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Progressive Type-II hybrid censored schemes based on maximum product spacing with application to Power Lomax distribution

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ABSTRACT

In this paper, progressive Type-II hybrid censoring based on maximum product spacing method is introduced. Parameters estimation for the Power Lomax (PL) distribution are discussed under the progressive Type-II hybrid censoring based on maximum likelihood estimation and maximum product spacing method. A comparison studies with classical methods as maximum likelihood is discussed. Also, asymptotic confidence intervals and bootstrap confidence intervals are obtained. A numerical study using two real data and Monte Carlo simulation are performed to compare between the different methods.

1. Introduction

Lomax distribution is a well-known model. Many forms and names of Lomax distribution are appeared in the literature and it was used in a wide range of scientifically applications, where it was found that it is compliant in lifetime models such as life testing and it has many applications in survival analysis. Lomax [1] introduced the Lomax distribution to model exceedances over thresholds. This distribution is also known as Pareto distribution of the second type or generalized Pareto distribution, for details see [2–5].

A new extension of the Lomax distribution is proposed by considering the power transformation $X = T^{\beta}$, where the random variable T follows Lomax distribution with parameters α, λ. The power Lomax (PL) distribution is obtained by Rady et al. [6]. A random variable X is said to have PL distribution, if its cumulative density function (cdf) is given by

$$F\left(x;\alpha,\beta,\lambda\right) = 1-\lambda^{\alpha}\left(\lambda+x^{\beta}\right)^{-\alpha};\alpha,\beta,\lambda>0, \tag{1.1}$$

and the probability density function (pdf) of PL distribution is given by

$$f\left(x;\alpha,\beta,\lambda\right) = \alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1}. \tag{1.2}$$

Many authors studied the PL distribution as Abdul-Moniem [7], Essam [8] and Hassan and Nassr [9].

In censored samples, the most important used censoring schemes are Type-I and Type-II censoring. A mixture of Type-I and Type-II schemes is known as the hybrid censoring scheme, which was first introduced by Epstein [10]. The three conventional censoring schemes all have the drawbacks that they do not allow for removal of units at points other than the terminal of the experiment. In general, censoring scheme called progressive censoring schemes are placed on a life testing experiment and m is a predetermined number of units to be failed, for more information see Balakrishnan [11] and

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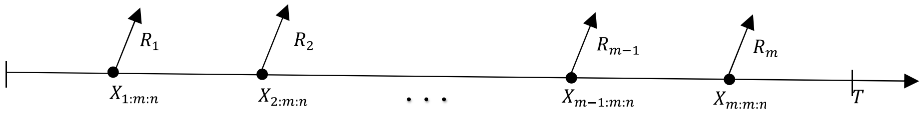

Fig. 1. The schematic representation of Case I where experiment terminates before time T.

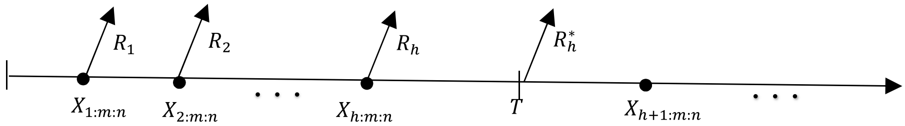

Fig. 2. The schematic representation of Case II where experiment terminates after time T.

Almetwally et al. [12]. The progressively hybrid censoring scheme, has its favorable position, to be extremely mainstream in the reliability and life-testing over the last few years, which was introduced by Kundu and Joarder [13].

Many authors have discussed inference under progressive Type-II hybrid censoring using different lifetime distributions. Claudio et al. [14] discussed parameter estimation of PL distribution based on progressive Type-II hybrid censoring scheme. For examples, see [15–20].

The maximum product spacing estimation (MPS) technique was presented by Cheng and Amin [21] and autonomously talked about by Ranneby [22] as an alternative method to maximum likelihood estimation (MLE) technique for continuous distributions, for example see [23]. Ng et al. [24] discussed progressive Type-II censored samples using the MPS method as an alternative method. For more example, see [25,26]. Basu et al. [27] introduced MPS method based on hybrid censoring scheme. Almetwally et al. [28] introduced the adaptive Type-II progressive censoring schemes using MPS method.

The aim of this paper is to introduce the parameter estimation for the PL distribution under progressive Type-II hybrid censoring scheme (PTIIHC) based on MPS method. The comparison between MLE and MPS methods is proposed in this study. The optimal censoring scheme depend on PTIIHC using two different methods for optimality criteria (mean squared error (MSE) is also discussed Bias). The interval estimation of PL distribution parameters are obtained by using asymptotic and bootstrap confidence intervals based on PTIIHC scheme. To evaluate the performance of the estimators, extensive simulation techniques is carried out. Applications of two real data sets are introduced to confirm the validity of the model and the schemes.

The paper is organized as follows: In Section 2, we introduce progressive Type-II hybrid censoring scheme using MPS technique. Parameter estimation of the PL distribution under PTIIHC scheme are derived in Section 3, while in Section 4 the confidence intervals are discussed. In Section 5, Monte Carlo simulation study to compare the performance of the parameters estimation for the different methods are presented. In Section 6, applications of two real data sets are studied. Finally, the results and conclusion of the current study are discussed in Section 7.

2. Progressive type-II hybrid censored schemes of MPS

Based on the observed sample $x_{1:m:n} < \cdots < x_{m:m:n}$ from a progressive Type-II hybrid censoring scheme, the MPS under progressive Type-II hybrid censoring scheme will be introduced depending on Cheng and Amin [21], Ng et al. [24], Almetwally et al. [28] and Kundu and Joarder [13].

The two cases of the Type-II progressive hybrid censoring scheme are Cases I and II, respectively, which will be discussed as following

Case I: If $X_{m:m:n} < T$, the progressive sample $\{X_{1:m:n}, \ldots, X_{m:m:n}\}$ is described as shown in Fig. 1 as follows

The product spacing in this case is as follows

$$S(\Psi) = C_1 \prod_{i=1}^{m} D_{i:m:n} \left(1 - F(x_{i:m:n}; \Psi)\right)^{R_i}, \tag{2.1}$$

where C_1 is a constant that it doesn't depend on parameters and Ψ is a vector of parameters, $D_{i:m:n} = \begin{cases} D_{1:m:n} = F(x_{1:m:n}; \Psi) \\ D_{i:m:n} = F(x_{i:m:n}; \Psi) - F(x_{(i-1):m:n}; \Psi); \; i = 2 \ldots m \\ D_{(m+1):m:n} = 1 - F(x_{m:m:n}; \Psi), \end{cases}$ and Eq. (2.1) can be referred as MPS under progressive Type-II censoring scheme which was introduced by Ng et al. [24] and had been used by Singh et al. [25] and Almetwally and Almongy [26].

Case II: if $X_{h:m:n} < T < X_{h+1:m:n}$, the progressive censoring sample $\{X_{1:m:n}, \ldots, X_{h:m:n}\}$, is described as shown in Fig. 2 as follows

Table 1
Special cases.

T	Scheme	Cases	$S(\Psi)$
$X_{m:m:n} < T$	Any different scheme R_i	MPS under progressive Type-II censoring scheme. Ng et al. [24].	Eq. (2.1)
$X_{m:m:n} < T$	$R_1 = \cdots = R_{m-1} = 0$, $R_m = n - m$	MPS function under Type-II censoring scheme.	$C_1 \left(1 - F(x_{m:n}; \Psi)\right)^{n-m} \prod_{i=1}^{m} D_{i:n}$,
$X_{m:m:n} < T$	$R_1 = \cdots = R_m = 0$	MPS function under complete censoring sample. Cheng and Amin [21]	$\prod_{i=1}^{m+1} (D_{i:m:n})$
$X_{h:m:n} < T < X_{h+1:m:n}$	$R_1 = \cdots = R_{m-1} = 0$, $R_m = n - m$	MPS under Type-II hybrid censoring scheme. Basu et al. [27]	$C_2 \left(1 - F(T; \Psi)\right)^{n-h} \prod_{i=1}^{h} [D_{i:n}]$

Eq. (2.2) is referred as MPS under Type-II progressive hybrid censoring scheme in case II as follows

$$S(\Psi) = C_2 \prod_{i=1}^{h} \left[D_{i:m:n} \left(1 - F(x_{i:m:n}; \Psi)\right)^{R_i}\right] \left(1 - F(T; \Psi)\right)^{R_h^*}, \tag{2.2}$$

where $R_h^* = n - h - \sum_{i=1}^{h} R_i$.
Table 1 shows the special cases of MPS under Type-II progressive hybrid censoring scheme.

