynamic flow at high Mach numbers. For dynamic pressures less than the flutter boundary, random pressure fluctuations due to turbulent boundary layer control the panel response. At this regime, the panel response can be determined using linear sonic fatigue analysis techniques and is usually in the small displacement region. As the dynamic pressure increases, the panel stiffness is increased by the aerodynamic loading such that the first mode natural frequency increases while the second mode natural frequency decreases. At the flutter boundary, the two modes coalesce and the panel starts experiencing an aerodynamic instability, i.e. flutter.

Thin plates are a commonly used form of structural components especially in aerospace vehicles, such as high-speed aircraft, rockets, and space crafts, which are subjected to thermal loads due to aerodynamic and/or solar radiation heating. This results in a temperature distribution over the surface and thermal gradient through the thickness of the plate. The presence of these thermal fields along with aerodynamic flow results in a flutter instability at lower dynamic pressure or a larger flutter oscillation amplitudes at the same dynamic pressure, which in turn affects the plate fatigue life performance. Accordingly, it is important to consider the interactive effect of both flutter and thermal load.

A vast amount of literature exists on panel flutter using different aerodynamic theories to model the aerodynamic pressure. Mei et al. (1999) presented a review on the various analytical methods and experimental results of supersonic and hypersonic panel flutter. Liaw (1997) studied the geometrically nonlinear supersonic flutter of thin laminated composite plate structures subjected to thermal loads. Abdel-Motagaly et al. (1999) investigated the effect of arbitrary flow direction on the large amplitude supersonic flutter of moderately thick composite panels, using the von Karman strain-displacement relation to account for large amplitude limit-cycle oscillations. The nonlinear finite element formulation introduced by Mei (1977) was the basis on which Dixon and Mei (1993), and Xue and Mei (1993) built their finite element models to analyze the flutter boundaries, the limit-cycle oscillations, and the thermal problems. Dixon and Mei (1993) presented a nonlinear flutter analysis of thin composite panels using finite element method. The governing equations of motion were formulated using the principle of virtual work, while the eigen value problem was solved by utilizing the linearized updated mode with nonlinear time function (LUM/NTF) approximation. Xue and Mei (1993) presented an incremental finite element frequency domain solution for the nonlinear flutter response of thin isotropic panels under combined thermal and aerodynamic loads. Dongi et al. (1996) investigated the nonlinear flutter control of flat and slightly curved panels in high supersonic flow using piezoelectric actuators.

Ibrahim et al. (2008) investigated the nonlinear flutter performance for temperature-dependent functionally-graded material panels under combined aerodynamic and thermal loads. To account for the temperature dependence of material properties, the thermal strain is stated as an integral quantity of thermal expansion coefficient with respect to temperature.

1.3. Shape memory alloy applications

Shape memory alloys (SMAs) have a unique ability to completely recover large pre-strains (up to 10% elongation) when heated above certain characteristic temperature called the austenite finish temperature. The austenite start temperature for Nitinol, a type of SMA, can be anywhere between -50 °C and 170 °C by varying the nickel content. During the shape recovery process, a large tensile recovery stress occurs if the SMA is restrained. SMAs are also characterized by the superelasticity phenomenon which will not be considered in our study. Cross et al. (1969) measured the recovery stress of Nitinol at various pre-strain values and Young’s modulus versus temperatures. Both the recovery stresses and Young’s modules of SMA exhibit nonlinear temperature-dependent properties.

Birman (1997) presented a comprehensive review on the literature concerning SMA up to 1997. Jia and Rogers (1992) formulated a mechanical model for composites with shape memory alloy embedded fibers using the micromechanical behavior of the highly nonlinear SMA and adopting the classical lamination plate theory. Tawfik et al. (2002) proposed a novel concept in enhancing the thermal buckling and aeroelastic behavior of plates through embedding SMA fibers in it. They investigated the response of thin SMA hybrid composite plates under the combined action of aerodynamic and thermal loadings utilizing an incremental method along with a nonlinear finite element model based on von Karman strain-displacement relation and the classical lamination plate theory. Park et al. (2004) utilized an incremental technique to study the nonlinear vibration behavior of thermally buckled composite plates embedded with SMA fibers. Roh et al. (2004) provided a finite element formulation for the thermal post-buckling response of thin SMA hybrid composite shell panels using an incremental method on the basis of the layer wise theory and von Karman nonlinear displacement-strain relationships. Guo (2005) offered an efficient finite element method for predicting the buckling temperature, post-buckling deflection, and flutter characteristics for thin SMA hybrid composites, considering nonlinear temperature dependent material properties of SMA through stating the thermal strain as an integral quantity of thermal expansion coefficient with respect to temperature. Gilat and
Aboudi (2004) derived micromechanically established constitutive equations for unidirectional composites with SMA fibers embedded in polymeric metallic matrices. These equations were subsequently employed to analyze the nonlinear behavior of infinitely wide composite plates that are subjected to the sudden application of thermal loading. Ibrahim et al. (2006) extended the work presented in Guo (2005) on the thermal buckling of SMA hybrid composite plate by including the shear deformation effect in the formulation to make it capable of handling moderately thick plates. Moreover, a frequency domain solution for predicting flutter boundaries at elevated temperatures was presented.

