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The Exponentiated Generalized Alpha Power Exponential Distribution: Properties and Applications

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Abstract

In this paper, we first introduce the exponentiated generalized alpha power family of distributions to extend several other distributions. We use the new family and develop a new distribution, called the exponentiated generalized alpha power exponential (EGAPEx) distribution. The proposed EGAPEx distribution provides greater flexibility in modeling data from a practical point of view. The new model includes the exponential; alpha power exponential; alpha power generalized exponential, generalized exponential, standardized generalized exponential and exponentiated generalized exponential distributions as a special cases. This distribution exhibits four hazard rate shapes such as L shaped, increasing, decreasing, and upside-down bathtub. Some statistical properties of the EGAPEx distribution are obtained. The model parameters are obtained by maximum likelihood, maximum product spacing, and Bayesian estimation methods. In addition, we have obtained approximate confidence intervals, two bootstrap confidence intervals, and Bayes credible intervals. A Monte Carlo Simulation is performed to compare the different methods of estimation. We illustrate the performance of the proposed distribution through two real data sets; one is related to economic data and another is failure time data and the data sets show the proposed distribution is superior in its ability to model the data sets as compared to the exponentiated generalized exponential, alpha power generalized exponential, alpha power exponential, generalized exponential and exponential distributions.

Key Words: Exponentiated Generalized distribution; Alpha Power distribution; Maximum Likelihood Estimation; Maximum Product Spacing; Bayesian estimation; Reliability Analysis.

Mathematical Subject Classification: 62F10, 62F15, 62E10, 62E15, 62F40

1. Introduction

The concept of developing new statistical distributions is not new in the statistical literature. The pioneering work of Pearson (1895) using the system of differential equation approach, set the trend for generating statistical distributions. Thereafter, numerous methods have been employed by various authors to generate a family of distributions. In this regard, readers may refer to the works of Burr (1942), Johnson (1949), Hastings et al. (1947), Tukey (1960), Azzalini (1985, 1986), Mudholkar and Srivastava (1993), Marshall and Olkin (1997), Eugene et al. (2002); Cooray and Ananda (2005); Ferreira and Steel (2006), Alzaatreh et al. (2013); and so on.
Of late, Cordeiro et al. (2013) proposed an important method wherein two new parameters were added to a continuous distribution resulting in a new distribution called the Exponentiated Generalized (EG) Family of distributions. The two extra parameters in the EG family of distributions can control both tail weights of the EG density function.

Let $G(x)$ and $g(x)$ be the cumulative distribution function (cdf) and probability density function (pdf) of a given random variable. Then, the cdf and pdf of the EG family class are, respectively, given by:

$$F(x) = \left[1 - (1 - G(X))^{\theta}\right]^{\beta}, \beta, \theta > 0, \tag{1}$$

and

$$f(x) = \theta\beta\left[1 - \left(1 - G(X)\right)^{\theta}\right]^{\beta-1}\left(1 - G(X)\right)^{\theta-1} g(x). \tag{2}$$

The EG family of densities (2) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. In this regard, readers may refer to the works of Cordeiro and Lemonte (2014), Andrade et al. (2015), Aryal and Elbatal (2015), and De Andrade et al. (2016), Elgarhy et al. (2018), and Reyad et al. (2019).
Mahdavi and Kundu (2017) proposed a transformation of the baseline cdf by adding a new parameter to obtain a family of distributions. The proposed method is called alpha power (AP) transformation. If $G(x)$ is the cdf of any distribution, then the cumulative distribution function(x) is given by:

$$W(x) = \begin{cases} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1} & if\ \alpha > 0, \alpha \neq 1 \\ G(x) & if\ \alpha = 1 \end{cases}, \tag{3}$$

and the corresponding pdf takes the form

$$w(x) = \begin{cases} \dfrac{\ln(\alpha)\, g(x)\alpha^{G(x)}}{\alpha - 1} & if\ \alpha > 0, \alpha \neq 1 \\ g(x) & if\ \alpha = 1 \end{cases} \tag{4}$$

It is clear that for $\alpha \neq 1$, $W(x)$ is a weighted version of g(x), where the weight function is u(x) = $\alpha^{G(x)}$.
This approach of generalization has been adopted frequently by several researchers in recent times. The applicability of this kind of generalized distribution can be found in the works of Dey et al. (2017a, 2017b, 2018, 2019), Nassar et al. (2017), Elbatal et al. (2018), Hassan et al. (2019), Basheer (2019, Eghwerido et al. (2020) amongst others in diverse areas like analysis of ozone data, earthquake data, failure time data, etc.

In this paper, our first objective is to obtain a new distribution, called the exponentiated generalized alpha power exponential (EGAPEx) distribution that contains some special submodels. One important feature of the new model is that the failure rate function can be L-shaped, increasing, decreasing, and upside-down bathtub. Owing to its flexibility of the failure rate function, this new model provides a good alternative to many existing lifetime distributions in modeling positive real data sets in practice. In particular, it can be an alternative to the well-known distributions like exponentiated generalized exponential (EGEx), Alpha Power Generalized Exponential (APGEx), Alpha Power Exponential (APEx), Generalized Exponential (GEx), and exponential (Ex) distributions in modeling economic and failure data. Besides, we provide some statistical properties of the proposed distribution. The second objective is to evaluate and study the behavior of maximum likelihood estimators (MLE) and maximum product spacing estimators (MPSE) as classical estimators. The third objective is to construct approximate confidence intervals (ACIs) and two bootstrap confidence intervals (BCIs) of the model parameters. The fourth objective is to obtain Bayes estimators of the unknown parameters of the model using the squared error loss function under the assumptions of independent gamma priors. Besides, we present a Metropolis-Hastings (MH) algorithm to compute the Bayes estimates and the associated credible intervals. Although the utility and usefulness of an estimator are subject to the area of study, in practice users look for their desired estimator under the different settings of parameters and sample sizes. As it is tedious to compare the performances of these estimators theoretically, we conduct extensive simulations for assessing the performances of the said estimators in terms of their bias and mean squared error (MSE). The novelty of this study is that thus far no study has been carried out on EGAPEx distribution using all these estimation methods.

The layout of the paper is as follows. In Section 2, we provide the genesis of the EGAPEx distribution and also plots for its HRF and PDF. In Section 3, we obtain some properties of this distribution. In Section 4, two classical estimation methods and the Bayesian estimation method of the unknown parameters are presented. In Section 5, asymptotic confidence intervals, two bootstrap confidence intervals (percentile and t bootstraps), and Bayes credible intervals are provided. In Section 6, we perform a simulation study to evaluate the performance of the aforementioned estimation methods. The potentiality of the EGAPEx distribution is also illustrated through two real data sets in Section 7. Finally, in Section 8, we provide some conclusions.

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2. Model description of EGAP Family

The CDF of the EGAP family is obtained by replacing $F(x)$ in Eq. (1) by the EG family and Eq. (3) of the AP class, thus, we have

$$F_{EGAP}(x) = \begin{cases} \left[1-\left(1-\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right)^{\theta}\right]^{\beta} & ;\ \alpha,\beta,\theta>0,\alpha\neq 1 \\ \left[1-(1-G(X))^{\theta}\right]^{\beta} & ;\ \alpha=1 \end{cases}, \tag{5}$$

The pdf of the EGAP family is obtained by replacing pdf of Eq. (2) by the EG family and Eq. (4) of the AP class, we have

$$f_{EGAP}(x) = \begin{cases} \theta\beta\left[1-\left(1-\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right)^{\theta}\right]^{\beta-1}\left(1-\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right)^{\theta-1}\dfrac{\ln(\alpha)\,g(x)\alpha^{G(x)}}{\alpha-1} \\ \theta\beta\left[1-\left(1-G(X)\right)^{\theta}\right]^{\beta-1}\left(1-G(X)\right)^{\theta-1}g(x) & ;\ \alpha=1 \end{cases}. \tag{6}$$

Figure 1 displays plots of the pdf of the $uniform(0,1)$ based on EGAP, then EGAPU distribution for some values of parameters as follows

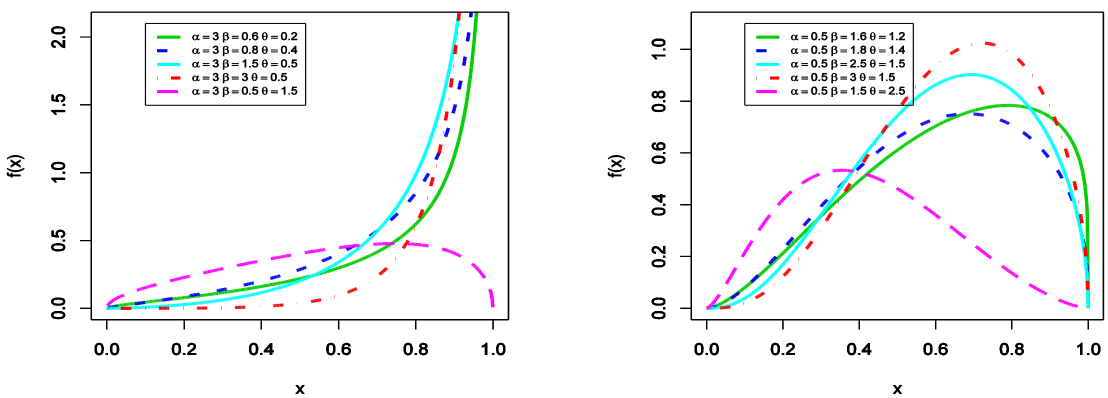

Figure 1: Plots of the EGAPU pdf for some values of parameters.

In Figure 1, we note the new family can have symmetrical, n-shaped, increasing and decreasing densities.

Let xu= Q(u), for 0 < u < 1 denote the quantile function of EGAP family. Then

$$x_u = G^{-1}\left[\frac{1}{\ln(\alpha)}\ln\left(1+(\alpha-1)\left[1-\left(1-u^{1/\beta}\right)^{1/\theta}\right]\right)\right]. \tag{7}$$

In particular, the first three quantiles, Q1, Q2, and Q3 can be obtained by setting u=0.25, u= 0:5, and u=0:75 in Eq. (7), respectively.