3. Parameters estimation of the PL distribution

In this section, MLE and MPS estimation of the PL distribution parameters under PTIIHC have been introduced.

3.1. MLE method

By using Eqs. (1.1), (1.2), the likelihood function of PL distribution under PTIIHC can be written as follows

$$L(\Psi) \propto \alpha^h \beta^h \lambda^{h\alpha} \prod_{i=1}^{h} x_{i:h:n}^{\beta-1} \left(\lambda + x_{i:h:n}^{\beta}\right)^{-\alpha-1} \left(\lambda^{\alpha} \left(\lambda + x_{i:h:n}^{\beta}\right)^{-\alpha}\right)^{R_i} \lambda^{\alpha R_h^*} \left(\lambda + T^{\beta}\right)^{-\alpha R_h^*}. \tag{3.1}$$

The log-likelihood function of PL distribution, based on PTIIHC, is given by:

$$\begin{aligned} l(\Psi) \propto\ & h \ln(\alpha) + h \ln(\beta) + h\alpha \ln(\lambda) + (\beta - 1) \sum_{i=1}^{h} \ln(x_{i:h:n}) - \alpha R_h^* \ln\left(\lambda + T^{\beta}\right) \\ & - \sum_{i=1}^{h} \ln\left(\lambda + x_{i:h:n}^{\beta}\right) [(\alpha + 1) + \alpha R_i] + \alpha \ln(\lambda) \sum_{i=1}^{h} R_i + \alpha R_h^* \ln(\lambda). \end{aligned} \tag{3.2}$$

The partial derivatives of Eq. (3.2) with respect to the unknown parameters are given as follows:

$$\frac{\partial l(\Psi)}{\partial \alpha} = \frac{h}{\alpha} + h \ln(\lambda) - \sum_{i=1}^{h} \ln\left(\lambda + x_{i:h:n}^{\beta}\right) [1 + R_i] + \ln(\lambda) \sum_{i=1}^{h} R_i + R_h^* \ln(\lambda) - R_h^* \ln\left(\lambda + T^{\beta}\right),$$

$$\frac{\partial l(\Psi)}{\partial \beta} = \frac{h}{\beta} + \sum_{i=1}^{h} \ln(x_{i:h:n}) - \sum_{i=1}^{h} \frac{\ln(x_{i:h:n})\, x_{i:h:n}^{\beta} [(\alpha + 1) + \alpha R_i]}{\left(\lambda + x_{i:h:n}^{\beta}\right)} - \alpha R_h^* \frac{T^{\beta} \ln(T)}{\left(\lambda + T^{\beta}\right)},$$

and

$$\frac{\partial l(\Psi)}{\partial \lambda} = \frac{h\alpha}{\lambda} - \sum_{i=1}^{h} \frac{[(\alpha + 1) + \alpha R_i]}{\lambda + x_{i:h:n}^{\beta}} - \alpha R_h^* \frac{1}{\left(\lambda + T^{\beta}\right)} + \frac{\alpha}{\lambda} \sum_{i=1}^{h} R_i + \frac{\alpha}{\lambda} R_h^*.$$

The MLEs $\widehat{\Psi}$ of the model parameters are the solution of non-linear equations after setting them equal to zero. These equations are very difficult to be solved, so iterative procedures are used.

3.2. MPS method

By using Eqs. (1.1), (1.2) and (2.2) the MPS based on PTIIHC scheme for PL distribution can be written as:

$$G(\Psi) \propto \left(1-\lambda^{\alpha}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha}\right)\left(\lambda^{\alpha}\left(\lambda+x_{h:h:n}^{\beta}\right)^{-\alpha}\right)\lambda^{\alpha}\prod_{i=2}^{h}\left[\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}-\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\right] \lambda^{\alpha R_h^*}\left(\lambda+T^{\beta}\right)^{-\alpha R_h^*}\prod_{i=1}^{h}\left(\lambda^{\alpha}\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}\right)^{R_i}, \tag{3.3}$$

The log-maximum product spacing function of PL distribution, based on PTIIHC, is given by:

$$\begin{aligned}\ln G(\Psi) \propto\ &+\ln\left(1-\lambda^{\alpha}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha}\right)-\alpha\ln\left(\lambda+x_{h:h:n}^{\beta}\right)+2\alpha\ln\lambda+\alpha R_h^* ln(\lambda)-\alpha R_h^*\ln\left(\lambda+T^{\beta}\right)\\&+\sum_{i=2}^{h}\ln\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}-\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\right)-\alpha\sum_{i=1}^{h}R_i\ln\left(\lambda+x_{i:h:n}^{\beta}\right)\\&+\alpha\ln(\lambda)\sum_{i=1}^{h}R_i.\end{aligned} \tag{3.4}$$

The partial derivatives of Eq. (3.4) with respect to the unknown parameters are given as follows:

$$\begin{aligned}\frac{\partial\ln G(\Psi)}{\partial\alpha}=\ &\frac{\lambda^{\alpha}\left(\ln\left(\lambda+x_{1:h:n}^{\beta}\right)-ln(\lambda)\right)}{\left(\left(\lambda+x_{1:h:n}^{\beta}\right)^{\alpha}-\lambda^{\alpha}\right)}-\ln\left(\lambda+x_{h:h:n}^{\beta}\right)+2\ln(\lambda)-R_h^*\ln\left(\lambda+T^{\beta}\right)+\ln(\lambda)\sum_{i=1}^{h}R_i\\&-\sum_{i=1}^{h}R_i\ln\left(\lambda+x_{i:h:n}^{\beta}\right)\\&+\sum_{i=2}^{h}\frac{\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}\ln\left(\lambda+x_{i:h:n}^{\beta}\right)\right)-\left(\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\ln\left(\lambda+x_{(i-1):h:n}^{\beta}\right)\right)}{\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}-\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\right)},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln G(\Psi)}{\partial\beta}=\ &\frac{\alpha\lambda^{\alpha}x_{1:h:n}^{\beta}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha-1}\ln(x_{1:h:n})}{\left(1-\lambda^{\alpha}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha}\right)}-\frac{\alpha x_{h:h:n}^{\beta}\ln(x_{h:h:n})}{\left(\lambda+x_{h:h:n}^{\beta}\right)}-\alpha\sum_{i=1}^{h}R_i\frac{\ln(x_{i:h:n})\,x_{i:h:n}^{\beta}}{\left(\lambda+x_{i:h:n}^{\beta}\right)}\\&-\alpha R_h^*\frac{T^{\beta}\ln(T)}{\left(\lambda+T^{\beta}\right)}\\&-\alpha\sum_{i=2}^{n}\frac{\left(x_{i:h:n}^{\beta}\ln(x_{i:h:n})\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha-1}\right)-\left(x_{(i-1):h:n}^{\beta}\ln\left(x_{(i-1):h:n}\right)\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha-1}\right)}{\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}-\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ln G(\Psi)}{\partial\lambda}=\ &\frac{-\alpha\lambda^{\alpha-1}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha-1}\left(\left(\lambda+x_{1:h:n}^{\beta}\right)-\lambda\right)}{\left(1-\lambda^{\alpha}\left(\lambda+x_{1:h:n}^{\beta}\right)^{-\alpha}\right)}+\alpha\sum_{i=2}^{n}\frac{\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha-1}\right)-\left(\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha-1}\right)}{\left(\left(\lambda+x_{i:h:n}^{\beta}\right)^{-\alpha}-\left(\lambda+x_{(i-1):h:n}^{\beta}\right)^{-\alpha}\right)}\\&-\frac{\alpha}{\lambda+x_{n:h:n}^{\beta}}+\frac{2\alpha}{\lambda}-\alpha\sum_{i=1}^{h}R_i\frac{1}{\lambda+x_{i:h:n}^{\beta}}-\alpha R_h^*\frac{1}{\left(\lambda+T^{\beta}\right)}+\frac{\alpha}{\lambda}\sum_{i=1}^{h}R_i+\frac{\alpha}{\lambda}R_h^*.\end{aligned}$$

Again, The MPS $\widehat{\Psi}$ of the model parameters are the solution of those non-linear equations after setting them equal zero. These equations are very difficult to be solved, so iterative procedures are used.