This work is an extension of the work presented in Ibrahim et al. (2006) by including the nonlinear flutter behavior of SMA hybrid composite plates. The nonlinear flutter and thermal buckling behavior of an SMA hybrid composite plate under combined thermal and aerodynamic loads are investigated utilizing the constrained shape recovery application of trained Nitinol fibers. In this study, the SMA is activated using the aerodynamic heating created by the presence of shock waves. To account for the highly nonlinear temperature dependence of material properties, the thermal strain is stated as an integral quantity of thermal expansion coefficient with respect to temperature (Ibrahim et al., 2008). This method saves a lot of the computational efforts exerted when applying the incremental technique found in the majority of the published work (Tawfik et al., 2002). Numerical results are provided to show the effect of thermal field, SMA volume fraction and prestrain value on the post-buckling behavior and nonlinear flutter characteristics of such plates.

2. Finite element formulation of thermal post-buckling of SMA-embedded plates

In this section, the equations of motion for moderately thick composite plates impregnated with SMA fibers will be derived. Plate strain is assumed following the first order shear deformation theory and the nonlinear deflections are assumed to follow the von Karman strain displacement relations. The material properties of the matrix material as well as the SMA fibers are assumed to be changing with temperature.

2.1. The displacement-nodal displacement relation

The nodal degrees of freedom vector of a nine-noded rectangular plate element can be written as (Park et al., 2004):

\[
\theta = \begin{bmatrix} w_b \n r \phi_x \n r \phi_y \n r u \n r v \end{bmatrix}^T = \begin{bmatrix} \{w_b\} \n r \{\phi_x\} \n r \{\phi_y\} \n r \{u\} \n r \{v\} \end{bmatrix}
\]  

(1)

where \(w_b\) is the transverse displacement, \(\phi_x\) and \(\phi_y\) are the rotations of the transverse normal about the x and y axes respectively, \(u\) and \(v\) are the membrane displacements in the x and y directions respectively, \(\{w_b\}\) is the nodal transverse displacement vector, \(\{\phi_x\}\) is the nodal rotation of the transverse normal vector, and \(\{w_m\}\) is the nodal membrane displacement vector.

The displacement-nodal displacement relation can be presented in terms of interpolation function matrices \([N_u], [N_{\phi_x}], [N_{\phi_y}], [N_u] and [N_v]\) as:

\[
w = [N_u]\{w_b\}, \phi_x = [N_{\phi_x}]\{w_b\}, \phi_y = [N_{\phi_y}]\{w_b\}, \\quad \text{and} \quad u = [N_u]\{w_m\}, \quad v = [N_v]\{w_m\}
\]  

(2)

2.2. The nonlinear strain-displacement relation

The inplane strains and curvatures, based on von Karman large deflection and first-order shear deformable plate theory, are given by Reddy (1999):

\[
\begin{bmatrix} \varepsilon_x \\
 r \varepsilon_y \\
 r \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_x}{\partial y} \\
 r \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} \\
 r \frac{\partial \phi_x}{\partial y} & \frac{\partial \phi_y}{\partial x} \end{bmatrix} \left[ \frac{1}{2} \left( \frac{\partial w_m}{\partial y} + \frac{\partial w_m}{\partial x} \right)^2 \right] z = \begin{bmatrix} \frac{\partial \phi_x}{\partial x} \\
 r \frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} \end{bmatrix}
\]  

(3)

or in compact form

\[
\{ \varepsilon \} = \{ \varepsilon_m \} + \{ \varepsilon_\theta \} + z \{ \kappa \}
\]  

(4)

Parameters \(u, v\) and \(w\) are displacements in x, y and z directions, respectively. \(\varepsilon_m, \varepsilon_\theta\), and \(z\kappa\) are the membrane linear strain vector, the membrane nonlinear strain vector, and the bending strain vector, respectively. Meanwhile, the transverse shear strain vector can be expressed as

\[
\begin{bmatrix} \gamma_{yz} \\
 r \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \phi_x \\
 r \phi_y \end{bmatrix} + \left[ \frac{\partial w_m}{\partial x} \right] \begin{bmatrix} \frac{\partial \phi_x}{\partial y} \\
 r \frac{\partial \phi_y}{\partial y} \end{bmatrix}
\]  

(5)

2.3. Constitutive equations

The constitutive equations of a traditional composite plate impregnated with SMA fibers are derived. Every layer of the composite matrix has an arbitrary orientation angle \(\theta\) and principal material directions 1, 2 and 3. The SMA fiber is embedded in the 1-direction, and assumed uniformly distributed within each layer.

The material properties of the laminated composite plate embedded with SMA fibers can be written as (Guo, 2005),

\[
E_1 = E_{1m}V_m + E_sV_s
\]  

(6)

\[
E_2 = \frac{E_{2m}E_s}{(E_{2m}V_s + E_sV_m)}
\]  

(7)

\[
G_{12} = \frac{G_{12m}G_s}{(G_{12m}V_s + G_sV_m)}
\]  

(8)
$G_{23} = G_{23m}V_m + G_sV_s$

$v_{12} = v_{12m}V_m + v_sV_s$

$2 = \frac{E_{1m}x_{1m}V_m + E_2x_sV_s}{E_1}$

$\rho = \rho_mV_m + \rho_sV_s$

where the subscripts ‘m’ and ‘s’ mean the composite matrix and SMA fiber, respectively. $E$, $G$, $v$, $\alpha$ and $\rho$ are Young’s modulus, the shear modulus, Poisson’s ratio, the thermal expansion coefficient, and the material density, respectively. In addition, $V_m$ and $V_s$ are the volume fractions of the composite matrix and SMA fibers, respectively.