2.1. The EGAPEx distribution

In this subsection, we apply the EGAP family to the exponential distribution. The generated distribution is called the EGAPEx distribution and it is represented by the random variable $X \sim EGAPEx(\alpha,\beta,\theta,\lambda)$. The pdf and cdf of the exponential distribution are $g(x;\ \lambda)=\lambda e^{-\lambda x};\ \lambda>0, x>0$ and $G(x)=1-e^{-\lambda x}$, respectively.

Using Eq. (6) and the exponential pdf, the density function of EGAPEx distribution becomes

$$f(x;\Omega) = \theta\beta\left[1-\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta}\right]^{\beta-1}\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta-1}\frac{\ln(\alpha)\,\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}}}{\alpha-1}, \tag{8}$$

where $\Omega = (\alpha, \beta, \theta, \lambda)$, $\Omega > 0$ and $\boldsymbol{\alpha} \neq \mathbf{1}$. By using Eq. (5) and the exponential cdf, the cdf of EGAPEx distribution takes the form

$$F(x;\Omega) = \left[1 - \left(1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}\right)^{\theta}\right]^{\beta}. \tag{9}$$

Using pdf, the binomial series and exponential series, the EGAPEx density can be expressed as a linear combination of exponential densities, namely

$$f(x;\Omega) = \mathfrak{P}(\alpha, \beta, \theta)\, g\big(x;\ \lambda(p+1)\big), \tag{10}$$

where $\mathfrak{P}(\alpha,\beta,\theta) = \frac{\theta\beta\ln(\alpha)}{(p+1)(\alpha-1)} \sum_{k=0}^{\infty}\binom{\beta-1}{k} \sum_{j=0}^{\infty}\binom{\theta(k+1)-1}{j}(1-\alpha)^{j} \sum_{h=0}^{\infty}\binom{j}{h} \sum_{h=0}^{\infty}\frac{\big((h+1)\ln(\alpha)\big)^{q}}{q!} \sum_{p=0}^{\infty}\binom{q}{p}(-1)^{k+j+h+p}$

It can be denoted by exponential density with scale parameter $\lambda(p+1)$. This formula is very useful to obtain various structural properties of the EGAPEx distribution directly from the exponential distribution.

Plots of the PDF of X of the EGAPEx distribution for selected parameter values of $\alpha, \beta, \theta, \lambda$ are shown in Figure 2. Figure 2 shows the EGAPEx density can be right-skewed, almost symmetrical, and unimodal shaped, where λ is responsible for the tapered form, and the higher the value of λ, the greater the tapered form of the distribution. While higher the value of the parameter α, the shape of the density is closer to normal distribution.

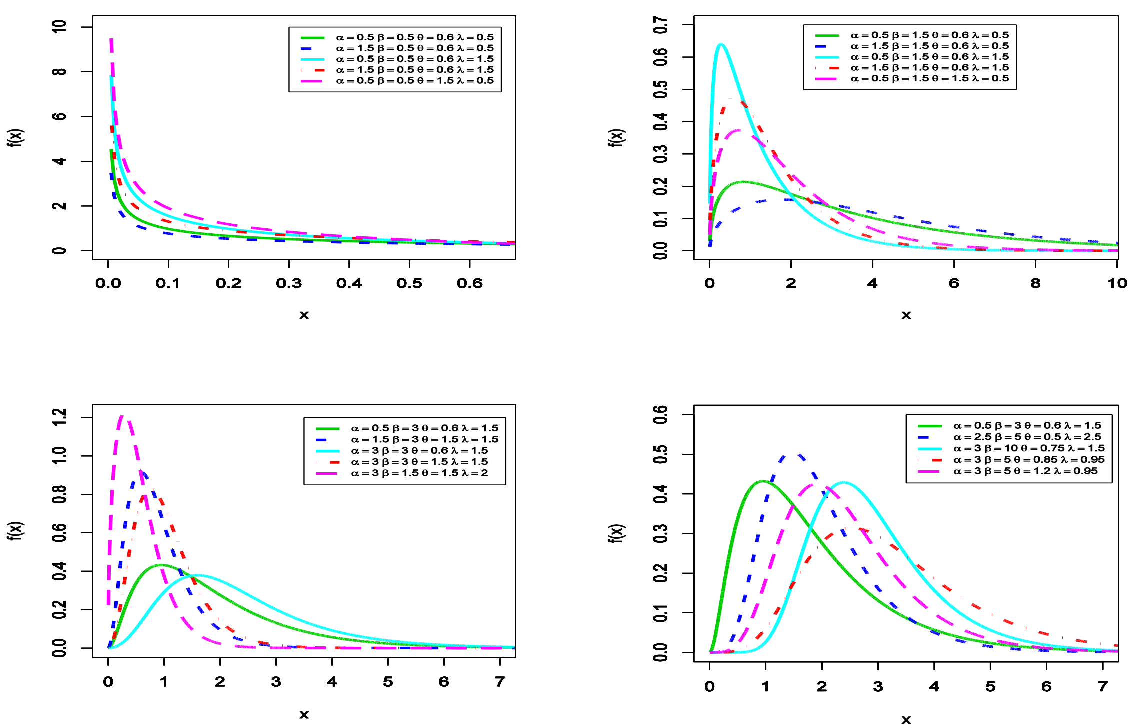

Figure 2: Plots of the EGAPEx pdf for some values of the parameters $\alpha, \beta, \theta, \lambda$

2.2. EGAPEx Sub-Models

From Eq. (5) pdf and Eq. (6) cdf of EGAPEx distribution, we study the sub-models of EGAPEx distribution. Table 1 summarizes the sub-models of the EGAPEx distribution.

Table 1: Sub-models of the EGAPEx distribution.

Distributions	α	β	θ	λ
Exponentiated Generalized Exponential (EGEx)	1	β	θ	λ
Generalized Exponential (GE)	1	1	θ	λ
Standardized Generalized Exponential (GE)	1	1	θ	1
Alpha Power Generalized Exponential (APGEx)	α	1	θ	λ
Alpha Power Exponential (APEx)	α	1	1	λ
Exponential (Ex)	1	1	1	λ

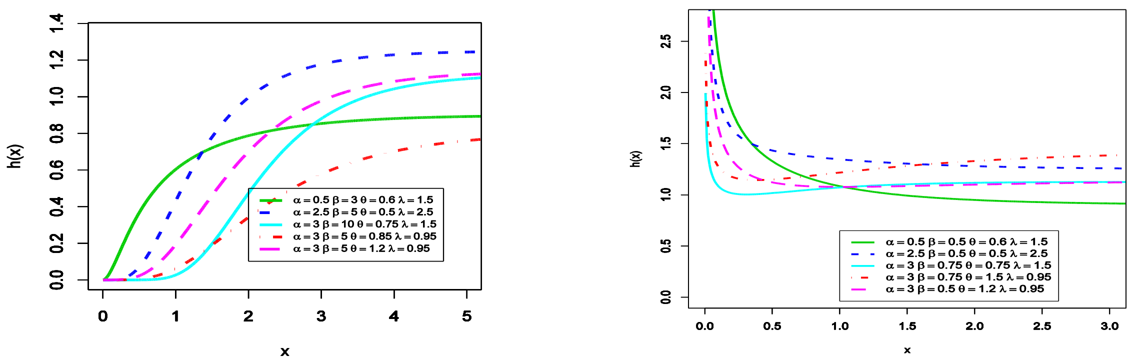

Figure 3: Plots of the EGAPEx hazard rate for some values of the parameters $\alpha, \beta, \theta, \lambda$

Figure 3 illustrates that the EGAPEx HRF can be L-shaped, increasing, decreasing and upside-down bathtub shapes. These features make the EGAPEx distribution very competitive over exponential, generalize exponential and alpha power exponential distributions as these distributions can model phenomena showing only monotonic failure rates which is a major weakness because most empirical life systems have a bathtub or unimodal shapes for their HRF and therefore EGAPEx distribution seems to be more flexible for analyzing lifetime, economic data, etc.

3. Statistical Properties

In this section, we study some statistical properties of the EGAPEx distribution such as quantile function, moments, moment generating function and stress-strength reliability.

3.1. Quantile function

If X has $EGAPEx(\Omega)$ distribution, then the quantile of a random variable X, is given by

$$x_u = \frac{-1}{\lambda} ln\left[1 - \frac{1}{\ln(\alpha)} \ln\left(1 + (\alpha - 1)\left[1 - \left(1 - u^{1/\beta}\right)^{1/\theta}\right]\right)\right]. \quad (11)$$

3.2. Moments, Moment Generating Functions and Related Measures

In this subsection, we will present the r^{th} moments of EGAPEx distribution. Moments are important in any statistical analysis. Moments are useful to determine the properties of distribution such as measures of central tendency, variance, skewness, and kurtosis. Let X be a random variable having the exponential distribution with rate parameter λ. The ordinary moments of X is $E(X^r) = \frac{\Gamma(r+1)}{\lambda r}$, the moment generating functions of X is $M_X(t) = \frac{\lambda}{(\lambda - t)}; t < \lambda$, and the characteristic function of X is $\varphi_X(t) = \frac{\lambda}{(\lambda - it)}$.