4. Confidence interval

In this section, we propose different confidence intervals. One is based on the asymptotic distribution of Ψ and the other is bootstrap confidence intervals.

Table 2
Parameter estimation of PL distribution under different censoring schemes for case 1.

n	m	$R^{(i)}$	$t=500$	$T<xm$ MLE Bias	$T<xm$ MLE MSE	$T<xm$ MPS Bias	$T<xm$ MPS MSE	$T>xm$ MLE Bias	$T>xm$ MLE MSE	$T>xm$ MPS Bias	$T>xm$ MPS MSE
100	70	1	$\hat{\alpha}$	−0.1610	0.0351	−0.1336	0.0557	−0.2474	0.0618	−0.2388	0.0577
			$\hat{\beta}$	0.2373	0.3546	0.0829	0.1218	0.6317	1.0729	0.3754	0.4969
			$\hat{\lambda}$	−0.2648	0.5970	−0.1399	0.4269	−0.6776	0.4668	−0.6583	0.4423
	90		$\hat{\alpha}$	−0.0652	0.0193	−0.0350	0.0219	−0.1726	0.0317	−0.1609	0.0279
			$\hat{\beta}$	0.1246	0.1874	0.0382	0.0823	0.2798	0.2820	0.1744	0.1524
			$\hat{\lambda}$	−0.0747	0.4122	0.0254	0.3517	−0.5311	0.3089	−0.5023	0.2799
	70	2	$\hat{\alpha}$	0.0202	0.0814	0.0634	0.1551	−0.0064	0.0198	0.0035	0.0165
			$\hat{\beta}$	0.1548	0.2259	0.0596	0.1171	0.1256	0.1585	0.0514	0.0713
			$\hat{\lambda}$	0.2281	2.1872	0.3603	3.7214	0.0754	0.4533	0.0673	0.3438
	90		$\hat{\alpha}$	0.0118	0.0301	0.0480	0.0418	−0.0095	0.0109	0.0027	0.0103
			$\hat{\beta}$	0.0935	0.0884	0.0232	0.0550	0.0771	0.0538	0.0217	0.0261
			$\hat{\lambda}$	0.1169	0.6561	0.2237	0.8270	0.0097	0.2165	0.0221	0.1900
	70	3	$\hat{\alpha}$	−0.1114	0.0254	−0.0816	0.0288	−0.2132	0.0469	−0.2021	0.0424
			$\hat{\beta}$	0.2026	0.2627	0.0809	0.1204	0.4887	0.7367	0.2923	0.3044
			$\hat{\lambda}$	−0.1564	0.5497	−0.0466	0.9103	−0.6157	0.3984	−0.5900	0.3691
	90		$\hat{\alpha}$	−0.0330	0.0244	0.0001	0.0335	−0.1274	0.0195	−0.1143	0.0164
			$\hat{\beta}$	0.1125	0.1301	0.0291	0.0492	0.2051	0.1721	0.1190	0.0816
			$\hat{\lambda}$	0.0028	0.5417	0.1068	0.7240	−0.4051	0.2190	−0.3741	0.1935
200	140	1	$\hat{\alpha}$	−0.1615	0.0281	−0.1476	0.0242	−0.2504	0.0630	−0.2455	0.0606
			$\hat{\beta}$	0.0999	0.0427	0.0393	0.0238	0.4109	0.3268	0.2915	0.1756
			$\hat{\lambda}$	−0.3171	0.1848	−0.2546	0.1620	−0.6960	0.4870	−0.6841	0.4709
	180		$\hat{\alpha}$	−0.0682	0.0095	−0.0504	0.0081	−0.1777	0.0325	−0.1708	0.0301
			$\hat{\beta}$	0.0509	0.0183	0.0099	0.0126	0.2066	0.0674	0.1537	0.0423
			$\hat{\lambda}$	−0.1200	0.1376	−0.0563	0.1411	−0.5599	0.3262	−0.5407	0.3055
	140	2	$\hat{\alpha}$	0.0054	0.0144	0.0295	0.0176	−0.0053	0.0067	0.0019	0.0064
			$\hat{\beta}$	0.0567	0.0295	0.0130	0.0197	0.0487	0.0218	0.0169	0.0149
			$\hat{\lambda}$	0.0752	0.3182	0.1514	0.3695	0.0229	0.1548	0.0290	0.1394
	180		$\hat{\alpha}$	0.0068	0.0101	0.0284	0.0123	−0.0018	0.0047	0.0055	0.0046
			$\hat{\beta}$	0.0308	0.0148	−0.0052	0.0114	0.0268	0.0121	0.0007	0.0096
			$\hat{\lambda}$	0.0597	0.2098	0.1288	0.2440	0.0184	0.1058	0.0290	0.0990
	140	3	$\hat{\alpha}$	−0.1117	0.0164	−0.0945	0.0136	−0.2179	0.0481	−0.2113	0.0453
			$\hat{\beta}$	0.0887	0.0415	0.0350	0.0256	0.3162	0.1900	0.2300	0.1070
			$\hat{\lambda}$	−0.2083	0.1669	−0.1418	0.1601	−0.6435	0.4210	−0.6263	0.3998
	180		$\hat{\alpha}$	−0.0359	0.0080	−0.0165	0.0079	−0.1305	0.0185	−0.1227	0.0166
			$\hat{\beta}$	0.0413	0.0165	0.0027	0.0120	0.1423	0.0380	0.1004	0.0242
			$\hat{\lambda}$	−0.0442	0.1561	0.0217	0.1711	−0.4318	0.2134	−0.4105	0.1953

4.1. Asymptotic confidence intervals (ACI)

In this section, the asymptotic confidence intervals by using MLE and MPS method based on PTIIHC scheme will be shown. The interval estimation of the parameters requires variance–covariance matrix, where it is the approximate inverse of Fisher information matrix. The diagonal elements are the variances of the parameters and the off-diagonal elements are the covariance between the parameters. An approximate 95% two side confidence intervals for (Ψ) is

$$\widehat{\Psi} \pm Z_{0.025}\sqrt{VC_{ii}} \tag{4.1}$$

According to Eq. (3.2) and assuming the regularity conditions are satisfied. Let

$$I_{ij}(\Psi) = \left(\frac{\partial^2 l(\Psi)}{\partial \Psi_i \partial \Psi_j}\right), i, j = 1, 2, 3.$$

We obtain $I(\widehat{\Psi})$ which is the observed second partial derivatives matrix and it is defined as:

$$I_{11}(\Psi) = \frac{-h}{\alpha^2}.$$

Table 3
Parameter estimation of PL distribution under different censoring schemes in case 2.