For the $k^{th}$ composite lamina impregnated with SMA fibers, the stress-strain relations for the first-order shear deformable plate theory may be expressed as follows (Guo, 2005),

$$\{\sigma\}^k = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^k = \begin{bmatrix} \tilde{Q}(T) & \gamma_x^k \\ \gamma_y^k & \gamma_z^k \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z + \gamma_{xy} \end{bmatrix}^k$$

$$- V_m^k \int_{\text{ref}}^{T} \tilde{Q}(\tau) \begin{bmatrix} \alpha(\tau)^k_m \\ \alpha(\tau)^k_n \end{bmatrix} d\tau$$

$$\{\tau\}^k = \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix}^k = \begin{bmatrix} \tilde{Q}_{44}(T) & \tilde{Q}_{45}(T) & \tilde{Q}_{55}(T) \end{bmatrix}^k \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

where $\{\sigma\}$, $\{\sigma\}_r$ and $\{\tau\}$ are the inplane stress vector, the SMA recovery stress vector at the given temperature $T$, and the transverse shear vector, respectively. In addition, $\{\alpha\}_m$, $[Q]$ and $[\tilde{Q}]_m$ are the thermal expansion coefficient vector of the composite matrix, the transformed reduced stiffness matrix of the SMA embedded lamina, and the transformed reduced stiffness matrix of the composite matrix, respectively.

Integrating equations (14) and (15) over the plate thickness, the constitutive equation can be obtained as

$$\begin{bmatrix} \{N\} \\ \{M\} \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{bmatrix} \{\epsilon_m\} \\ \{\epsilon_n\} \end{bmatrix} + \begin{bmatrix} \{N^T\} \\ \{M^T\} \end{bmatrix} + \begin{bmatrix} \{N_r\} \\ \{M_r\} \end{bmatrix}$$

where

$$\{N\} = \begin{bmatrix} R_{yz} \\ R_{xz} \end{bmatrix} = \begin{bmatrix} A_{44} \\ A_{45} \\ A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = [A]^T \{\gamma\}$$

$$\begin{bmatrix} \epsilon_m \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} \epsilon_{ln} + \{\epsilon_0\} \end{bmatrix}$$

$$\begin{bmatrix} \{N^T\} \\ \{M^T\} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} [\tilde{Q}(\tau)]^k_m \{\alpha(\tau)^k_m\} d\tau \\ \times (1, z) dz \end{bmatrix}$$

where $[A]$, $[A]^T$, $[B]$ and $[D]$ are the laminate stiffness matrices, membrane, coupling and bending respectively. $\{N\}$, $\{M\}$ and $\{R\}$ are the resultant vectors of the inplane force, moment, and transverse shear force. In addition, $\{N^T\}$ and $\{M^T\}$ are the inplane thermal load and thermal bending moment, respectively, while $\{N_r\}$ and $\{M_r\}$ are the inplane SMA recovery load and SMA recovery bending moment.

### 2.4. Aerodynamic loading

The first-order quasi-steady piston theory for supersonic flow states that (Tawfik et al., 2002):

$$P_a = - \frac{\rho_s v^2}{2}, \beta = \sqrt{M_{\infty}^2 - 1}, \alpha_o = \frac{(D_{110})^\frac{1}{2}}{\rho_{a}}, g = \frac{\rho_s v(M_{\infty}^2 - 2)}{\rho_{a} \beta}, \lambda = \frac{2 g a^3}{\beta D_{110}}$$

where $P_a$ is the aerodynamic loading, $v$ is the velocity of airflow, $M_{\infty}$ is the Mach number, $q$ is the dynamic pressure, $\rho_{a}$ is the air mass density, $g_{a}$ is non-dimensional aerodynamic damping, $\lambda$ is non-dimensional aerodynamic pressure, $D_{110}$ is the first entry of the flexural stiffness matrix $D(1, 1)$ when all the fibers of the composite layers are aligned in the airflow x-direction, and $a$ is the panel length.

### 2.5. Governing equations

By using the principle of virtual work and equations (4), (5), (16) and (17), the governing equation of thermal...
post-buckling and nonlinear flutter of a plate embedded with SMA fibers can be derived as follows

$$\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} = 0 \quad (19)$$

The internal virtual work $\delta W_{\text{int}}$ is given as

$$\delta W_{\text{int}} = \int_A \left( \left\{ \delta \epsilon_{\text{m}} + \epsilon_{\text{e}} \right\}^T \left\{ N \right\} + \left\{ \delta \kappa \right\}^T \left\{ M \right\} + \left\{ \delta \phi \right\}^T \left\{ R \right\} \right) dA$$

$$= \left\{ \delta \theta \right\}^T \left( \left\{ k \right\} + \left\{ k_{\text{e}} \right\} - \left\{ k_{\text{m}} \right\} + \frac{1}{2} \left\{ n_1 \right\} + \frac{1}{3} \left\{ n_2 \right\} \right)$$

$$\left( \theta \right) - \left( \delta \theta \right) \left\{ \left\{ p_{\text{r}} \right\} - \left\{ p_{\text{t}} \right\} \right\} \quad (20)$$

where $\left\{ \theta \right\}$ is the nodal displacement vector of the element; $\delta$ is a shear correction coefficient; $\left\{ k \right\}$, $\left\{ k_{\text{e}} \right\}$, $\left\{ k_{\text{m}} \right\}$ and $\left\{ k_{\text{m}} \right\}$ are the linear, shear, thermal and recovery stress stiffness matrices; $\left\{ n_1 \right\}$ and $\left\{ n_2 \right\}$ are the first- and second-order nonlinear stiffness matrices, respectively. In addition, $\left\{ p_{\text{r}} \right\}$ and $\left\{ p_{\text{t}} \right\}$ are the thermal load vector and the recovery stress load vector, respectively.