Let X be a random variable having the EGAPEx distribution with vector parameter Ω, and by using Eq. (8) and the ordinary moments of the exponential distribution, the r^{th} ordinary moment of EGAPEx distribution is as follows

$$E(x^n) = \mathfrak{P}(\alpha,\beta,\theta)\frac{\Gamma(r+1)}{\lambda(p+1)r}$$

By using the moment generating functions of the exponential distribution, the moment generating functions of EGAPEx is as follows

$$M_X(t) = \mathfrak{P}(\alpha,\beta,\theta)\frac{\lambda(p+1)}{(\lambda(p+1)-t)}; t < \lambda(p+1)$$

By using the characteristic function of exponential distribution, the characteristic function of EGAPEx is as follows

$$\mathfrak{F}_X(t) = \mathfrak{P}(\alpha,\beta,\theta)\frac{\lambda(p+1)}{(\lambda(p+1)-it)}$$

The cumulant-generating function of EGAPEx distribution is as follows

$$K(t) = \log(\mathfrak{F}_X(t)) = \log\left(\mathfrak{P}(\alpha,\beta,\theta)\frac{\lambda(p+1)}{(\lambda(p+1)-it)}\right)$$

3.3. Survival and Hazard Rate Function

The survival function (reliability function) of EGAPEx distribution is given by

$$S(x;\Omega) = 1 - \left[1-\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta}\right]^{\beta}$$

The hazard rate function (failure rate) of a lifetime random variable X with EGAPEx distribution is given by

$$h(x;\Omega) = \frac{\theta\beta\left[1-\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta}\right]^{\beta-1}\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta-1}\frac{\ln(\alpha)\,\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}}}{\alpha-1}}{1-\left[1-\left(1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right)^{\theta}\right]^{\beta}}$$

3.4. Stress-Strength Reliability

Let X and Y are the independent strength and stress random variables observed from EGAPEx. Then, the stress-strength reliability R is defined as;

$$R = P(Y<X) = \int_{x=0}^{\infty}\left\{\int_{y=0}^{x} f(y;\Omega_2)\,dy\right\} f(x;\Omega_1)\,dx.$$
$$= \int_0^{\infty} f(x;\Omega_1)F(x;\Omega_2)\,dx = \frac{\mathfrak{G}(\alpha,\beta,\theta)}{(p+1)} \tag{12}$$

where $\mathfrak{G}(\alpha,\beta,\theta) = \frac{\theta\beta\ln(\alpha)}{(\alpha-1)}\sum_{k=0}^{\infty}\binom{2\beta-1}{k}\sum_{j=0}^{\infty}\binom{\theta(k+1)-1}{j}(1-\alpha)^j\sum_{h=0}^{\infty}\binom{j}{h}\sum_{h=0}^{\infty}\frac{((h+1)\ln(\alpha))^q}{q!}\sum_{p=0}^{\infty}\binom{q}{p}(-1)^{k+j+h+p}$

4. Parameter Estimation

In this section, we adopt the maximum likelihood estimation (MLE) method, maximum product spacing estimation (MPS) method and Bayesian estimation method to estimate the model parameters. Many papers used these methods based on complete samples as Almetwally and Almongy (2019b), Basheer et al. (2021), Almongy et al. (2021) Almetwally (2022a,b), while based on censored samples see Almetwally et al. (2018,2019), Almetwally and Almongy (2019b), Alshenawy et al. (2021), and Almongy et al. (2021), while in the case of estimate fuzzy reliability model see Sabry et al. (2021), and Riad et al. (2022).

4.1. Maximum Likelihood Estimation

The likelihood function is the most frequently used method of parameter estimation. The likelihood function of the EGAPEx distribution is given by

$$\mathrm{L}(\Omega)=\prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\alpha^{1-e^{-\lambda x_i}}\right\}\left(\frac{\theta\beta\ln(\alpha)\,\lambda}{\alpha-1}\right)^{n}e^{-\lambda\sum_{i=1}^{n}x_i}, \tag{13}$$

where $\varphi(x_i,\alpha,\lambda)=\frac{\alpha^{1-e^{-\lambda x_i}}-1}{\alpha-1}$, and the log likelihood function is given as

$$\begin{aligned} l(\Omega)=n\ln\left(\frac{\theta\beta\ln(\alpha)\,\lambda}{\alpha-1}\right)-\lambda\sum_{i=1}^{n}x_i+\ln(\alpha)\sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)+(\theta-1)\sum_{i=1}^{n}\ln\left(1-\varphi(x_i,\alpha,\lambda)\right)\\ +(\beta-1)\sum_{i=1}^{n}\ln\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]. \end{aligned} \tag{14}$$

The likelihood equations for the model parameters are

$$\frac{\partial\ln l(\Omega)}{\partial\alpha}=\frac{n}{\hat{\alpha}\,ln(\hat{\alpha})}-\frac{n}{\hat{\alpha}-1}+\frac{1}{\hat{\alpha}}\sum_{i=1}^{n}\left(1-e^{-\hat{\lambda}x_i}\right)+\left(\hat{\theta}-1\right)\sum_{i=1}^{n}\varrho\left(x_i,\hat{\alpha},\hat{\lambda}\right)-\left(\hat{\beta}-1\right)\sum_{i=1}^{n}\varsigma\left(x_i,\hat{\alpha},\hat{\vartheta},\hat{\lambda}\right)=\mathbf{0},$$

$$\frac{\partial\ln l(\Omega)}{\partial\beta}=\frac{n}{\hat{\beta}}+\sum_{i=1}^{n}\ln\left[1-\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)^{\hat{\theta}}\right]=\mathbf{0},$$

$$\frac{\partial\ln l(\Omega)}{\partial\theta}=\frac{n}{\hat{\theta}}+\sum_{i=1}^{n}\ln\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)-\left(\hat{\beta}-1\right)\sum_{i=1}^{n}\frac{ln\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)^{\hat{\theta}}}{1-\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)^{\hat{\theta}}}=\mathbf{0},$$

and

$$\begin{aligned}\frac{\partial\ln l(\Omega)}{\partial\lambda}=\frac{n}{\hat{\lambda}}-\sum_{i=1}^{n}x_i+\ln(\hat{\alpha})\sum_{i=1}^{n}x_ie^{-\hat{\lambda}x_i}-\frac{\left(\hat{\theta}-1\right)\ln(\hat{\alpha})}{(\hat{\alpha}-1)}\sum_{i=1}^{n}\frac{\hat{\alpha}^{1-e^{-\hat{\lambda}x_i}}x_ie^{-\hat{\lambda}x_i}}{1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)}\\ +\left(\hat{\beta}-1\right)\frac{\hat{\theta}\ln(\hat{\alpha})}{(\hat{\alpha}-1)}\sum_{i=1}^{n}\frac{\hat{\alpha}^{1-e^{-\hat{\lambda}x_i}}x_ie^{-\hat{\lambda}x_i}\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)^{\hat{\theta}-1}}{1-\left(1-\varphi\left(x_i,\hat{\alpha},\hat{\lambda}\right)\right)^{\hat{\theta}}}=\mathbf{0},\end{aligned}$$

where

$$\varrho(x_i,\alpha,\lambda)=\frac{\frac{\varphi(x_i,\alpha,\lambda)}{\alpha-1}+\frac{e^{-\lambda x_i}-1}{(\alpha-1)\alpha^{e^{-\lambda x_i}}}}{1-\varphi(x_i,\alpha,\lambda)},$$

$$\varsigma(x_i,\alpha,\vartheta,\lambda)=\frac{\theta\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\varrho(x_i,\alpha,\lambda)}{1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}}.$$

These equations cannot be solved explicitly, so numerical methods like the Conjugate Gradients algorithms can be used to calculate the MLE of Ω.

4.2. Maximum Product Spacing

The maximum product spacing (MPS) method was proposed by Cheng and Amin (1983). The geometric mean of the differences is given as

$$G=\left(\prod_{i=1}^{n+1}D_i\right)^{\frac{1}{n+1}}, \tag{15}$$

where G is defined as the geometric mean of the product spacing function. Also, the difference Di is defined as

$$D_i=\begin{cases}D_1=F(x_1,\Omega)\\ D_i=F(x_i,\Omega)-F(x_{i-1},\Omega);\ i=2\dots n,\\ D_{n+1}=1-F(x_n,\Omega),\end{cases}$$

such that $\sum D_i=1$, then the product spacing function for EGAPEx distribution is

$$G(\Omega)=\left(\left[1-\left(1-\varphi(x_1,\alpha,\lambda)\right)^{\theta}\right]^{\beta}\left(1-\left[1-\left(1-\varphi(x_n,\alpha,\lambda)\right)^{\theta}\right]^{\beta}\right)\right)^{\frac{1}{n+1}} \tag{16}$$

$$\left(\prod_{i=2}^{n}\left(\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta}-\left[1-\left(1-\varphi(x_{i-1},\alpha,\lambda)\right)^{\theta}\right]^{\beta}\right)\right)^{\frac{1}{n+1}}.$$

The natural logarithm of the product spacing function is

$$\begin{aligned}\ln G(\Omega)=\frac{1}{n+1}\Bigg[&\beta\ln\left(1-\left(1-\varphi(x_1,\alpha,\lambda)\right)^{\theta}\right)+\ln\left(1-\left[1-\left(1-\varphi(x_n,\alpha,\lambda)\right)^{\theta}\right]^{\beta}\right)\\&+\sum_{i=2}^{n}\ln\left(\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta}-\left[1-\left(1-\varphi(x_{i-1},\alpha,\lambda)\right)^{\theta}\right]^{\beta}\right)\Bigg].\end{aligned}\tag{17}$$

To estimate of the unknown parameters, we differentiate Equation (17) partially with respect to the parameter Ω and equate them to zero. The estimators $\breve{\Omega}$ of Ω can be obtained as the solution of the following equations.