n	m	$R^{(i)}$	$t = 500$	$T < xm$				$T > xm$			
				MLE		MPS		MLE		MPS	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
100	70	1	$\hat{\alpha}$	−0.3309	0.1152	−0.3093	0.1028	−0.4146	0.1730	−0.4044	0.1648
			$\hat{\beta}$	0.5183	1.0795	0.2626	0.5256	1.0871	2.8479	0.6959	1.3622
			$\hat{\lambda}$	−0.4687	0.3089	−0.4095	0.2825	−0.6817	0.4698	−0.6640	0.4471
	90		$\hat{\alpha}$	−0.1599	0.0462	−0.1291	0.0419	−0.2959	0.0919	−0.2824	0.0843
			$\hat{\beta}$	0.2269	0.2368	0.0841	0.1409	0.5222	0.5301	0.3501	0.2977
			$\hat{\lambda}$	−0.2105	0.2877	−0.1404	0.2732	−0.5430	0.3189	−0.5179	0.2938
	70	2	$\hat{\alpha}$	0.1602	1.8356	0.2016	15.2788	0.0093	0.0814	0.0071	0.0587
			$\hat{\beta}$	0.2335	0.4927	0.0879	0.2787	0.2152	0.4505	0.1046	0.2886
			$\hat{\lambda}$	0.4971	1.1768	0.5666	0.9086	0.0847	0.6282	0.0545	0.4215
	90		$\hat{\alpha}$	0.0421	0.0788	0.0807	0.0930	0.0097	0.0367	0.0144	0.0328
			$\hat{\beta}$	0.1164	0.1572	0.0033	0.1101	0.1109	0.1353	0.0265	0.0985
			$\hat{\lambda}$	0.1530	0.6458	0.2295	0.7382	0.0636	0.2829	0.0567	0.2376
	70	3	$\hat{\alpha}$	−0.2513	0.0756	−0.2236	0.0657	−0.3614	0.1335	−0.3485	0.1245
			$\hat{\beta}$	0.4239	0.7751	0.2129	0.4011	0.8625	1.8804	0.5669	0.9202
			$\hat{\lambda}$	−0.3566	0.2678	−0.2902	0.2596	−0.6257	0.4051	−0.6031	0.3792
	90		$\hat{\alpha}$	−0.0829	0.0400	−0.0484	0.0415	−0.2173	0.0553	−0.2030	0.0495
			$\hat{\beta}$	0.1737	0.1956	0.0459	0.1264	0.3712	0.3336	0.2341	0.1975
			$\hat{\lambda}$	−0.0783	0.3130	−0.0041	0.3573	−0.4140	0.2216	−0.3883	0.2012
200	140	1	$\hat{\alpha}$	−0.2922	0.0909	−0.2726	0.0814	−0.4231	0.1795	−0.4172	0.1746
			$\hat{\beta}$	0.2170	0.1861	0.0955	0.1184	0.8512	1.1474	0.6511	0.7124
			$\hat{\lambda}$	−0.3631	0.2229	−0.3005	0.2096	−0.7009	0.4930	−0.6907	0.4791
	180		$\hat{\alpha}$	−0.1255	0.0339	−0.0977	0.0317	−0.3115	0.0990	−0.3036	0.0942
			$\hat{\beta}$	0.1085	0.0850	0.0252	0.0624	0.4755	0.3270	0.3766	0.2249
			$\hat{\lambda}$	−0.1341	0.1904	−0.0642	0.2156	−0.5808	0.3467	−0.5655	0.3296
	140	2	$\hat{\alpha}$	0.0364	0.0734	0.0781	0.0975	−0.0013	0.0210	−0.0003	0.0190
			$\hat{\beta}$	0.1054	0.1115	0.0211	0.0859	0.0903	0.0784	0.0382	0.0629
			$\hat{\lambda}$	0.1324	0.5490	0.2269	0.7131	0.0274	0.1639	0.0195	0.1415
	180		$\hat{\alpha}$	0.0325	0.0499	0.0698	0.0656	0.0025	0.0146	0.0054	0.0137
			$\hat{\beta}$	0.0572	0.0686	−0.0146	0.0565	0.0492	0.0528	0.0052	0.0443
			$\hat{\lambda}$	0.1101	0.3776	0.1947	0.4909	0.0260	0.1116	0.0250	0.1021
	140	3	$\hat{\alpha}$	−0.2073	0.0543	−0.1820	0.0472	−0.3726	0.1400	−0.3653	0.1346
			$\hat{\beta}$	0.1901	0.1799	0.0821	0.1196	0.6731	0.7050	0.5272	0.4652
			$\hat{\lambda}$	−0.2503	0.1856	−0.1813	0.1866	−0.6538	0.4322	−0.6401	0.4152
	180		$\hat{\alpha}$	−0.0613	0.0300	−0.0299	0.0327	−0.2353	0.0589	−0.2269	0.0550
			$\hat{\beta}$	0.0847	0.0765	0.0069	0.0591	0.3310	0.1865	0.2531	0.1295
			$\hat{\lambda}$	−0.0387	0.2181	0.0362	0.2630	−0.4626	0.2345	−0.4458	0.2196

$$I_{22}(\Psi) = \frac{-h}{\beta^2} + \sum_{i=1}^{h} \frac{\ln(x_{i:h:n})^2 x_{i:h:n}^{\beta}\left[(\alpha+1)+\alpha R_i\right]}{\left(\lambda + x_{i:h:n}^{\beta}\right)}\left[-1 + \frac{x_{i:h:n}^{\beta}}{\left(\lambda + x_{i:h:n}^{\beta}\right)}\right] + \alpha R_h^{*} \frac{T^{\beta}\ln(T)^2}{\left(\lambda + T^{\beta}\right)}\left[-1 + \frac{T^{\beta}}{\left(\lambda + T^{\beta}\right)}\right].$$

$$I_{33}(\Psi) = \frac{-h\alpha}{\lambda^2} + \sum_{i=1}^{h} \frac{\left[(\alpha+1)+\alpha R_i\right]}{\left(\lambda + x_{i:h:n}^{\beta}\right)^2} + \frac{\alpha R_h^{*}}{\left(\lambda + T^{\beta}\right)^2} - \frac{\alpha}{\lambda^2}\sum_{i=1}^{h} R_i - \frac{\alpha}{\lambda^2} R_h^{*}.$$

$$I_{12}(\Psi) = -\sum_{i=1}^{h} \frac{\ln(x_{i:h:n})\, x_{i:h:n}^{\beta}\,(1+R_i)}{\left(\lambda + x_{i:h:n}^{\beta}\right)} - R_h^{*}\frac{T^{\beta}\ln(T)}{\left(\lambda + T^{\beta}\right)}.$$

$$I_{23}(\Psi) = \sum_{i=1}^{h} \frac{x_{i:h:n}^{\beta}\ln(x_{i:h:n})\left[(\alpha+1)+\alpha R_i\right]}{\left(\lambda + x_{i:h:n}^{\beta}\right)^2} + \frac{\alpha T^{\beta} R_h^{*}\ln(T)}{\left(\lambda + T^{\beta}\right)^2}.$$

$$I_{13}(\Psi) = \frac{h}{\lambda} - \sum_{i=1}^{h} \frac{(1+R_i)}{\lambda + x_{i:h:n}^{\beta}} - \frac{R_h^{*}}{\left(\lambda + T^{\beta}\right)} + \frac{1}{\lambda}\sum_{i=1}^{h} R_i + \frac{1}{\lambda}R_h^{*}.$$

Table 4
CI of PL distribution under different censoring schemes when $n = 100$ in case 1.