On the other hand, the external virtual work $\delta W_{\text{ext}}$ is given as (Park et al., 2004)

$$\delta W_{\text{ext}} = \int_A \left( - I_0 \left( \left\{ \delta u \right\}^T \left\{ \dot{u} \right\} + \left\{ \delta v \right\}^T \left\{ \dot{v} \right\} \right) + \left\{ \delta w \right\}^T \left\{ \ddot{w} \right\} \right) dA$$

$$- I_2 \left( \left\{ \delta \phi_x \right\}^T \left\{ \ddot{\phi}_x \right\} + \left\{ \delta \phi_y \right\}^T \left\{ \ddot{\phi}_y \right\} \right) + \left\{ \delta w \right\}^T \dot{P}_a$$

$$= - \left\{ \delta \theta \right\}^T \left\{ m \right\} \left\{ \ddot{\theta} \right\} - \left\{ \delta \theta \right\}^T \left\{ \frac{E}{\alpha^2} \right\} \left\{ g \right\} \left\{ \ddot{g} \right\}$$

$$= - \left\{ \delta \theta \right\}^T \left\{ m \right\} \left\{ \ddot{\theta} \right\} - \left\{ \delta \theta \right\}^T \left\{ \frac{E}{\alpha^2} \right\} \left\{ g \right\} \left\{ \ddot{g} \right\}$$

$$+ \left\{ \delta \theta \right\}^T \left\{ \frac{E}{\alpha^2} \right\} \left\{ a_u \right\} \left\{ \theta \right\} \quad (21)$$

where $(I_0, I_2) = \int_{-h/2}^{h/2} \rho(1, z^2) dz$ with $h$ denotes the plate thickness, $\left\{ m \right\}$ is the element mass matrix, $\left\{ g \right\}$ is the element aerodynamic damping matrix, and $\left\{ a_u \right\}$ is the element aerodynamic influence matrix.

By substituting equations (20) and (21) into (19), and assembling the element equations of motion to the system level by summing up the contributions from all elements and applying the boundary conditions, the governing equations for an SMA hybrid composite panel, under combined aerodynamic and thermal loads, can be obtained as

$$\begin{pmatrix} M_b & 0 & 0 & W_b \\ 0 & M_b & 0 & W_b \\ 0 & 0 & M_n & W_n \end{pmatrix} + \frac{E}{\alpha^2_0} \begin{pmatrix} G_b & 0 & 0 & \{ W_{\text{th}} \} \\ 0 & G_b & 0 & \{ W_{\text{th}} \} \\ 0 & 0 & G_n & \{ W_{\text{th}} \} \end{pmatrix} \begin{pmatrix} W_b \\ W_b \\ W_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & W_b \\ 0 & K_b & K_{\text{th}} & K_{\text{de}} \\ 0 & K_{\text{de}} & K_{\text{m}} & K_{\text{th}} \end{pmatrix} \begin{pmatrix} \{ \theta \} \\ \{ \theta \} \\ \{ \theta \} \end{pmatrix}$$

$$+ \begin{pmatrix} a_u & 0 & 0 & 0 \\ 0 & a_u & 0 & 0 \\ 0 & 0 & a_u & 0 \end{pmatrix} \begin{pmatrix} \{ \theta \} \\ \{ \theta \} \\ \{ \theta \} \end{pmatrix}$$

$$= \begin{pmatrix} W_{\text{th}} \\ W_{\text{th}} \\ W_{\text{th}} \end{pmatrix} \quad (22)$$

3. Solution procedures

The solution of the governing equation (23) is assumed to be as follows

$$\{ W \} = \{ W_{\text{p}} \} + \{ W_{\text{r}} \} \quad (24)$$

where $\{ W_{\text{p}} \}$ is the time-independent particular solution which means the large thermal deflection, and $\{ W_{\text{r}} \}$ is the time-dependent homogenous solution.

Substituting equation (24) into the governing equation (23),
Equation (25) presents the general equation for the thermal buckling and nonlinear flutter of an SMA hybrid composite plate under combined aerodynamic and thermal loads. The subscripts "s" and "t" indicate that the relevant matrix depends on the static or dynamic displacements, respectively.

Separating static and dynamic terms of equation (25), the following two equations can be obtained,

\[
[M] \{ \ddot{W}_s \} + \frac{g_s}{\omega_s} [G] \{ \dot{W}_s \} + \left( [K] + [K_s] - [K_T] \right) \{ W_s \} + \left( \frac{\dot{\lambda}}{\alpha_s} [A_s] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{ W_t \} + \{ P_T \} - \{ P_r \} = 0
\]  

(26)

\[
\begin{align*}
[M] \{ \ddot{W}_t \} &+ \frac{g_s}{\omega_s} [G] \{ \dot{W}_t \} + \left( [K] + [K_s] - [K_T] \right) \{ W_s \} \\
&+ \left( \frac{\dot{\lambda}}{\alpha_s} [A_s] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{ W_t \} \\
&+ \left( [N1]_s + [N2]_s + [N3]_s \right) \{ W_r \} = 0
\end{align*}
\]  

(27)

3.1. Static aero-thermal buckling

This section presents the solution procedure for the aero-thermal post-buckling deflection of symmetrically-laminated composite panels embedded with SMA fibers using Newton-Raphson method.