$$\begin{aligned}\frac{\partial\ln G(\Omega)}{\partial\alpha}=\frac{1}{n+1}\Bigg[&\beta\varsigma(x_1,\hat{\theta},\hat{\alpha},\hat{\lambda})\left[1-\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}+\frac{\beta\varsigma(x_n,\hat{\theta},\hat{\alpha},\hat{\lambda})\left(1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right)\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{1-\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}\\&+\beta\sum_{i=2}^{n}\frac{\varsigma(x_i,\hat{\theta},\hat{\alpha},\hat{\lambda})\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}-\varsigma(x_{i-1},\hat{\alpha},\hat{\theta},\hat{\lambda})\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}-\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}\Bigg]=\mathbf{0},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln G(\Omega)}{\partial\beta}=\frac{1}{n+1}\Bigg[&\ln\left[1-\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]-\frac{\ln\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}{1-\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}\\&+\sum_{i=2}^{n}\frac{\ln\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}-\ln\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}{\left(\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}-\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}\right)}\Bigg]=\mathbf{0},\end{aligned}$$

$$\begin{aligned}&\frac{\partial\ln G(\Omega)}{\partial\theta}\\&=\frac{1}{n+1}\Bigg[\frac{-\hat{\beta}\,ln\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}}{\left(1-\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right)}+\frac{\hat{\beta}\,\ln\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{1-\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}\\&+\beta\sum_{i=2}^{n}\frac{\ln\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}-\ln\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{\left(\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}-\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}\right)}\Bigg]=\mathbf{0}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial\ln G(\Omega)}{\partial\lambda}\\&=\frac{1}{n+1}\Bigg[\frac{\hat{\theta}\ln(\hat{\alpha})}{(\hat{\alpha}-1)}\sum_{i=1}^{n}\frac{\hat{\alpha}^{1-e^{-\hat{\lambda}x_1}}x_1e^{-\hat{\lambda}x_1}\left(1-\varphi(x_1,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}-1}}{1-\varphi(x_1,\hat{\alpha},\hat{\lambda})}-\frac{\hat{\beta}\hat{\theta}\ln(\hat{\alpha})}{(\hat{\alpha}-1)}\sum_{i=1}^{n}\frac{\hat{\alpha}^{1-e^{-\hat{\lambda}x_n}}x_ne^{-\hat{\lambda}x_n}\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}-1}\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{1-\left[1-\left(1-\varphi(x_n,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}}\\&+\frac{\hat{\beta}\hat{\theta}\ln(\hat{\alpha})}{(\hat{\alpha}-1)}\sum_{i=2}^{n}\frac{\hat{\alpha}^{1-e^{-\hat{\lambda}x_i}}x_ie^{-\hat{\lambda}x_i}\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}-1}\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}-\hat{\alpha}^{1-e^{-\hat{\lambda}x_{i-1}}}x_{i-1}e^{-\hat{\lambda}x_{i-1}}\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}-1}\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}-1}}{\left(\left[1-\left(1-\varphi(x_i,\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}-\left[1-\left(1-\varphi(x_{i-1},\hat{\alpha},\hat{\lambda})\right)^{\hat{\theta}}\right]^{\hat{\beta}}\right)}\Bigg]=\mathbf{0}.\end{aligned}$$

These equations cannot be solved explicitly, so numerical methods like the conjugate gradients algorithms can be used to calculate the MPS of Ω.

4.3. Bayesian Estimation

In this section, we present a Bayesian estimation of the unknown parameters $\alpha, \beta, \theta \; and \; \lambda$ of EGAPEx distribution. As a powerful and valid alternative to classical estimation, Bayesian analysis is a natural way to combine the observed information with the prior information which makes it very valuable in reliability, lifetime study and other associated fields where one of the major challenges is the limited availability of data. It is assumed that $\Omega = (\alpha, \beta, \theta, \lambda)$ is independently distributed as gamma distribution which is a conjugate prior to the EGAPEx distribution. Assumed that $\Omega_j \sim Gamma(a_j, b_j); j = 1,2,3,4$, then the joint prior density of Ω can be written as

$$g(\Omega) \propto \prod_{j=1}^{4} \Omega_j^{a_j-1} e^{-\frac{\Omega_j}{b_j}}; \quad a_j, b_j > 0, \tag{18}$$

here all the hyper parameters a_j, b_j are known and non-negative.
Based on the likelihood function (5) and the joint prior density (18), the joint posterior of Ω is

$$g(\Omega|x) \propto e^{-\frac{\alpha}{b_1}} e^{-\frac{\beta}{b_2}} e^{-\frac{\theta}{b_3}} \left(\frac{\theta^{a_3-1}\beta^{a_2-1}\ln(\alpha)\,\lambda^{a_4-1}}{\alpha-1}\right)^n e^{-\lambda\left(\sum_{i=1}^n x_i + \frac{1}{b_4}\right)} \prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\alpha^{a_1-e^{-\lambda x_i}}\right\}, \tag{19}$$

The Markov chain Monte Carlo (MCMC) can be used to generate posterior samples from the posterior density function (19) which in turn computes the Bayes estimates of the unknown parameters. To generate posterior samples from (19), we can use the fully conditional posterior distributions of the parameters and are given by

$$g_1(\alpha|\beta,\theta,\lambda,x) \propto e^{-\frac{\alpha}{b_1}}\left(\frac{\ln(\alpha)}{\alpha-1}\right)^n \prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\alpha^{a_1-e^{-\lambda x_i}}\right\},$$

$$g_2(\beta|\alpha,\theta,\lambda,x) \propto e^{-\frac{\beta}{b_2}}(\beta^{a_2-1})^n \prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\right\},$$

$$g_3(\theta|\alpha,\beta,\lambda,x) \propto e^{-\frac{\theta}{b_3}}(\theta^{a_3-1})^n \prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\right\},$$

and

$$g_4(\lambda|\alpha,\beta,\theta,x) \propto (\lambda^{a_4-1})^n e^{-\lambda\left(\sum_{i=1}^n x_i + \frac{1}{b_4}\right)} \prod_{i=1}^{n}\left\{\left[1-\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta}\right]^{\beta-1}\left(1-\varphi(x_i,\alpha,\lambda)\right)^{\theta-1}\alpha^{a_1-e^{-\lambda x_i}}\right\}.$$

Since the full conditional posterior distributions do not have simple forms, we adopt a Metropolis–Hastings within Gibbs sampling approach to generate random samples from the conditional densities of the parameters and use them to obtain the Bayes credible intervals and point Bayes point estimates based on the squared error loss function.

The Metropolis-Hastings algorithm generates samples to estimate the parameters of EGAPEx distribution as follows:

1. By using initial values $\Omega_l^{\langle 0\rangle}; l = 1, \ldots 4$, where satisfying $\pi(\Omega_l^{\langle 0\rangle}) > 0$.
2. After using the initial values, to get candidate point Ω^* from proposal $q(\Omega^*)$, here we use the normal

distribution as proposal distribution.

3. When $t = 0$ to *N*, given the candidate point Ω^*, we calculate acceptance probability as shown

$$\forall_l = min\left(1, \frac{L(\Omega|x)\pi(\Omega_l^*)\, q(\Omega_l)}{L_1(\Omega|x)\pi(\Omega_l) q(\Omega_l^*)}\right)$$

4. Simulate a number *u* from the uniform (0,1) distribution, $\forall_l^{\langle t+1\rangle} = \begin{cases}\Omega_l^* & if u \leq \forall_l \\ \Omega_l^{\langle t\rangle} & otherwise\end{cases}$.

5. Repeat steps 2 - 4 $(t+1)$ times until we get N iteration of the MCMC process.
6. Repeat these steps l times to get Bayes estimate of Ω_l.

By using the generated random samples from the above Gibbs sampling technique, the approximate Bayes estimate of the parameters under the squared error loss function can be obtained as $\tilde{\Omega}_l = \sum_{i=1}^{N} \frac{\Omega_l^{\langle i \rangle}}{N}$, where N is the number of iterations in the MCMC process, which equals 10,000.

5. Confidence Interval (CI)

In this section, we discuss the asymptotic confidence intervals, Bayes credible intervals, and bootstrap confidence intervals of the parameters of EGAPEx distribution.

5.1. Asymptotic Confidence Intervals

Since the MLEs of the unknown parameters of the model cannot be derived in closed form, it is not easy to derive the exact distributions of the MLEs. Using the asymptotic distribution of the MLEs, the approximate confidence intervals for unknown parameters can be obtained by inverting the Fisher matrix F, which consists of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the MLEs of the parameters.
Assume some regularity conditions to be satisfied, a $100(1-\gamma)$ % approximate confidence intervals for EGAPEx distribution parameters are given respectively, as follows

$\hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\nu_{11}},\ \hat{\beta} \pm z_{\frac{\gamma}{2}}\sqrt{\nu_{22}},\ \hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\nu_{33}}$ and $\hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\nu_{44}}$

where ν_{11},ν_{22},ν_{33}, and ν_{44} are the elements on the main diagonal of the variance - covariance matrix of EGAPEx distribution and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail $\frac{\gamma}{2}$.

5.2. Bayes Credible Intervals

Using Chen and Shao (1999) procedure, we obtain Bayes credible intervals of the parameters of EGAPEx distribution α, β, θ and λas follow:

- Arrange $\tilde{\alpha}^{\langle j \rangle}$ as $\left(\tilde{\alpha}^{\langle 1 \rangle} < \tilde{\alpha}^{\langle 2 \rangle} < \cdots < \tilde{\alpha}^{\langle N \rangle}\right)$, $\tilde{\beta}^{\langle j \rangle}$ as $\left(\tilde{\beta}^{\langle 1 \rangle} < \tilde{\beta}^{\langle 2 \rangle} < \cdots < \tilde{\beta}^{\langle N \rangle}\right)$, $\tilde{\theta}^{\langle j \rangle}$ as $\left(\tilde{\theta}^{\langle 1 \rangle} < \tilde{\theta}^{\langle 2 \rangle} < \cdots < \tilde{\theta}^{\langle N \rangle}\right)$, and $\tilde{\lambda}^{\langle j \rangle}$ as $\left(\tilde{\lambda}^{\langle 1 \rangle} < \tilde{\lambda}^{\langle 2 \rangle} < \cdots < \tilde{\lambda}^{\langle N \rangle}\right)$, where N is the number of periods in the MCMC process.
- The $100(1-\gamma)\%$ symmetric credible intervals of Ω become$\left(\tilde{\alpha}^{\langle N\frac{\gamma}{2} \rangle}, \tilde{\alpha}^{\langle N(1-\frac{\gamma}{2}) \rangle}\right)$, $\left(\tilde{\beta}^{\langle N\frac{\gamma}{2} \rangle}, \tilde{\beta}^{\langle N(1-\frac{\gamma}{2}) \rangle}\right)$, $\left(\tilde{\theta}^{\langle N\frac{\gamma}{2} \rangle}, \tilde{\theta}^{\langle N(1-\frac{\gamma}{2}) \rangle}\right)$, and $\left(\tilde{\lambda}^{\langle N\frac{\gamma}{2} \rangle}, \tilde{\lambda}^{\langle N(1-\frac{\gamma}{2}) \rangle}\right)$.