	CI			$T < xm$						$T > xm$					
				MLE			MPS			MLE			MPS		
n	m	$R^{(i)}$		L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB
100	70	1	$\hat{\alpha}$	0.3752	0.0123	0.0119	0.7354	0.0299	0.0234	0.0976	0.0031	0.0031	0.0928	0.0032	0.0032
			$\hat{\beta}$	2.1418	0.0698	0.0695	1.4842	0.0448	0.0421	3.2196	0.1029	0.1019	2.5963	0.0764	0.0748
			$\hat{\lambda}$	2.8469	0.1052	0.0926	2.7576	0.1025	0.0908	0.3422	0.0107	0.0109	0.3526	0.0120	0.0121
	90		$\hat{\alpha}$	0.4812	0.0151	0.0151	0.5334	0.0177	0.0177	0.1705	0.0052	0.0052	0.1630	0.0055	0.0055
			$\hat{\beta}$	1.6261	0.0534	0.0509	1.2019	0.0373	0.0378	1.7703	0.0580	0.0571	1.4754	0.0458	0.0442
			$\hat{\lambda}$	2.5009	0.0786	0.0794	2.4718	0.0689	0.0649	0.6415	0.0204	0.0203	0.6230	0.0207	0.0210
	70	2	$\hat{\alpha}$	1.1163	0.0354	0.0341	1.4809	0.0584	0.0482	0.5518	0.0175	0.0176	0.4938	0.0153	0.0153
			$\hat{\beta}$	1.7625	0.0545	0.0545	1.4166	0.0428	0.0430	1.4819	0.0458	0.0453	1.1020	0.0335	0.0318
			$\hat{\lambda}$	5.7308	0.1872	0.1809	7.3005	0.2599	0.2484	2.6238	0.0812	0.0804	2.2924	0.0713	0.0713
	90		$\hat{\alpha}$	0.6791	0.0217	0.0219	0.7428	0.0241	0.0245	0.4074	0.0127	0.0127	0.3861	0.0124	0.0123
			$\hat{\beta}$	1.1070	0.0357	0.0348	0.9858	0.0293	0.0284	0.8583	0.0276	0.0281	0.6833	0.0200	0.0199
			$\hat{\lambda}$	3.1434	0.0970	0.0970	3.3502	0.1043	0.1040	1.8244	0.0589	0.0586	1.6948	0.0532	0.0528
	70	3	$\hat{\alpha}$	0.4463	0.0139	0.0138	0.5544	0.0195	0.0183	0.1512	0.0050	0.0049	0.1450	0.0046	0.0046
			$\hat{\beta}$	1.8465	0.0585	0.0577	1.4452	0.0430	0.0431	2.7675	0.0911	0.0901	2.0317	0.0592	0.0605
			$\hat{\lambda}$	2.8423	0.0951	0.0943	3.6275	0.1294	0.1222	0.5442	0.0178	0.0178	0.5430	0.0188	0.0191
	90		$\hat{\alpha}$	0.5988	0.0200	0.0202	0.6851	0.0230	0.0227	0.2240	0.0069	0.0069	0.2141	0.0072	0.0071
			$\hat{\beta}$	1.3440	0.0438	0.0435	0.9462	0.0278	0.0279	1.4145	0.0470	0.0454	1.1048	0.0322	0.0315
			$\hat{\lambda}$	2.8866	0.0908	0.0898	3.2065	0.1014	0.1031	0.9193	0.0292	0.0291	0.8765	0.0288	0.0285
200	140	1	$\hat{\alpha}$	0.1762	0.0056	0.0056	0.1773	0.0060	0.0060	0.0648	0.0021	0.0021	0.0611	0.0021	0.0021
			$\hat{\beta}$	0.7091	0.0225	0.0227	0.6463	0.0185	0.0189	1.5586	0.0512	0.0499	1.3001	0.0357	0.0361
			$\hat{\lambda}$	1.1383	0.0363	0.0351	1.1601	0.0369	0.0368	0.1984	0.0062	0.0062	0.2026	0.0070	0.0068
	180		$\hat{\alpha}$	0.2730	0.0085	0.0085	0.2746	0.0090	0.0093	0.1178	0.0035	0.0035	0.1127	0.0039	0.0038
			$\hat{\beta}$	0.4913	0.0155	0.0155	0.4799	0.0146	0.0146	0.6168	0.0201	0.0200	0.5895	0.0173	0.0173
			$\hat{\lambda}$	1.3767	0.0437	0.0441	1.3928	0.0455	0.0458	0.4417	0.0131	0.0131	0.4317	0.0144	0.0142
	140	2	$\hat{\alpha}$	0.4694	0.0143	0.0142	0.4833	0.0167	0.0166	0.3195	0.0103	0.0103	0.3060	0.0095	0.0098
			$\hat{\beta}$	0.6360	0.0202	0.0202	0.5920	0.0169	0.0170	0.5466	0.0173	0.0169	0.5051	0.0144	0.0143
			$\hat{\lambda}$	2.1928	0.0690	0.0687	2.2327	0.0735	0.0736	1.5407	0.0492	0.0495	1.4538	0.0482	0.0480
	180		$\hat{\alpha}$	0.3939	0.0126	0.0125	0.3994	0.0137	0.0132	0.2689	0.0077	0.0079	0.2580	0.0089	0.0086
			$\hat{\beta}$	0.4623	0.0150	0.0150	0.4540	0.0138	0.0138	0.4190	0.0133	0.0132	0.4098	0.0123	0.0123
			$\hat{\lambda}$	1.7810	0.0571	0.0570	1.8011	0.0601	0.0605	1.2739	0.0391	0.0393	1.2183	0.0410	0.0409
	140	3	$\hat{\alpha}$	0.2473	0.0080	0.0079	0.2508	0.0085	0.0086	0.0952	0.0031	0.0031	0.0908	0.0033	0.0033
			$\hat{\beta}$	0.7196	0.0227	0.0226	0.6662	0.0186	0.0186	1.1767	0.0372	0.0376	0.9987	0.0285	0.0284
			$\hat{\lambda}$	1.3783	0.0464	0.0466	1.4010	0.0441	0.0441	0.3258	0.0105	0.0104	0.3249	0.0107	0.0106
	180		$\hat{\alpha}$	0.3224	0.0103	0.0101	0.3222	0.0101	0.0100	0.1532	0.0045	0.0045	0.1469	0.0050	0.0048
			$\hat{\beta}$	0.4770	0.0151	0.0152	0.4678	0.0142	0.0142	0.5220	0.0165	0.0166	0.5087	0.0150	0.0152
			$\hat{\lambda}$	1.5397	0.0497	0.0500	1.5544	0.0507	0.0503	0.6432	0.0199	0.0199	0.6209	0.0201	0.0203

The elements of the variance–covariance matrix are obtained from MLE

$$VC = Var.Cov\left(\widehat{\Psi}\right) = \begin{bmatrix} -I_{11} & & \\ -I_{12} & -I_{22} & \\ -I_{13} & -I_{23} & -I_{33} \end{bmatrix}^{-1}. \tag{4.2}$$

We can use the same algorithm to obtained second partial derivatives matrix of estimators for MPS method.

4.2. Bootstrap confidence interval

Here, we construct two parametric bootstrap confidence intervals for Ψ as follow:

4.2.1. Normal approximation bootstrap method (NB)

The confidence interval of the mean of a measurement variable is commonly estimated on the assumption that the statistic follows a normal distribution, and that the variance is therefore independent of the mean. This is known as a normal approximation confidence interval. Here, we construct parametric bootstrap confidence interval for Ψ by use percentile bootstrap confidence interval as follow:

(1) Compute the MLE of Ψ based on PTIIHC with different schemes.

Table 5
CI of PL distribution under different censoring schemes in case 2.