Introducing the function \{ \Psi(W) \} to equation (26),

\[
\{ \Psi(W_s) \} = \left( [K] + [K_s] - [K_T] + [K_r] \right) \{ W_s \} + \left( \frac{\dot{\lambda}}{\alpha_s} [A_s] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{ W_t \} - \{ P_T \} + \{ P_r \} = 0
\]  

(28)

Equation (28) can be written in the form of a truncated Taylor series expansion as

\[
\{ \Psi(W_s + \delta W) \} = \{ \Psi(W_s) \} + \frac{d\{ \Psi(W_s) \}}{d(W_s)} \{ \delta W \} \approx 0
\]  

(29)

where

\[
\frac{d\{ \Psi(W_s) \}}{d(W_s)} = \left( [K] + [K_s] - [K_T] + [K_r] \right) \{ W_s \} + \left( \frac{\dot{\lambda}}{\alpha_s} [A_s] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{ W_t \} = [K_{\tan}]
\]  

(30)

Thus, the Newton-Raphson iteration procedure for the determination of the post-buckling deflection can be expressed as follows:

\[
\{ \Psi(W_s) \}_{i+1} = \left( [K] + [K_s] - [K_T] + [K_r] \right) \{ W_s \} + \left( \frac{\dot{\lambda}}{\alpha_s} [A_s] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{ W_t \} - \{ P_T \} + \{ P_r \}
\]  

\[
[K_{\tan}] \{ \delta W \}_{i+1} = - \{ \Psi(W_s) \}_{i}
\]  

\[
\{ W_s \}_{i+1} = \{ W_s \}_{i} + \{ \delta W \}_{i+1}
\]

Convergence occurs in the above procedure when the maximum value of \{ \delta W \}_{i+1} becomes less than a given tolerance \epsilon_{tol}, i.e. max | \{ \delta W \}_{i+1} | \leq \epsilon_{tol}.

3.2. Flutter boundaries at elevated temperatures

In this section, the procedure of determining the critical non-dimensional dynamic pressure under the presence of thermal loads is presented. Equation (27) can be reduced for the solution of the linear (pre-buckling and pre-flutter) problem to the following equation,

\[
\begin{pmatrix}
M_b & 0 & 0 & 0 & W_{h_b} \\
0 & M_\phi & 0 & 0 & W_{\phi_h} \\
0 & 0 & M_m & 0 & W_{m_h} \\
0 & 0 & 0 & 0 & W_{m_0} \\
\end{pmatrix}
+ \frac{g_s}{\omega_s}
\begin{pmatrix}
G_b & 0 & 0 & 0 & W_{b_0} \\
0 & G_\phi & 0 & 0 & W_{\phi_0} \\
0 & 0 & G_m & 0 & W_{m_0} \\
0 & 0 & 0 & 0 & W_{m_0} \\
\end{pmatrix}
= 0
\]  

(31)
Applying a new notation for the bending degree of freedom by combining the shear and bending degrees-of-freedom as,

$$\{W_B\} = \begin{bmatrix} W_b \\ W_\phi \end{bmatrix}$$

(32)

Separating equations (31) into membrane and transverse directions results in the following transverse dynamic equation,

$$\begin{bmatrix} M_b & 0 \\ 0 & M_\phi \end{bmatrix} \{\ddot{W}_B\} + \frac{g_n}{\omega_n} \begin{bmatrix} G_b & 0 \\ 0 & 0 \end{bmatrix} \{W_B\} = \begin{bmatrix} K_B & K_{sb} \\ K_{sb} & K_{\phi \phi} \end{bmatrix} \{W_B\} + \begin{bmatrix} K_Tb & 0 \\ 0 & 0 \end{bmatrix} \{W_B\}$$

(33)

Or in a compact form as:

$$\begin{bmatrix} M_B \end{bmatrix} \{\ddot{W}_B\} + \frac{g_n}{\omega_n} \begin{bmatrix} G_B \end{bmatrix} \{\dot{W}_B\} + \begin{bmatrix} [K_B] + [K gs] - [K_Tb] \\ [K_{gs}] \end{bmatrix} \{W_B\} + \begin{bmatrix} K_{gs} \\ 0 \end{bmatrix} \{K_{gs} \} \{W_B\}\} = 0$$

(34)

Note that the terms related to $[N2]$ and $[N1_{mb}]$ are dropped as they depend on $\{W_B\}$, which is essentially zero before the occurrence of buckling or flutter, while $[N1_{mm}]$ terms are kept as they depend on $\{W_m\}$ which might have non-zero values depending on the inplane boundary conditions (Tawfik et al., 2002).

Now, assuming the deflection function of the transverse displacement $\{W_B\}$, to be in the form,

$$\{W_B\} = \tilde{c} \{\Phi_B\} e^{i\Omega t}$$

(35)

where $\Omega = \alpha + i\omega$ is the complex panel motion parameter ($\alpha$ is the damping ratio and $\omega$ is the frequency), $\tilde{c}$ is the amplitude of vibration, and $\{\Phi_B\}$ is the mode shape (Xue, 1991).