5.3. Bootstrap Method

Here, we construct percentile bootstrap and bootstrap-t- bootstrap confidence interval of the parameters of EGAPEx distribution as follows where
the estimators of MLE $\Psi^1 = \hat{\Omega} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}\right)$,
the estimators of MPS$\Psi^2 = \breve{\Omega} = \left(\breve{\alpha}, \breve{\beta}, \breve{\theta}, \breve{\lambda}\right)$ and
the estimators of Bayesian$\Psi^3 = \tilde{\Omega} = \left(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{\lambda}\right)$.

1) Percentile Bootstrap Confidence Interval (PB)

i. Compute the estimators for parameters of EGAPEx distribution$\Psi^l = (\alpha^l, \beta^l, \theta^l, \lambda^l)$; l= 1,2,3.

ii. Generate a bootstrap sample using $\Psi^l$to obtain the bootstrap estimate of α, say α^{lb}, β, say β^{lb}, θ, say θ^{lb}, and λ, say λ^{lb} using the bootstrap sample.

iii. Repeat step (ii) B times to have $\left(\alpha^{lb(1)}, \alpha^{lb(2)}, \dots, \alpha^{lb(B)}\right)$, $\left(\beta^{lb(1)}, \beta^{lb(2)}, \dots, \beta^{lb(B)}\right)$, $\left(\theta^{lb(1)}, \theta^{lb(2)}, \dots, \theta^{lb(B)}\right)$ and $\left(\lambda^{lb(1)}, \lambda^{lb(2)}, \dots, \lambda^{lb(B)}\right)$.

iv. Arrange $\left(\alpha^{lb(1)}, \alpha^{lb(2)}, \dots, \alpha^{lb(B)}\right)$,$\left(\beta^{lb(1)}, \beta^{lb(2)}, \dots, \beta^{lb(B)}\right)$, $\left(\theta^{lb(1)}, \theta^{lb(2)}, \dots, \theta^{lb(B)}\right)$ and $\left(\lambda^{lb(1)}, \lambda^{lb(2)}, \dots, \lambda^{lb(B)}\right)$ in ascending order as $\left(\alpha^{lb[1]}, \alpha^{lb[2]}, \dots, \alpha^{lb[B]}\right)$, $\left(\beta^{lb[1]}, \beta^{lb[2]}, \dots, \beta^{lb[B]}\right)$, $\left(\theta^{lb[1]}, \theta^{lb[2]}, \dots, \theta^{lb[B]}\right)$ and $\left(\lambda^{lb[1]}, \lambda^{lb[2]}, \dots, \lambda^{lb[B]}\right)$.

v. A two sided $100\,(1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters $\alpha^l, \beta^l, \theta^l$ and λ^l are given by $\left\{\alpha^{lb[B\gamma/2]}, \alpha^{lb[B(1-\gamma/2)]}\right\}$, $\left\{\beta^{lb[B\gamma/2]}, \beta^{lb[B(1-\gamma/2)]}\right\}$, $\left\{\theta^{lb[B\gamma/2]}, \theta^{lb[B(1-\gamma/2)]}\right\}$ and $\left\{\lambda^{lb[B\gamma/2]}, \lambda^{lb[B(1-\gamma/2)]}\right\}$.

2) Bootstrap-t Confidence Interval (BT)

i. Same as the steps (i-ii) in Boot-p

ii. Compute the t-statistic of Ψ^l as $T = (\Psi^{lb} - \Psi^l)/\sqrt{V(\Psi^{lb})}$, where $V(\Psi^{lb})$ is a variance of Ψ^{lb}.

iii. Repeat step (ii) B times and obtain $T^{(1)}, T^{(2)}, \dots, T^{(B)}$.

iv. Arrange $T^{(1)}, T^{(2)}, \dots, T^{(B)}$ in ascending order as $\mathrm{T}[1], \mathrm{T}[2], \dots, \mathrm{T}[B]$.

v. A two sided $100\,(1-\gamma)\%$ bootstrap-t confidence interval for the unknown parameters $\alpha, \beta, \theta,$ andλ are given by

$\left\{\alpha^l + T_1[B\,\gamma/2]\sqrt{V(\alpha^l)},\ \alpha^l + T_1[B(1-\gamma/2)]\sqrt{V(\alpha^l)}\right\}$,$\left\{\beta^l + T_2[B\,\gamma/2]\sqrt{V(\beta^l)},\ \beta^l + T_2[B(1-\gamma/2)]\sqrt{V(\beta^l)}\right\}$,$\left\{\theta^l + T_3[B\,\gamma/2]\sqrt{V(\theta^l)},\ \theta^l + T_3[B(1-\gamma/2)]\sqrt{V(\theta^l)}\right\}$ and $\left\{\lambda^l + T_4[B\,\gamma/2]\sqrt{V(\lambda^l)},\ \lambda^l + T_4[B(1-\gamma/2)]\sqrt{V(\lambda^l)}\right\}$.

6. Simulation Study

In this section, we conduct a Monte Carlo simulation analysis to decide the best estimation method that can be used to estimate the EGAPEx distribution parameters because it is not possible to compare the output of the different estimators derived in the previous sections theoretically. Using the EGAPEx distribution and different actual parameters, we produce 10000 random samples of sizes 50, 100, and 200. The bias of estimates, as well as bias, the mean square error (MSE), Relative efficacy (RE), and the length of the confidence interval, are calculated for each combination of different parameter combinations as follows:
For MLE: Bias $=\hat{\Omega} - \Omega$, $MSE = Mean(\hat{\Omega} - \Omega)^2$, and L.CI=upper CI($\hat{\Omega}$) - lower CI($\hat{\Omega}$).
For MPS: Bias $=\breve{\Omega} - \Omega$, $MSE = Mean(\breve{\Omega} - \Omega)^2$, and L.CI=upper CI($\breve{\Omega}$) - lower CI($\breve{\Omega}$).
For Bayesian: Bias $=\tilde{\Omega} - \Omega$, $MSE = Mean(\tilde{\Omega} - \Omega)^2$, and L.CCI=upper credible CI – lower credible CI.

$$RE_1 = \frac{MSE(\hat{\Omega})_{MLE}}{MSE(\breve{\Omega})_{MPS}}, RE_2 = \frac{MSE(\hat{\Omega})_{MLE}}{MSE(\tilde{\Omega})_{Bayesian}}$$

Solving four nonlinear equations yields all of the estimators. This is a well-known problem in the literature, and estimates can be obtained using numerical techniques. We first used the maximum likelihood approach to estimate the parameters, maximizing the log-likelihood function using the conjugate-gradient method. The average value of maximum likelihood estimates was used as the initial value for MPS and Bayesian estimation methods. In Tables 2-6, these values are shown for the various estimation methods.

R-program version 3.5.3 was used to conduct the simulation analysis. As we can see from these tables, the bias, MSE, and length of CI decrease as the sample size grow, implying that the various methods have consistent estimators. In most cases, Bayesian estimates outperform other estimates in terms of minimum MSE and CI length, followed by MPS and MLE.

Table 2: Bias, MSE, and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when $n = 50$