	CI			$T < xm$						$T > xm$					
				MLE			MPS			MLE			MPS		
n	m	Schemes		L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB	L.ACI	L.NB	L.PB
100	70	1	$\hat{\alpha}$	0.2945	0.0097	0.0097	0.3104	0.0108	0.0108	0.1334	0.0044	0.0044	0.1281	0.0044	0.0044
			$\hat{\beta}$	3.5316	0.1130	0.1128	2.9062	0.0867	0.0840	5.0623	0.1613	0.1575	4.0661	0.1217	0.1208
			$\hat{\lambda}$	1.1714	0.0384	0.0373	1.2696	0.0434	0.0421	0.2794	0.0091	0.0091	0.2921	0.0096	0.0096
	90		$\hat{\alpha}$	0.5631	0.0186	0.0187	0.5921	0.0195	0.0192	0.2573	0.0084	0.0084	0.2498	0.0081	0.0081
			$\hat{\beta}$	1.6886	0.0563	0.0558	1.5773	0.0442	0.0444	1.9898	0.0677	0.0668	1.8136	0.0494	0.0494
			$\hat{\lambda}$	1.9349	0.0630	0.0610	1.9074	0.0629	0.0608	0.6084	0.0200	0.0195	0.6014	0.0200	0.0197
	70	2	$\hat{\alpha}$	5.0932	0.6332	0.3949	15.2683	0.6188	0.3899	1.1183	0.0350	0.0345	0.9519	0.0304	0.0304
			$\hat{\beta}$	2.5961	0.0880	0.0887	2.1870	0.0675	0.0644	2.4935	0.0859	0.0846	2.1772	0.0689	0.0675
			$\hat{\lambda}$	3.9040	0.5658	0.5021	3.0692	0.4635	0.4034	3.0908	0.0956	0.0953	2.5675	0.0773	0.0778
	90		$\hat{\alpha}$	1.0887	0.0346	0.0351	1.1146	0.0358	0.0356	0.7509	0.0240	0.0240	0.7033	0.0221	0.0221
			$\hat{\beta}$	1.4865	0.0487	0.0492	1.4143	0.0389	0.0394	1.3754	0.0462	0.0443	1.3111	0.0384	0.0384
			$\hat{\lambda}$	3.0940	0.0998	0.0962	3.1708	0.0998	0.0983	2.0710	0.0651	0.0651	1.9057	0.0624	0.0619
	70	3	$\hat{\alpha}$	0.2945	0.0097	0.0097	0.3104	0.0108	0.0108	0.2099	0.0067	0.0067	0.2041	0.0070	0.0069
			$\hat{\beta}$	3.5316	0.1130	0.1128	2.9062	0.0867	0.0840	4.1812	0.1326	0.1296	3.3305	0.0928	0.0931
			$\hat{\lambda}$	1.1714	0.0384	0.0373	1.2696	0.0434	0.0421	0.4581	0.0146	0.0144	0.4667	0.0157	0.0156
	90		$\hat{\alpha}$	0.5631	0.0186	0.0187	0.5921	0.0195	0.0192	0.3514	0.0117	0.0119	0.3418	0.0110	0.0111
			$\hat{\beta}$	1.6886	0.0563	0.0558	1.5773	0.0442	0.0444	1.7352	0.0594	0.0576	1.6190	0.0448	0.0447
			$\hat{\lambda}$	1.9349	0.0630	0.0610	1.9074	0.0629	0.0608	0.8785	0.0275	0.0274	0.8555	0.0290	0.0287
200	140	1	$\hat{\alpha}$	0.2922	0.0095	0.0096	0.3113	0.0104	0.0104	0.0886	0.0029	0.0029	0.0851	0.0029	0.0029
			$\hat{\beta}$	1.4621	0.0456	0.0461	1.4178	0.0408	0.0404	2.5505	0.0806	0.0781	2.3068	0.0641	0.0637
			$\hat{\lambda}$	1.1836	0.0377	0.0380	1.2918	0.0434	0.0433	0.1643	0.0051	0.0051	0.1696	0.0059	0.0057
	180		$\hat{\alpha}$	0.5281	0.0163	0.0165	0.5565	0.0194	0.0187	0.1722	0.0051	0.0051	0.1670	0.0057	0.0056
			$\hat{\beta}$	1.0613	0.0326	0.0323	1.0577	0.0322	0.0325	1.2456	0.0390	0.0398	1.2289	0.0358	0.0360
			$\hat{\lambda}$	1.6286	0.0523	0.0508	1.7337	0.0561	0.0561	0.3784	0.0112	0.0111	0.3736	0.0120	0.0120
	140	2	$\hat{\alpha}$	1.0533	0.0325	0.0324	1.1439	0.0379	0.0370	0.5681	0.0189	0.0187	0.5399	0.0165	0.0164
			$\hat{\beta}$	1.2427	0.0384	0.0383	1.2306	0.0346	0.0338	1.0394	0.0323	0.0328	1.0241	0.0300	0.0300
			$\hat{\lambda}$	2.8592	0.0887	0.0890	3.0956	0.1028	0.1016	1.5843	0.0507	0.0502	1.4813	0.0492	0.0508
	180		$\hat{\alpha}$	0.8669	0.0272	0.0262	0.9296	0.0301	0.0300	0.4735	0.0136	0.0134	0.4560	0.0154	0.0155
			$\hat{\beta}$	1.0024	0.0314	0.0313	1.0025	0.0299	0.0300	0.8805	0.0285	0.0284	0.8695	0.0263	0.0264
			$\hat{\lambda}$	2.3710	0.0760	0.0741	2.5550	0.0854	0.0838	1.3062	0.0408	0.0404	1.2501	0.0433	0.0422
	140	3	$\hat{\alpha}$	0.4172	0.0135	0.0138	0.4407	0.0149	0.0149	0.1334	0.0045	0.0044	0.1299	0.0045	0.0045
			$\hat{\beta}$	1.4872	0.0474	0.0479	1.4254	0.0397	0.0400	1.9688	0.0626	0.0630	1.8429	0.0506	0.0510
			$\hat{\lambda}$	1.3752	0.0443	0.0449	1.4685	0.0489	0.0484	0.2716	0.0086	0.0086	0.2749	0.0090	0.0090
	180		$\hat{\alpha}$	0.6349	0.0196	0.0198	0.6676	0.0223	0.0219	0.2316	0.0069	0.0068	0.2253	0.0074	0.0073
			$\hat{\beta}$	1.0325	0.0313	0.0312	1.0312	0.0312	0.0314	1.0881	0.0338	0.0341	1.0813	0.0312	0.0315
			$\hat{\lambda}$	1.8254	0.0588	0.0587	1.9313	0.0647	0.0654	0.5621	0.0169	0.0172	0.5493	0.0187	0.0184

(2) Generated a bootstrap samples using Ψ to obtain the bootstrap estimate of Ψ say $\widehat{\Psi}^b$ using the bootstrap sample.
(3) Repeat step (2) B times to have $\left(\Psi^{b(1)},\ \Psi^{b(2)},\ldots,\Psi^{b(B)}\right)$.
(4) A two side $100\,(1-\gamma)\%$ normal bootstrap confidence intervals for the unknown parameters Ψ are given by $\left\{\overline{\widehat{\Psi}^b} \pm Z_{0.025}\frac{s_{\widehat{\Psi}^b}}{\sqrt{b}}\sqrt{\left(1-\frac{b}{B}\right)}\right\}$.

4.2.2. Percentile bootstrap confidence interval (PB)

Here, we construct parametric bootstrap confidence interval for Ψ by use percentile bootstrap confidence interval as follow:

(1) Repeat step (1, 2, 3) in normal bootstrap.
(2) Arrange $\left(\Psi^{b(1)},\ \Psi^{b(2)},\ldots,\Psi^{b(B)}\right)$ in ascending order as $\left(\Psi^{b[1]},\ \Psi^{b[2]},\ldots,\Psi^{b[B]}\right)$.
(3) A two side $100\,(1-\gamma)\%$ percentile bootstrap confidence intervals for the unknown parameters Ψ are given by $\left\{\widehat{\Psi}^{b[B\gamma/2]},\widehat{\Psi}^{b[B(1-\gamma/2)]}\right\}$.

5. Simulation study

In this section; a Monte Carlo simulation is done to estimate the parameters of PL distribution based on PTIIHC for MLE and MPS methods. Using R language as follows:

Step 1: Generate 10 000 random samples of size 100 and 200 from the PL distribution based on PTIIHC.

Step 2: Using the quantile $x_i = \left(\left(\frac{\lambda^{\alpha}}{1-u_i}\right)^{\frac{1}{\alpha}} - \lambda\right)^{\frac{1}{\beta}}$; $0 < u_i < 1$, where x are distributed as PL for different parameters $\Psi = (\alpha, \beta, \lambda)$, Two sets of parameters values are selected as; case-I is $\Psi = (0.3, 0.6, 0.75)$ and case-II is $\Psi = (0.5, 1.5, 0.75)$.

Step 3: In PTIIHC, the effective of sample sizes (failure items) m are selected based on two levels of censoring as 70% and 90%, according to Hassan and Abd-Allah [5]. Selected T in case-I is 500 and case-II is 10, and sets of different samples schemes.