Substituting equation (35) into (34), the generalized eigenvalue problem can be obtained as

$$\tilde{c}[-\kappa [M_B] + \tilde{K}] \{\Phi_B\} e^{i\Omega t} = 0$$

(36)

where $[M_B] = \omega_n^2 [M_B]$, $\kappa$ is the non-dimensional eigenvalue, given by

$$\kappa = - \left( \frac{\Omega^2}{\omega_n^2} \right) - \frac{g_n}{\omega_n} \frac{\Omega}{\omega_n}$$

(37)

and

$$\tilde{K} = [K_B] + [K gs] - [K_Tb] + \frac{\lambda}{a^3} [A_{gs}] + [N1_{mb}]$$

(38)

From equation (36) we can write the generalized eigenvalue problem

$$[-\kappa [M_B] + \tilde{K}] \{\Phi_B\} = 0$$

(39)

where $\kappa$ is the eigenvalue and $\{\Phi_B\}$ is the mode shape, with the characteristic equation written as

$$[-\kappa [M_B] + \tilde{K}] = 0$$

(40)

Given that the values of $\kappa$ are real for all values of $\lambda$ below the critical value, an iterative solution can be utilized to determine the critical non-dimensional dynamic pressure $\lambda_{cr}$ (Xue, 1991).

### 3.3. The Limit-Cycle Oscillation Amplitude

In this section, the nearly harmonic limit-cycle oscillation amplitude will be determined for a fluttering SMA hybrid composite plate at temperatures less than the buckling temperature ($\{W_B\}_r = 0$), i.e. the non-periodic and chaotic limit-cycle oscillation will not be considered in this section. Following the same procedure outlined in the previous section, with the only difference that nonlinear stiffness terms that depend on the transverse dynamic displacement $\{W_B\}_r$ will be included to end up with the following equation,

$$\begin{bmatrix} M_B & 0 \\ 0 & M_m \end{bmatrix} \begin{bmatrix} \ddot{W}_B \\ \ddot{W}_m \end{bmatrix} + \frac{g_n}{\omega_n} \begin{bmatrix} G_B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{W}_B \\ \dot{W}_m \end{bmatrix} = \begin{bmatrix} [K_B] + [K_{gs}] - [K_{Tb}] \\ [K_{gs}] \end{bmatrix} \begin{bmatrix} W_B \\ W_m \end{bmatrix} + \begin{bmatrix} K_{gs} \\ 0 \end{bmatrix} \{K_{gs} \} \{W_B\}_r$$

(41)

Separating the membrane displacement equation and the transverse displacement equation from equation (41),

$$\begin{bmatrix} M_B \end{bmatrix} \{\ddot{W}_B\}_r + \frac{g_n}{\omega_n} \begin{bmatrix} G \end{bmatrix} \{\dot{W}_B\}_r + \begin{bmatrix} [K_B] + [K_{gs}] - [K_{Tb}] + [K_{gs}] \\ [K_{gs}] \end{bmatrix} \{W_B\}_r + \begin{bmatrix} \frac{1}{2} [N1_{mb}] \{W_B\}_r + \frac{1}{3} [N2_B] \{W_B\}_r \\ [N1_{mb}] \{W_m\}_r \end{bmatrix} = 0$$

(42)
where \([N^{1}_{nmB}]\) is the transpose of \([N^{1}_{mB}]\). Neglecting the inplane and shear inertia terms will not bring significant error, because inplane and shear natural frequencies are 2 to 3 order of magnitude higher than bending ones (Guo, 2005). Therefore the inplane displacement vector \(\{W_{m}\}_i\) can be expressed in terms of the bending displacement vector \(\{W_{B}\}_i\) as

\[
\{W_{m}\}_i = -\frac{1}{2} [K_m]^{-1} [N^{1}_{mB}]_i \{W_{B}\}_i \tag{44}
\]

Substituting equation (44) into (42),

\[
[M_B] \{\ddot{W}_{B}\}_i + \frac{g_B}{\omega_o} [G] \{\dot{W}_{B}\}_i + \left( [K_B]_{x} - [K_{TB}] + [K_B] \right) \{W_{B}\}_i + \left( \frac{1}{2} [N^{1}_{nmB}]_i + \frac{1}{3} [N^{2}_{B}]_i \right) \{W_{B}\}_i \nonumber \\
= \frac{1}{4} [N^{1}_{Bm}] [K_m]^{-1} [N^{1}_{mB}]_i \{W_{B}\}_i = 0 \tag{45}
\]

It can be shown that 0,

\[
[N^{1}_{nmB}]_i \{W_{B}\}_i = -\frac{1}{2} [N^{1}_{Bm}]_i [K_m]^{-1} [N^{1}_{mB}]_i \{W_{B}\}_i \tag{46}
\]

This can be used to write the bending equation in its final form to be,

\[
[M_B] \{\ddot{W}_{B}\}_i + \frac{g_B}{\omega_o} [G] \{\dot{W}_{B}\}_i + \left( [K_B]_{x} - [K_{TB}] + [K_B] \right) \{W_{B}\}_i + \frac{\lambda}{\alpha} [A_{B}] \{W_{B}\}_i \nonumber \\
= \frac{1}{3} [N^{2}_{B}]_i - \frac{1}{2} [N^{1}_{Bm}]_i [K_m]^{-1} [N^{1}_{mB}]_i \{W_{B}\}_i = 0 \tag{47}
\]