Actuals values					MLE		MPS		Bayesian			
α	β	θ	λ		Bias1	MSE	Bias2	MSE	Bias	MSE	RE_1	RE_2
0.75	1.25	1.2	1.5	$\hat{\alpha}$	0.6705	2.9029	0.6335	2.8836	0.1052	0.2283	101%	1272%
				$\hat{\beta}$	0.1228	0.2131	-0.0413	0.1400	0.0400	0.0680	152%	313%
				$\hat{\theta}$	0.1016	0.3557	-0.0617	0.2146	0.1040	0.2063	166%	172%
				$\hat{\lambda}$	0.1149	0.3150	-0.0316	0.3629	0.0520	0.2953	87%	107%
1.5	1.25	1.2	1.5	$\hat{\alpha}$	0.6083	3.8214	0.6853	4.5316	-0.1484	0.4569	84%	836%
				$\hat{\beta}$	0.2101	0.2927	0.0242	0.1820	0.0844	0.0840	161%	348%
				$\hat{\theta}$	0.2607	0.5033	-0.0289	0.2459	0.1131	0.1903	205%	264%
				$\hat{\lambda}$	-0.0460	0.2556	0.0137	0.3304	0.0013	0.2371	77%	108%
1.5	1.25	0.5	1.5	$\hat{\alpha}$	0.0503	1.0917	0.3323	1.1970	-0.1529	0.3373	91%	324%
				$\hat{\beta}$	0.2225	0.2461	-0.0237	0.1170	0.1052	0.0933	210%	264%
				$\hat{\theta}$	0.2012	0.2173	0.0720	0.2361	0.2031	0.1228	92%	177%
				$\hat{\lambda}$	-0.1813	0.2085	-0.1250	0.2015	-0.1548	0.1961	103%	106%
1.5	1.5	2.5	1.5	$\hat{\alpha}$	0.8470	5.0995	0.7003	5.1997	-0.2628	0.5336	98%	956%
				$\hat{\beta}$	0.1379	0.2565	-0.0407	0.1815	0.1654	0.1876	141%	137%
				$\hat{\theta}$	-0.0055	0.3588	-0.3387	0.3880	-0.0300	0.2632	92%	136%
				$\hat{\lambda}$	0.0783	0.2792	0.0622	0.5998	0.0526	0.2447	47%	114%
2.5	1.5	2.5	0.75	$\hat{\alpha}$	0.1873	2.8673	0.1096	2.5720	-0.1594	1.1990	111%	239%
				$\hat{\beta}$	0.1670	0.2308	-0.0232	0.1555	0.1854	0.1859	148%	124%
				$\hat{\theta}$	0.0440	0.1777	-0.2283	0.2545	-0.0554	0.1833	70%	97%
				$\hat{\lambda}$	0.0061	0.0584	0.0235	0.1091	0.0323	0.0293	54%	199%
1.5	1.25	1.2	0.75	$\hat{\alpha}$	0.1755	1.5901	0.2140	1.6898	-0.1641	0.3931	94%	404%
				$\hat{\beta}$	0.2121	0.2663	0.0254	0.1550	0.1216	0.0987	172%	270%
				$\hat{\theta}$	0.0872	0.1747	-0.0871	0.1153	-0.0577	0.1628	152%	107%
				$\hat{\lambda}$	0.0126	0.0724	0.0132	0.1032	0.0217	0.0614	70%	118%
1.5	1.25	1.2	0.4	$\hat{\alpha}$	0.0947	1.0227	0.1269	1.0537	-0.2042	0.3211	97%	319%
				$\hat{\beta}$	0.1448	0.1496	-0.0216	0.0933	0.1299	0.1135	160%	132%
				$\hat{\theta}$	0.0596	0.0859	-0.0723	0.0750	-0.0767	0.0722	115%	119%
				$\hat{\lambda}$	0.0041	0.0172	0.0092	0.0310	0.0052	0.0126	55%	136%
0.5	1.25	1.2	0.4	$\hat{\alpha}$	0.3224	0.8970	0.2705	0.9285	0.1737	0.1739	97%	516%
				$\hat{\beta}$	0.1642	0.1906	-0.0299	0.1041	0.0564	0.0933	183%	204%
				$\hat{\theta}$	-0.1514	0.1543	-0.0783	0.0972	-0.1475	0.1495	159%	103%
				$\hat{\lambda}$	0.1639	0.1158	0.0063	0.0577	0.1252	0.1065	201%	109%
1.5	2.5	1.2	1.5	$\hat{\alpha}$	0.3319	2.3942	0.5047	2.9124	-0.0384	0.4910	82%	488%
				$\hat{\beta}$	0.4766	1.1620	0.0365	0.7660	0.1073	0.3592	152%	324%
				$\hat{\theta}$	0.3350	0.5057	0.1249	0.2666	0.1882	0.2793	190%	181%
				$\hat{\lambda}$	-0.1611	0.1858	-0.1304	0.2437	0.0247	0.1312	76%	142%

Table 3: Bias, MSE and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when $n = 100$

Actuals values					MLE		MPS		Bayesian			
α	β	θ	λ		Bias	MSE	Bias	MSE	Bias	MSE	RE_1	RE_2
0.75	1.25	1.2	1.5	$\hat{\alpha}$	0.3513	1.2206	0.2951	1.1492	0.0100	0.1462	106%	835%
				$\hat{\beta}$	0.0703	0.1019	-0.0189	0.0790	0.0376	0.0413	129%	247%
				$\hat{\theta}$	0.0236	0.1343	-0.0224	0.1016	0.0752	0.1141	132%	118%
				$\hat{\lambda}$	0.0779	0.1944	-0.0720	0.2386	-0.0238	0.1863	81%	104%
1.5	1.25	1.2	1.5	$\hat{\alpha}$	0.2230	1.4221	0.2874	1.6845	-0.1510	0.2531	84%	562%
				$\hat{\beta}$	0.1067	0.1290	0.0003	0.0921	0.0518	0.0463	140%	279%
				$\hat{\theta}$	0.1007	0.1570	-0.0471	0.1009	0.0895	0.1266	156%	124%
				$\hat{\lambda}$	-0.0359	0.1206	0.0150	0.1679	-0.0257	0.1138	72%	106%
1.5	1.25	0.5	1.5	$\hat{\alpha}$	0.0524	0.7136	0.2766	0.8378	-0.1471	0.2626	85%	272%
				$\hat{\beta}$	0.1043	0.1019	-0.0322	0.0693	0.0657	0.0453	147%	225%
				$\hat{\theta}$	0.1315	0.1133	0.0630	0.0752	0.1746	0.0951	151%	119%
				$\hat{\lambda}$	-0.1130	0.1523	-0.0505	0.1488	-0.1068	0.1225	102%	124%
1.5	1.5	2.5	1.5	$\hat{\alpha}$	0.5087	2.8735	0.3627	2.6425	-0.1891	0.3818	109%	753%
				$\hat{\beta}$	0.0619	0.1263	-0.0331	0.1066	0.0770	0.0667	118%	189%
				$\hat{\theta}$	-0.0088	0.1990	-0.2272	0.2079	-0.0115	0.1365	96%	146%
				$\hat{\lambda}$	0.0272	0.1497	0.0176	0.3444	0.0174	0.1091	43%	137%
2.5	1.5	2.5	0.75	$\hat{\alpha}$	0.0912	1.4561	0.1118	1.6823	-0.2710	0.5444	87%	267%
				$\hat{\beta}$	0.0776	0.1028	-0.0351	0.0819	0.0701	0.0532	126%	193%
				$\hat{\theta}$	0.0226	0.1294	-0.2069	0.2511	-0.0349	0.1146	52%	113%
				$\hat{\lambda}$	0.0044	0.0332	0.0419	0.0671	0.0108	0.0108	49%	308%
1.5	1.25	1.2	0.75	$\hat{\alpha}$	0.1541	1.1054	0.2314	1.3205	-0.1032	0.2479	84%	446%
				$\hat{\beta}$	0.0987	0.1129	-0.0088	0.0813	0.0573	0.0408	139%	277%
				$\hat{\theta}$	0.0734	0.1386	-0.0747	0.1154	-0.0528	0.1275	120%	109%
				$\hat{\lambda}$	0.0047	0.0576	0.0485	0.0998	0.0133	0.0502	58%	115%
1.5	1.25	1.2	0.4	$\hat{\alpha}$	0.0495	0.5572	0.0591	0.5649	-0.1580	0.1996	99%	279%
				$\hat{\beta}$	0.0681	0.0694	-0.0232	0.0550	0.0709	0.0407	126%	170%
				$\hat{\theta}$	0.0357	0.0508	-0.0718	0.0458	-0.0134	0.0452	111%	112%
				$\hat{\lambda}$	-0.0013	0.0084	0.0136	0.0185	0.0014	0.0058	45%	144%
0.5	1.25	1.2	0.4	$\hat{\alpha}$	0.2042	0.4829	0.1458	0.4827	0.1172	0.1234	100%	391%
				$\hat{\beta}$	0.0832	0.1025	-0.0264	0.0630	0.0301	0.0478	163%	214%
				$\hat{\theta}$	-0.1307	0.1124	-0.0387	0.0697	-0.1106	0.1037	161%	108%
				$\hat{\lambda}$	0.1153	0.0706	-0.0044	0.0429	0.0914	0.0677	164%	104%
1.5	2.5	1.2	1.5	$\hat{\alpha}$	0.1920	1.2139	0.3375	1.4913	-0.0836	0.2863	81%	424%
				$\hat{\beta}$	0.2286	0.5152	-0.0451	0.3865	0.0678	0.1626	133%	317%
				$\hat{\theta}$	0.1794	0.2082	0.0419	0.1273	0.1027	0.1613	164%	129%
				$\hat{\lambda}$	-0.1066	0.0975	-0.0459	0.1139	0.0099	0.0820	86%	119%

Table 4: Bias, MSE and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when $n = 200$