- Scheme 1: $R^{(1)} = R_2 = \cdots = R_{m-1} = 0,\ and\ R_m = n - m$. It is type-II scheme
- Scheme 2: $R^{(2)} = n - m\ and\ R_2 = R_3 = \cdots = R_{m-1} = 0$.
- Scheme 3: $R^{(3)} = \frac{n-m}{2}, R_2 = \cdots = R_{m-2} = 0,\ and\ R_m = \frac{n-m}{2}$.

Step 4: The MLE and MPS of the model parameters are obtained by solving the non-linear equation based on PTIIHC with different schemes.

Step 5: The Bias, mean square errors (MSE) and length of confidence interval of the parameters are obtained.

Step 6: The numerical results of parameters estimation of PL distribution under different censoring schemes are listed in Tables 2 and 3. Also, the results of confidence intervals are illustrated through Tables 4 and 5.

From simulation results of parameters estimation for model PL distribution based on PTIIHC with different schemes, we note that: the Bias and MSE for all estimators of parameters decrease when the sample size (n) increase in all cases. The efficiency of the model increases when the effective sample size (m) increases. Three different samples schemes were applied on PTIIHC with different schemes and that to get to the most effective scheme, the efficiency is the best for scheme 3. The results prove that, the efficiency of the MPS over MLE, then MPS is good alternative estimators of MLE. In confidence interval, the efficiency of the bootstrap methods over ACI.

6. Applications

We will apply the parameter estimation of PL distribution based on PTIIHC with different schemes for a two real data.

Real Data Set-I of economic data : which consists of 31 observations on response variable GDP growth of Egypt. The data are 10.01, 3.76, 9.91, 7.40, 6.09, 6.60, 2.65, 2.52, 7.93, 4.97, 5.70, 1.08, 4.43, 2.90, 3.97, 4.64, 4.99, 5.49, 4.03, 6.11, 5.37, 3.54, 2.37, 3.19, 4.09, 4.48, 6.85, 7.09, 7.16, 4.67 and 5.15.

In Table 6: Goodness of fit by Kolmogorov–Smirnov test were presented using MLE and different model criterion were also presented using MLE in real data set-I of economic data.

Based on the p-values, it is clear that the PL under complete sample fits the real data I. In Fig. 3: Plots of the fitted pdf and cdf of the PL Distribution for Fatigue: Life Data I, this figure, the results of Table 6 are confirmed.

Table 7 shows the censored sample based on PTIIHC with different schemes when $m = 16$ for real data I.

In Real Data Set-I of Economic Data: We discussed the parameter estimation of PL distribution based on PTIIHC with different schemes by using MLE and MPS method as shown in Table 8. We observed that the standard deviation of most parameters in MPS was lower than MLE. Also, we observed that when the experiment time limit of the PTIIHC model increases, the standard deviation value of most parameters for both methods is increased.

Real Data Set-II of survival times : which represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [29] and Hassan and Abd-Allah [5] used this data to estimate parameters of Power Lomax distribution. The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58 and 5.55.

In Table 9: Parameter estimation under complete sample and Goodness of fit by Kolmogorov–Smirnov test were presented using MLE and different model criterion were also presented using MLE in real data set-II of survival times. In Fig. 4: Plots of the fitted pdf and cdf of the PL Distribution for Fatigue: Life Data II, this figure, the results of Table 9 are confirmed.

Table 6
Statistics measures for PL distribution under complete sample of the real data I.

	$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)	$\hat{\lambda}$ (SE)	K–S (P-valu)	AIC	BIC	CAIC	HQIC	W	A
MLE	5.4337 (5.746)	2.904 (0.655)	757.115 (0.0498)	0.0582 (0.9997)	137.11	141.41	137.99	138.51	0.0151	0.1372

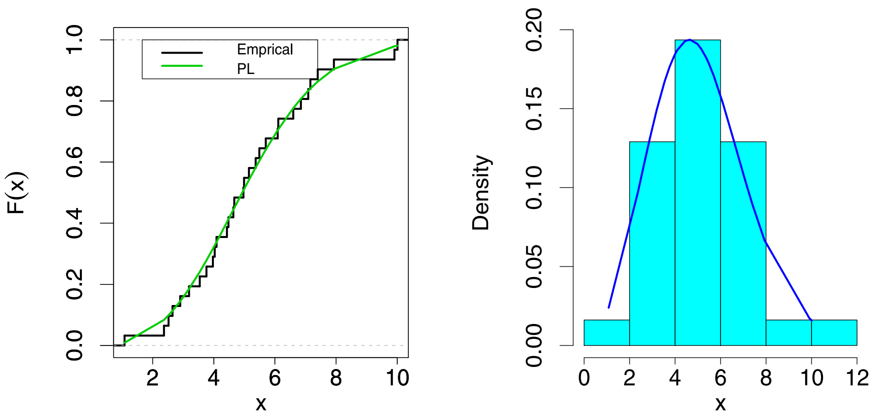

Fig. 3. Plots of the fitted pdf and cdf of the PL distribution for Fatigue: Life Data I.

Table 7
The sample order of PTIIHC with different schemes when $m = 16$ for real data I.

Schemes	$x_{i:m:n}$							
1	1	2	3	4	5	6	7	8
	1.08	2.52	2.65	2.9	3.76	3.97	4.43	4.64
	9	10	11	12	13	14	15	16
	4.97	5.7	6.09	6.6	7.4	7.93	9.91	10.01
2	1	2	3	4	5	6	7	8
	2.52	2.65	2.9	3.19	3.76	3.97	4.48	4.64
	9	10	11	12	13	14	15	16
	4.67	4.97	5.15	6.6	7.16	7.93	9.91	10.01
3	1	2	3	4	5	6	7	8
	2.52	2.65	2.9	3.76	3.97	4.43	4.64	4.97
	9	10	11	12	13	14	15	16
	4.99	5.49	6.09	6.6	7.4	7.93	9.91	10.01

Table 8
Estimation for PL distribution under PTIIHC with different schemes when $m = 16$ of real data I.

Scheme		$T = \max(x) + 1$		$T = \bar{x}$	
		MLE	MPS	MLE	MPS
1	$\hat{\alpha}$	0.2789	0.4139	3.7233	0.1220
	SE	(0.2805)	(0.5572)	(4.6555)	(0.0883)
	$\hat{\beta}$	2.6963	2.0154	2.2928	1.8869
	SE	(1.3008)	(1.0219)	(0.7879)	(0.8617)
	$\hat{\lambda}$	40.7247	23.7714	441.059	1.9124
	SE	(43.8121)	(24.5252)	(0.0409)	(1.6247)
2	$\hat{\alpha}$	0.4407	0.5199	0.3045	0.2425
	SE	(0.2580)	(0.2846)	(0.5127)	(0.2260)
	$\hat{\beta}$	6.1370	5.0797	4.3574	6.9082
	SE	(0.9596)	(0.7640)	(1.0937)	(1.0636)
	$\hat{\lambda}$	3319.1	1237.8	38593.0	2741.8
	SE	(0.00032)	(0.00015)	(0.0424)	(0.00087)
3	$\hat{\alpha}$	0.1706	0.15408	0.6356	0.64075
	SE	(0.09155)	(0.1359)	(2.1959)	(1.0614)
	$\hat{\beta}$	5.8844	5.9336	4.3932	3.1857
	SE	(1.1092)	(0.7214)	(2.9855)	(1.0659)
	$\hat{\lambda}$	1279.9	2590.5	1204.9	2542.4
	SE	(0.000946)	(0.000162)	(0.00256)	(0.02831)

In Real Data Set-II of survival times: We discussed the parameter estimation of PL distribution based on PTIIHC with different schemes by using MLE and MPS method as shown in Table 10. We observed that the standard deviation of most parameters in MPS was lower than MLE.