Assuming harmonic solution, a procedure similar to those described in Section 3.2 can be used to write the equation of motion in the form (Xue, 1991),

\[
[-\kappa[M_B] + [\bar{K}]] \{\Phi_B\} = \{0\} \tag{48}
\]

where

\[
\{W_{B}\}_i = e^{i\omega t} \{\Phi_B\}_i (\cos \omega t + i \sin \omega t) \tag{49}
\]

\[
[\bar{K}] = [K_B]_{x} - [K_{TB}] + [K_B] + \frac{\lambda}{\alpha} [A_{B}] + \frac{3}{8} [N^{1}_{Bm}]_i [K_m]^{-1} [N^{1}_{mB}]_i \tag{50}
\]

Since the nonlinear stiffness terms of the above equation depend on the amplitude of the vibration, an iterative scheme should be used. The following algorithm outlines the steps used in the iterative procedure (Xue, 1991),

1. Normalize the eigenvector \(\{\Phi\}_i\), obtained at the flutter point, using the maximum displacement.
2. Select a value for the amplitude \(\epsilon\).
3. Evaluate the linear and nonlinear stiffness terms.
4. Change the value of the non-dimensional aerodynamic pressure \(\lambda\).
5. Solve the eigenvalue problem for \(\kappa\).
6. If coalescence occurs proceed, else go to step 4.
7. Check the differences between the obtained eigenvector and the initial one, if small, proceed, else normalize the eigenvector as described in step 1 and go to step 3.
8. The obtained dynamic pressure corresponds to the initially given amplitude.
9. Go to step 2.

It should be noted that the above mentioned procedure is valid only for the case when panel flutter occurs while the plate is not buckled or when the dynamic pressure is high enough that the buckled plate becomes flat again; i.e. it does not cover the region of chaotic vibration.

4. Numerical results and discussions

Numerical analyses for the thermal post-buckling and nonlinear flutter of a laminated composite plate with and without SMA fibers are performed using the finite element method.

A uniform 6 × 6 finite element mesh of nine-noded elements is found adequate. The reduced order technique is used for integrating terms related to the transverse shear so as to avoid shear locking. In addition, 0%, 5%, 10% and 15% volume fractions and 1%, 3% and 5% initial pre-strain values of the SMA fibers are used. Modulus of elasticity and recovery stress of SMA were obtained from Cross et al. (1969).

4.1. Aero-thermal buckling analysis

Thermal post-buckling behavior of a traditional laminated composite panel with and without SMA fibers is studied. The dimensions of the plate are 0.381 × 0.305 × 0.0013 (m) and the stacking sequence is [0/-45/45/90]s. Clamped boundary conditions are assumed for all edges.

The SMA fibers utilized in the current simulation has \(G = 25.6\) GPa, \(\rho = 6450\) Kg/m³, \(v = 0.3\) and \(x = 10.26 \times 10^{-6}/^\circ C\). In addition, the composite matrix has \(E_1 = 155 (1 - 6.35 \times 10^{-3} \Delta T)\) GPa, \(E_2 = 8.07 (1 - 7.69 \times 10^{-4} \Delta T)\) GPa, \(G = 4.55 (1 - 1.09 \times 10^{-3} \Delta T)\) GPa, \(\rho = 1550\) Kg/m³, \(v = 0.22\), \(\alpha_1 = -0.07 \times 10^{-6} (1.069 \times 10^{-3} \Delta T)/^\circ C\), and \(\alpha_2 = 30.6 \times 10^{-6} (1 + 0.28 \times 10^{-4} \Delta T)/^\circ C\) where \(\Delta T\) is the temperature rise value. Uniform temperature rise was
applied to the panel, and the reference temperature is assumed to be 21°C in this study.

To validate the present formulation, it is found in Figure 1 that the current results are in a good agreement with the analytical solution (Paul, 1982) that was derived using 25-term series solution for an aluminum square panel with all edges clamped and dimensions 0.305 m x 0.305 m x 0.002 m. The non-dimensional temperature in the results is calculated by

\[ \tilde{T} = 12(1 + \epsilon) \frac{a \Delta T a^2}{\pi^2 h^2} \]

Figures 2 and 3 demonstrate the effectiveness of using the SMA-embedded plate to delay the buckling temperature and to reduce the thermal post-buckling deflection. It is seen in the Figures that, the increase of volume fraction and pre-strain values of SMA increase the performance of the SMA plate regarding the buckling problem.

4.2. Predicting panel flutter boundaries

The results of a clamped traditional composite plate subject to thermal and aerodynamic loads are presented in Figure 5 in terms of the critical buckling temperature boundary and linear flutter boundary. The area of the graph is divided into four regions. The flat region is where the plate is stable, i.e. neither buckling nor panel flutter occurs. The thermally buckled region is the case in which the thermal stresses overcome the plate stiffness and the aerodynamic stiffening action. In this region, the plate undergoes static instability. The third region is the flutter region, in which the plate
undergoes dynamic instability. The fourth region represents chaos, where thermal buckling and dynamic flutter occur simultaneously.

Thus, the wider the flat plate zone is, the more stable is the plate. It is the objective of the SMA fiber embeddings to increase the flat and stable plate region when the panel is subjected to combined thermal and aerodynamic loads.

Figures 6 and 7 demonstrate how the recovery stresses with a high volume fraction of SMA overcome the expected decreasing stiffness with temperature increase. Figure 6 illustrates the stability boundary for four different volume fractions with 1% pre-strain value.

It is seen in the Figure that the higher the volume fraction, the wider the flat and stable region. But, increasing the volume fraction should be optimized, as the SMA fibers are much heavier in weight than the composite matrix and increasing the volume fraction is also bounded by fiber de-bonding problems.