Actuals values					MLE		MPS		Bayesian			
α	β	θ	λ		Bias1	MSE	Bias2	MSE	Bias	MSE	RE_1	RE_2
0.75	1.25	1.2	1.5	$\hat{\alpha}$	0.2011	0.6064	0.1798	0.6032	-0.0028	0.0751	101%	807%
				$\hat{\beta}$	0.0367	0.0493	-0.0129	0.0431	0.0191	0.0202	114%	244%
				$\hat{\theta}$	0.0262	0.0861	0.0249	0.0674	0.0083	0.0416	128%	207%
				$\hat{\lambda}$	0.0328	0.1130	-0.0644	0.1670	0.0064	0.0697	68%	162%
1.5	1.25	1.2	1.5	$\hat{\alpha}$	0.2388	1.1194	0.2979	1.2094	-0.0336	0.1006	93%	1113%
				$\hat{\beta}$	0.0489	0.0509	-0.0133	0.0433	0.0100	0.0162	117%	314%
				$\hat{\theta}$	0.0905	0.1303	-0.0340	0.0919	0.0082	0.0438	142%	297%
				$\hat{\lambda}$	-0.0239	0.1032	0.0624	0.1529	0.0185	0.0528	67%	195%
1.5	1.25	0.5	1.5	$\hat{\alpha}$	0.0003	0.3797	0.1529	0.3450	-0.0905	0.1640	110%	232%
				$\hat{\beta}$	0.0637	0.0484	-0.0237	0.0327	0.0458	0.0239	148%	203%
				$\hat{\theta}$	0.0751	0.0476	0.0315	0.0303	0.0944	0.0429	157%	111%
				$\hat{\lambda}$	-0.0878	0.0889	-0.0371	0.0814	-0.0816	0.0712	109%	125%
1.5	1.5	2.5	1.5	$\hat{\alpha}$	0.3104	1.3989	0.2360	1.5026	-0.0178	0.0338	93%	4139%
				$\hat{\beta}$	0.0298	0.0544	-0.0205	0.0534	0.0003	0.0128	102%	425%
				$\hat{\theta}$	-0.0130	0.0748	-0.1554	0.1087	-0.0138	0.0260	69%	288%
				$\hat{\lambda}$	0.0157	0.0793	0.0202	0.1778	0.0013	0.0167	45%	474%
2.5	1.5	2.5	0.75	$\hat{\alpha}$	0.1835	1.4148	0.1649	1.5365	-0.0163	0.0302	92%	4679%
				$\hat{\beta}$	0.0387	0.0466	-0.0247	0.0458	0.0042	0.0143	102%	327%
				$\hat{\theta}$	0.0365	0.1399	-0.2154	0.2205	-0.0195	0.0253	63%	553%
				$\hat{\lambda}$	0.0027	0.0214	0.0624	0.0582	0.0081	0.0046	37%	460%
1.5	1.25	1.2	0.75	$\hat{\alpha}$	0.1412	0.7696	0.1979	0.8841	-0.0517	0.0927	87%	830%
				$\hat{\beta}$	0.0503	0.0471	-0.0108	0.0403	0.0192	0.0179	117%	263%
				$\hat{\theta}$	0.0397	0.0823	-0.0773	0.0795	-0.0300	0.0626	104%	131%
				$\hat{\lambda}$	0.0109	0.0393	0.0648	0.0687	0.0627	0.0346	57%	113%
1.5	1.25	1.2	0.4	$\hat{\alpha}$	0.0514	0.3562	0.0774	0.3544	-0.0478	0.0934	100%	381%
				$\hat{\beta}$	0.0354	0.0276	-0.0203	0.0231	0.0244	0.0152	119%	181%
				$\hat{\theta}$	0.0362	0.0394	-0.0443	0.0356	-0.0282	0.0180	111%	219%
				$\hat{\lambda}$	-0.0022	0.0051	0.0119	0.0080	0.0167	0.0021	64%	247%
0.5	1.25	1.2	0.4	$\hat{\alpha}$	0.1168	0.2352	0.0725	0.2382	0.0148	0.0459	99%	513%
				$\hat{\beta}$	0.0432	0.0445	-0.0150	0.0314	0.0252	0.0221	141%	202%
				$\hat{\theta}$	-0.0783	0.0684	0.0075	0.0501	-0.0794	0.0681	137%	100%
				$\hat{\lambda}$	0.0647	0.0374	-0.0171	0.0259	0.0663	0.0248	144%	151%
1.5	2.5	1.2	1.5	$\hat{\alpha}$	0.0632	0.3830	0.1259	0.4513	-0.0067	0.0276	85%	1386%
				$\hat{\beta}$	0.0974	0.1683	-0.0442	0.1558	-0.0080	0.0238	108%	707%
				$\hat{\theta}$	0.0691	0.0401	0.0088	0.0299	0.0026	0.0134	134%	299%
				$\hat{\lambda}$	-0.0578	0.0241	-0.0311	0.0270	0.0050	0.0183	89%	132%

Table 5: Asymptotic and bootstrap Confidence Intervals for MLE, MPS, and Bayesian of EGAPEx parameters with different Actuals Values when $n = 50$

Actuals values					MLE			MPS			Bayesian		
α	β	θ	λ		L.CI	LBP	LBT	L.CI	LBP	LBT	L.CCI	LBP	LBT
0.75	1.25	1.2	1.5	$\hat{\alpha}$	7.0741	0.8237	0.8077	6.9017	0.7957	0.7825	1.3044	0.3969	0.3782
				$\hat{\beta}$	1.7482	0.2047	0.1975	1.6097	0.2001	0.1997	0.4330	0.1520	0.1435
				$\hat{\theta}$	2.5375	0.3721	0.3607	2.4399	0.2240	0.2236	1.0937	0.3984	0.3969
				$\hat{\lambda}$	2.2802	0.3197	0.3079	2.1093	0.3130	0.3021	1.6174	0.2891	0.2847
1.5	1.25	1.2	1.5	$\hat{\alpha}$	7.2979	0.9854	1.0372	7.0985	0.9236	0.9026	1.7521	3.4223	2.7759
				$\hat{\beta}$	2.2814	0.2959	0.2958	2.0235	0.2527	0.2490	1.2641	1.8334	1.6004
				$\hat{\theta}$	2.1419	0.2843	0.2793	2.0221	0.2697	0.2517	1.2170	0.3510	0.3512
				$\hat{\lambda}$	1.8509	0.2657	0.2662	1.7039	0.2566	0.2480	1.1746	0.1736	0.1774
1.5	1.25	0.5	1.5	$\hat{\alpha}$	3.7469	0.5100	0.4875	3.6396	0.4576	0.4558	1.4222	0.2143	0.2182
				$\hat{\beta}$	1.4507	0.2081	0.2070	1.3451	0.1701	0.1743	0.6002	1.2652	1.0062
				$\hat{\theta}$	1.2188	0.2044	0.2030	1.2188	0.1018	0.1032	0.8868	0.1825	0.1885
				$\hat{\lambda}$	0.4984	0.0808	0.0826	0.4863	0.0806	0.0802	0.6946	0.3637	0.3467
1.5	1.5	2.5	1.5	$\hat{\alpha}$	7.2979	0.9854	1.0372	7.3597	1.0359	1.0262	1.7521	3.4223	2.7759
				$\hat{\beta}$	2.2814	0.2959	0.2958	2.2694	0.2527	0.2590	1.2641	1.8334	1.6004
				$\hat{\theta}$	2.1419	0.2843	0.2793	2.0986	0.2697	0.2517	1.2170	0.3510	0.3512
				$\hat{\lambda}$	1.8509	0.2657	0.2662	1.9855	0.3656	0.3680	1.1746	0.1736	0.1774
2.5	1.5	2.5	0.75	$\hat{\alpha}$	5.7927	0.9186	0.9151	5.6524	0.7230	0.7237	2.4159	0.5116	0.5164
				$\hat{\beta}$	1.8068	0.2816	0.2843	1.7653	0.1800	0.1813	0.6045	0.1069	0.1093
				$\hat{\theta}$	1.5421	0.2143	0.2206	1.6136	0.2290	0.2376	2.0197	0.3330	0.3214
				$\hat{\lambda}$	0.9033	0.1638	0.1626	0.8759	0.1460	0.1474	0.6732	0.1984	0.2000
1.5	1.25	1.2	0.75	$\hat{\alpha}$	4.7354	0.6417	0.6361	4.6085	0.6951	0.6850	1.4971	0.2911	0.2815
				$\hat{\beta}$	1.6913	0.2259	0.2300	1.4954	0.2135	0.2183	0.3405	4.2267	3.7733
				$\hat{\theta}$	1.5168	0.2157	0.2202	1.3854	0.1666	0.1639	1.1046	0.2664	0.2679
				$\hat{\lambda}$	1.1334	0.1094	0.1097	1.3896	0.1320	0.1295	1.0304	0.1548	0.1563
1.5	1.25	1.2	0.4	$\hat{\alpha}$	3.7469	0.5100	0.4875	3.9517	0.5764	0.5583	1.4222	0.2143	0.2182
				$\hat{\beta}$	1.4507	0.2081	0.2070	1.2988	0.1701	0.1743	0.6002	1.2652	1.0062
				$\hat{\theta}$	1.2188	0.2044	0.2030	1.0192	0.1018	0.1032	0.8868	0.1825	0.1885
				$\hat{\lambda}$	0.4984	0.0808	0.0826	0.4095	0.0860	0.0824	0.6946	0.3637	0.3467
0.5	1.25	1.2	0.4	$\hat{\alpha}$	3.2667	0.5483	0.5377	3.4585	0.6014	0.5898	0.7100	0.0812	0.0815
				$\hat{\beta}$	1.5788	0.2337	0.2313	1.2065	0.2029	0.2021	0.3644	0.1394	0.1396
				$\hat{\theta}$	1.4662	0.2102	0.2072	1.2114	0.1478	0.1476	0.7932	0.1356	0.1340
				$\hat{\lambda}$	1.2474	0.1845	0.1783	0.9579	0.1006	0.1005	0.6913	0.3306	0.3258
1.5	2.5	1.2	1.5	$\hat{\alpha}$	5.2380	0.7025	0.7126	6.7710	0.8382	0.8434	1.6644	0.2194	0.2118
				$\hat{\beta}$	3.8567	0.5442	0.5390	3.4287	0.4258	0.4214	1.0752	0.2319	0.2455
				$\hat{\theta}$	2.1754	0.3597	0.3572	2.0534	0.3252	0.3216	0.9780	0.2127	0.2055
				$\hat{\lambda}$	1.5140	0.2202	0.2252	1.9995	0.2481	0.2470	1.3391	0.2319	0.2308

Table 6: Asymptotic and bootstrap Confidence Intervals for MLE, MPS, and Bayesian of EGAPEx parameters with different Actuals Values when $n = 200$