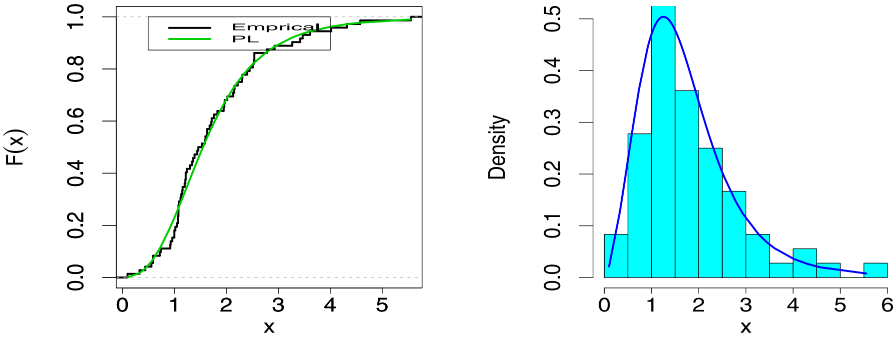

Fig. 4. Plots of the fitted pdf and cdf of the PL distribution for Fatigue: Life Data II.

Table 9
Statistics measures for PL model of the real data set-II.

	$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)	$\hat{\lambda}$ (SE)	-K-S (P-value)	AIC	BIC	CAIC	HQIC	W	A
MLE	1.8027 (1.1488)	2.5216 (0.4614)	6.4537 (4.3949)	0.08093 (0.7332)	193.69	200.52	194.04	196.41	0.0669	0.4076

Table 10
MLE for PL distribution under PTIIHC with different schemes of real data II.

m	Scheme	$T = \max(x) + 1$			$T = 1.5$		
		$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)	$\hat{\lambda}$(SE)	$\hat{\alpha}$ (SE)	$\hat{\beta}$ (SE)	$\hat{\lambda}$ (SE)
35	1	7.1297 (20.230)	2.647 (0.4544)	26.991 (78.001)	7.1297 (20.230)	2.6473 (0.4544)	26.991 (78.001)
	2	2.0205 (1.6388)	2.2967 (0.525)	6.104 (5.396)	9.7498 (28.122)	2.2028 (0.4710)	32.995 (97.267)
34	3	7.3797 (19.541)	2.396 (0.4079)	23.606 (64.462)	12.075 (49.121)	2.363 (0.423)	38.781 (160.38)

7. Conclusion

In this paper, the MPS under progressive Type-II hybrid censoring scheme was introduced. The parameter estimation for the PL distribution based on PTIIHC are introduced with different schemes based on the MPS and MLE methods. In parameter estimation under PTIIHC, the MPS method can be used as alternative method for the MLE method. In case of PL distribution based on PTIIHC, the estimators based on MPS method are better than the estimators based on the MLE. We can conclude that the MPS method is a good alternative method to the usual MLE method in many situation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] K.S. Lomax, Business failures: Another example of the analysis of failure data, J. Amer. Statist. Assoc. 49 (268) (1954) 847–852.
[2] A.S. Hassan, A.S. Al-Ghamdi, Optimum step stress accelerated life testing for Lomax distribution, J. Appl. Sci. Res. 5 (12) (2009) 2153–2164.
[3] S. Rezaei, R. Tahmasbi, M. Mahmoodi, Estimation of P [Y < X] for generalized Pareto distribution, J. Statist. Plann. Inference 140 (2) (2010) 480–494.
[4] D.E. Giles, H. Feng, R.T. Godwin, Bias-corrected maximum likelihood estimation of the parameters of the generalized pareto distribution, Comm. Statist. Theory Methods 45 (8) (2016) 2465–2483.
[5] A.S. Hassan, M. Abd-Allah, On the inverse power Lomax distribution, Ann. Data Sci. 6 (2) (2019) 259–278.
[6] E.H.A. Rady, W.A. Hassanein, T.A. Elhaddad, The power Lomax distribution with an application to bladder cancer data, SpringerPlus 5 (1) (2016) 1838.

[7] I.B. Abdul-Moniem, Order statistics from power lomax distribution, Int. J. Innov. Sci. Eng. Technol. 4 (2017) 1–4.
[8] A.A. Essam, On record values of the power lomax distribution, Adv. Appl. Stat. 50 (6) (2017) 497–515.
[9] A.S. Hassan, S.G. Nassr, Power Lomax poisson distribution: properties and estimation, J. Data Sci. 18 (2018) 105–128.
[10] B. Epstein, Estimation from life test data, Technometrics 2 (4) (1960) 447–454.
[11] N. Balakrishnan, Progressive censoring methodology: an appraisal (with discussions), TEST 16 (2007) 211–296.
[12] E.M. Almetwally, H.M. Almongy, A. El sayed Mubarak, Bayesian and maximum likelihood estimation for the Weibull generalized exponential distribution parameters using progressive censoring schemes, Pak. J. Stat. Oper. Res. 14 (4) (2018) 853–868.
[13] D. Kundu, A. Joarder, Analysis of Type-II progressively hybrid censored data, Comput. Statist. Data Anal. 50 (10) (2006) 2509–2528.
[14] C. Claudio, Kandza-Tadi, Leo, O. Romanus, Odhiambo, Parameter estimation of power lomax distribution based on Type-II progressively hybrid censoring scheme, Appl. Math. Sci. 12 (18) (2018) 879–891.
[15] C.T. Lin, H.K.T. Ng, P.S. Chan, Statistical inference of Type-II progressively hybrid censored data with Weibull lifetimes, Comm. Statist. Theory Methods 38 (10) (2009) 1710–1729.
[16] A.M. Salem, O.E. Abo-Kasem, Estimation for the parameters of the exponentiated Weibull distribution based on progressive hybrid censored samples, Int. J. Contemp. Math. Sci. 6 (2011) 1713–1724.
[17] J. Li, L. Ma, Inference for the generalized Rayleigh distribution based on progressively type-II hybrid censored data, J. Inf. Comput. Sci. 12 (3) (2015) 1101–1112.
[18] P.S. Chan, H.K.T. Ng, F. Su, Exact likelihood inference for the two-parameter exponential distribution under Type-II progressively hybrid censoring, Metrika 78 (6) (2015) 747–770.
[19] A.H. Abdel-Hamid, T.A. Abushal, Inference on progressive-stress model for the exponentiated exponential distribution under type-II progressive hybrid censoring, J. Stat. Comput. Simul. 85 (6) (2015) 1165–1186.
[20] T. Sen, R. Bhattacharya, Y.M. Tripathi, B. Pradhan, Inference and optimum life testing plans based on Type-II progressive hybrid censored generalized exponential data, Comm. Statist. Simulation Comput. (2019) 1–29.
[21] R.C.H. Cheng, N.A.K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, J. R. Stat. Soc. Ser. B Stat. Methodol. 39 (1983) 4–403.
[22] B. Ranneby, The maximum spacing method. An estimation method related to the maximum likelihood method, Scand. J. Stat. 9 (1984) 3–112.
[23] E.M. Almetwally, H.M. Almongy, Estimation method for new Weibull-Pareto distribution: Simulation and application, J. Data Sci. 17 (3) (2019) 610–630.
[24] H.K.T. Ng, L. Luo, Y. Hu, F. Duan, Parameter estimation of three-parameter Weibull distribution based on progressively type-II censored samples, J. Stat. Comput. Simul. 82 (11) (2012) 1661–1678.
[25] R.K. Singh, S.K. Singh, U. Singh, Maximum product spacings method for the estimation of parameters of generalized inverted exponential distribution under progressive Type II censoring, J. Stat. Manag. Syst. 19 (2) (2016) 219–245.
[26] E.M. Almetwally, H.M. Almongy, Maximum product spacing and Bayesian method for parameter estimation for generalized power Weibull distribution under censoring scheme, J. Data Sci. 17 (2) (2019) 407–444.
[27] S. Basu, S.K. Singh, U. Singh, Estimation of inverse lindley distribution using product of spacings function for hybrid censored data, Methodol. Comput. Appl. Probab. (2018) 1–18.
[28] E.M. Almetwally, H.M. Almongy, E.A. El-Sherpieny, Adaptive Type-II progressive censoring schemes based on maximum product spacing with application of generalized Rayleigh distribution, J. Data Sci. 17 (4) (2019) 767–793.
[29] T. Bjerkedal, Acquisition of resistance in guinea pies infected with different doses of virulent tubercle bacilli, Am. J. Hygiene 72 (1) (1960) 130–148.