Figure 7 shows that the 1% and 3% pre-strain curves coincide at the beginning of temperature rise, as they have the same recovery stress curve in the temperature rise range 21–43 °C. After a temperature rise of 22, the 3% curve separates from the 1% curve, as the recovery stress of the 3% pre-strain value starts to overcome the decreased stiffness due to thermal expansion. In the temperature rise range 21–43 °C, the 5% pre-strain curve, as shown in Figure 2, has a slower rate of recovery-stress rise with temperature than that of the 1% and 3% pre-strains. This makes the thermal stress dominate the SMA recovery stress during this temperature range, and hence, the critical dynamic pressure decreases steeply with temperature rise compared to the other cases. However, for temperatures above 43 °C, the 5% pre-strain curve has a higher rate of increase than that of temperatures lower than 43 °C, which temporarily makes the recovery stress dominate the thermal stress and make the critical...
dynamic pressure increase with temperature up to a temperature rise of 61°C. From 61°C to 76°C, the thermal expansion dominates the recovery stress, which makes the critical dynamic pressure decrease again with temperature. After 93°C, the critical dynamic pressure starts to increase up to the occurrence of chaos at a temperature rise of 110°C. But, it should be noted that the increase in the pre-strain value is bounded by the required fatigue performance of the plate, as high pre-strain values adversely affect the SMA fatigue life.

4.3. The limit cycle amplitude

In this section, the nearly harmonic limit-cycle oscillation amplitude will be determined for a fluttering composite plate with and without SMA fiber embeddings at temperatures less than the buckling temperature ($\{W_b\}_s = 0$), illustrating the effect of the temperature, the volume fraction and the pre-strain value on the nonlinear flutter of a SMA hybrid composite plate.

Figure 8 illustrates the nonlinear flutter behavior of a traditional clamped composite plate by presenting a full map of the variation of both the limit cycle amplitude as well as the post-buckling deflection with the dynamic pressure for different values of the temperature increase illustrating the distinction between the static and dynamic regions. It is seen in the Figure that the limit cycle amplitude increases with increasing the non-dimensional dynamic pressure at certain temperature rise. The stiffening effect of the air flow over the panel is also demonstrated through the decrease in the maximum lateral deflection by increasing the dynamic pressure value, while the temperature being constant.

Figure 9 illustrates the nonlinear flutter behavior of a SMA hybrid composite plate with 10% volume fraction and 3% pre-strain value through presenting the variation of the limit-cycle amplitude with the variation of the non-dimensional dynamic pressure and the value of the temperature increase. It is seen in the Figure that, for a certain non-dimensional dynamic pressure, the limit-cycle amplitude decreases with increasing the temperature. Because increasing the temperature results in the building up of the SMA recovery stress that gives way for a stiffer plate. But after a temperature rise of 40°C, the thermal stresses overcome the recovery stresses that make the limit-cycle amplitude increases with heating.

The variation of the limit-cycle amplitude with the SMA pre-strain value is presented in Figure 10. A SMA hybrid composite plate with SMA volume fraction of 10%, pre-strained by 1%, 3% and 5% and at a temperature increase of 60°C is studied here. It is seen in the Figure that, for a certain non-dimensional dynamic pressure, the limit-cycle amplitude decreases with increasing the pre-strain value. But, this decrease depends on how much is the difference between the recovery stresses of the different pre-strain values at the given temperature step, as the difference between them varies with temperature (see Figure 2).

The variation of the limit-cycle amplitude with the SMA volume fraction is presented in Figure 11. An SMA hybrid composite plate with SMA pre-strain value of 3% and volume fractions of 5%, 10% and 15%, and at a temperature rise of 60°C is studied here. It is seen in the Figure that, for a certain non-dimensional dynamic pressure, the limit-cycle amplitude decreases with increasing the volume fraction.

5. Conclusions

A new nonlinear finite element model based on the first-order shear deformable plate theory was presented for the analysis of nonlinear panel flutter and thermal buckling characteristics of SMA hybrid composite plate with highly nonlinear temperature-dependent material properties. The
constrained shape recovery feature of SMA fibers is utilized in enhancing the thermal buckling, post-buckling, and limit cycle oscillation characteristics of SMA hybrid composite plates. To account for the highly nonlinear temperature dependence of material properties, the thermal strain is stated as an integral quantity of thermal expansion coefficient with respect to temperature. Therefore, the present formulation saves a lot of the computational efforts exerted when applying the incremental techniques found in the majority of the published work. The solution procedures of the thermal buckling, and nonlinear panel flutter under thermal load were presented.

The numerical results of this study demonstrate a significant increase in the buckling temperature and a noteworthy reduction in the post-buckling deflection of the SMA-embedded plate. The combined aerodynamic-thermal loading problem was investigated in order to determine the effectiveness of SMA-embedded plates in increasing the flutter stability boundaries and in decreasing the limit-cycle amplitudes. The critical non-dimensional dynamic pressure has shown significant increase for the SMA-embedded plates.

The effectiveness of SMA volume fractions and pre-strain values has also been investigated.

Therefore, SMA can be useful to the application of the structures due to their ability to control the thermal deflection, flutter boundaries, and limit-cycle oscillation amplitudes. A further development of this work may be by investigating the behavior of the plate in the chaotic regime as well as its response to random pressure loading in the aero-thermal-loading regime.

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