Actuals values					MLE			MPS			Bayesian		
α	β	α	β		L.CI	LBP	LBT	L.CI	LBP	LBT	L.CCI	LBP	LBT
0.75	1.25	1.2	1.5	$\hat{\alpha}$	3.7610	0.4975	0.4903	3.5239	0.4194	0.4069	1.5484	0.2354	0.2477
				$\hat{\beta}$	1.8985	0.2721	0.2796	1.7032	0.2090	0.2113	0.9078	0.1915	0.1882
				$\hat{\theta}$	1.7976	0.2286	0.2344	1.6804	0.1858	0.1902	0.9978	1.4857	1.1814
				$\hat{\lambda}$	1.6229	0.2425	0.2430	1.5986	0.2346	0.2359	1.2127	0.2510	0.2471
1.5	1.25	1.2	1.5	$\hat{\alpha}$	4.7354	0.6417	0.6361	4.5735	0.6395	0.6385	1.4971	0.2911	0.2815
				$\hat{\beta}$	1.6913	0.2259	0.2300	1.5639	0.2135	0.2183	0.3405	4.2267	3.7733
				$\hat{\theta}$	1.5168	0.2157	0.2202	1.4895	0.1666	0.1639	1.1046	0.2664	0.2679
				$\hat{\lambda}$	1.1334	0.1094	0.1097	1.0270	0.0913	0.0930	1.0304	0.1548	0.1563
1.5	1.25	0.5	1.5	$\hat{\alpha}$	5.2380	0.7025	0.7126	5.2965	0.6838	0.6843	1.6644	0.2194	0.2118
				$\hat{\beta}$	3.8567	0.5442	0.5390	3.6599	0.4258	0.4214	1.0752	0.2319	0.2455
				$\hat{\theta}$	2.1754	0.3597	0.3572	2.0318	0.3252	0.3216	0.9780	0.2127	0.2055
				$\hat{\lambda}$	1.5140	0.2202	0.2252	1.4951	0.2148	0.2147	1.3391	0.1923	0.1920
1.5	1.5	2.5	1.5	$\hat{\alpha}$	4.5075	0.3252	0.3224	4.5953	0.3271	0.3220	0.4045	0.0365	0.0370
				$\hat{\beta}$	0.9934	0.0709	0.0687	0.9899	0.0721	0.0714	0.0925	0.0534	0.0529
				$\hat{\theta}$	1.0863	0.0785	0.0799	1.1720	0.0659	0.0670	0.3491	0.0296	0.0292
				$\hat{\lambda}$	1.0698	0.0749	0.0753	1.0716	0.1011	0.0998	0.2017	0.0211	0.0203
2.5	1.5	2.5	0.75	$\hat{\alpha}$	4.8702	0.3439	0.3541	4.7330	0.2873	0.2905	0.4071	0.0439	0.0437
				$\hat{\beta}$	0.9081	0.0676	0.0689	0.8912	0.0614	0.0622	0.0902	0.0239	0.0235
				$\hat{\theta}$	1.5203	0.1020	0.0989	1.6099	0.1125	0.1119	0.3375	0.0382	0.0382
				$\hat{\lambda}$	0.6356	0.0404	0.0423	0.6599	0.0617	0.0623	0.0456	0.0125	0.0119
1.5	1.25	1.2	0.75	$\hat{\alpha}$	3.2678	0.2357	0.2307	3.4098	0.2460	0.2435	0.4159	0.2984	0.2679
				$\hat{\beta}$	0.7801	0.0566	0.0572	0.7295	0.0546	0.0544	0.0976	0.4522	0.4418
				$\hat{\theta}$	1.1645	0.0797	0.0806	1.0987	0.0763	0.0774	0.2779	0.0227	0.0222
				$\hat{\lambda}$	0.8608	0.0588	0.0598	0.8608	0.0783	0.0821	0.1580	0.0128	0.0131
1.5	1.25	1.2	0.4	$\hat{\alpha}$	2.2994	0.1645	0.1698	2.3970	0.1740	0.1790	0.4264	0.1224	0.1126
				$\hat{\beta}$	0.6695	0.0490	0.0490	0.6209	0.0444	0.0447	0.0674	0.3282	0.2574
				$\hat{\theta}$	0.7986	0.0521	0.0516	0.7871	0.0518	0.0527	0.2900	0.0398	0.0395
				$\hat{\lambda}$	0.2801	0.0208	0.0212	0.2924	0.0283	0.0291	0.1161	0.0141	0.0140
0.5	1.25	1.2	0.4	$\hat{\alpha}$	1.8910	0.1211	0.1180	1.9648	0.1390	0.1385	0.1890	0.0175	0.0172
				$\hat{\beta}$	0.8395	0.0583	0.0576	0.7346	0.0527	0.0529	0.0653	0.0781	0.0755
				$\hat{\theta}$	0.9427	0.0620	0.0632	0.8080	0.0565	0.0558	0.2458	0.0267	0.0274
				$\hat{\lambda}$	0.6061	0.0439	0.0437	0.5618	0.0416	0.0416	0.1134	0.0137	0.0138
1.5	2.5	1.2	1.5	$\hat{\alpha}$	2.9053	0.2036	0.2088	3.1092	0.2116	0.2179	0.4102	0.0403	0.0399
				$\hat{\beta}$	1.7501	0.1186	0.1203	1.6794	0.1117	0.1128	0.2994	0.0293	0.0294
				$\hat{\theta}$	0.8366	0.0636	0.0633	0.7120	0.0507	0.0495	0.2013	0.0188	0.0191
				$\hat{\lambda}$	0.6388	0.0467	0.0467	0.7207	0.0488	0.0484	0.3162	0.0299	0.0294

7. Two Applications of Real Data

This section is presented to illustrate the potentiality of the EGAPEx distribution for two real data sets and compared with other competitive models, namely: the exponentiated generalized exponential (EGEx), Alpha Power Generalized Exponential (APGEx), Alpha Power Exponential (APEx), Generalized Exponential (GEx)and exponential (Ex) distributions. To compare the fits of all models, we consider the Kolmogorov Smirnov (K-S) statistic and its p-value, and some different criteria measures as Akaike information criterion (AIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC), Cram´er-von Mises (CVM) and Anderson-Darling (AD).

The first data: Almetwally et al. (2019) introduced the real data set consists of economic data set consists of 31 observations subjected to a GDP growth of Egypt. The data are given below: Data1: 10.01, 3.76, 9.91, 7.40, 6.09, 6.60, 2.65,2.52, 7.93, 4.97, 5.70, 1.08, 4.43, 2.90, 3.97, 4.64, 4.99, 5.49, 4.03, 6.11, 5.37, 3.54, 2.37, 3.19, 4.09, 4.48, 6.85, 7.09, 7.16, 4.67, 5.15.

The second data: Badar and Priest (1982) discussed the real data set of sample size 69 observed failure times, the data set is represented the data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows. Data 2: 0.562, 0.564, 0.729, 1.216, 1.474, 1.632, 1.816, 2.020, 2.317, 1.247, 1.490, 1.676, 1.824, 2.023, 2.334, 1.256, 1.503, 1.684, 1.836, 2.050, 2.340, 0.802, 1.271, 1.520, 1.685, 1.879, 2.059, 2.346, 0.950, 1.277, 1.522, 1.728, 1.883, 2.068, 2.378, 1.053, 1.305, 1.524, 1.740, 1.892, 2.071, 2.483, 1.111, 1.348, 1.551, 1.764, 1.934, 2.130, 2.835, 1.115, 1.313, 1.551, 1.761, 1.898, 2.098, 2.683, 1.194, 1.390,1.609,1.785,1.947,2.204,2.835,1.208,1.429,1.632,1.804,1.976,2.262.

Table 7: MLE, K-S and p-values for data 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	P-Value
EGAPEx	38.8065	2.6267	7.1558	0.1695	0.0486	0.99999
APEx	142.8432	-	-	0.4311	0.1460	0.4794
APGEx	170.8721	-	3.0280	0.2548	0.1288	0.6362
Ex	-	-	-	0.1948	0.3375	0.0012
GEx	-	-	8.3654	0.0233	0.3377	0.0012
EGEx	-	7.4348	14.2389	0.0358	0.0906	0.9411

Table 8: AIC, BIC, CAIC, HQIC, CVM and AD for data 1

Model	AIC	BIC	CAIC	HQIC	CVM	AD
EGAPEx	138.9972	143.5331	140.1356	140.5670	0.0143	0.1320
APEx	140.8450	143.7129	141.2735	141.7799	0.0153	0.1357
APGEx	140.5094	144.8114	141.3983	141.9118	0.0150	0.1349
Ex	165.4194	166.8534	165.5574	165.8869	0.0216	0.1838
GEx	167.4194	170.2874	167.8480	168.3543	0.0216	0.1838
EGEx	139.3125	143.6145	140.2014	140.7149	0.0362	0.2770

Table 9: MLE, K-S and p-values for data 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	K-S	P-Value
EGAPEx	134.8556	4.8093	56.8063	0.2587	0.0516	0.9929
APEx	193.9359	-	-	1.3186	0.2165	0.0031
APGEx	247.4593	-	3.6256	0.7610	0.2147	0.0034
Ex	-	-	-	0.5878	0.3926	0.0000
GEx	-	-	124.2706	0.0047	0.3926	0.0000
EGEx	-	16.6852	4.3895	0.4498	0.1030	0.4570

Table 10: AIC, BIC, CAIC, HQIC, CVM and AD for data 2

Model	AIC	BIC	CAIC	HQIC	CVM	AD
EGAPEx	107.8578	116.7942	108.4828	111.4032	0.0419	0.3165
APEx	140.6218	145.0900	140.8036	142.3945	0.0710	0.5052
APGEx	130.8831	137.5854	131.2523	133.5421	0.0510	0.3399
Ex	213.3396	215.5737	213.3993	214.2259	0.0849	0.6067
GEx	215.3396	219.8078	215.5214	217.1123	0.0849	0.6067
EGEx	117.0317	123.7340	117.4009	119.6907	0.1750	1.1913

The MLE of the parameters for EGAPEx distribution, with the values of the K-S statistic with p-values are reported in Tables 7-10. First, we check whether the considered data sets actually come EGAPEx distribution or not by goodness-of-fit test and compare the fits with the exponentiated generalized exponential (EGEx), alpha power generalized exponential (APGEx), alpha power exponential (APEx), generalized exponential (GEx), exponential (Ex) distributions. The EGAPEx distribution gives the lowest value for the K-S, AIC, BIC, CAIC, and HAIC statistics among all the fitted models to these data sets, also The EGAPEx distribution gives the highest value for the P-Value comparing another distribution.

8. Conclusion

In this paper, we introduced a new three-parameter model, called the exponentiated generalized alpha power family. We used the new family to discuss a new four-parameter model, called the EGAPEx distribution which extends the exponential distribution. The EGAPEx distribution is motivated by the wide utilization of the exponential distribution in life testing and provides more flexibility to analyze lifetime data. We obtain some mathematical properties such as the quantile function, moments, generating function, characteristic function, cumulant-generating function, hazard rate function, and stress–strength reliability. We provide some applications of EGAPEx distribution in the context of statistics. The parameter estimation of EGAPEx distribution is derived by MLE, MPS, and Bayesian. The methods of estimation are employed to estimate the model parameters and simulation results are provided to assess the model performance. Two real-life data proposed model provides a consistently better fit than the APEx, APGEx, EGEx, GEx and Ex distributions.

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