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Some Problems in Finsler and Teleparallel Geometries

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Ebtsam Hassan Taha Mohamed
M.Sc.
Department of Mathematics
Faculty of Science - Cairo University

Supervised by

Prof. Dr. Nabil Labib Youssef
Department of Mathematics
Faculty of Science
Cairo University

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Name of the candidate:

Ebtsam Hassan Taha Mohamed

Submitted to:

Faculty of Science - Cairo University

Supervision Committee:

Prof. Dr. Nabil Labib Youssef

Department of Mathematics

Faculty of Science

Cairo University

Prof. Dr. Hosny Ali Abdusalam

Chairman of Mathematics Department

Faculty of Science

Cairo University

Dedicated to the memory of
Late Waleed A. Elsayed

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Abstract

We study conformal transformations in the context of local absolute parallelism geometry and find out some new conformally invariant geometric objects.

We establish the existence of a sub-Riemannian structure on a parallelizable distribution (PD). Besides the Weitzenböck connection, we construct on PD a linear connection which corresponds to the Riemannian connection. Contrary to the Riemannian case, an explicit global expression for such a connection is given. We apply the obtained results to the spheres S3 and S7.

We solve the Finsler metrizability problem for a 2-dimensional non-flat spray using the integrability of the Berwald distribution. Moreover, we provide an algorithm to construct the Finsler function metrizing the spray. Various examples are given to show how our method is powerful and easy to handle.

We transform a non-conservative Lagrangian system to a conservative one using the notion of scalar deformation. This has been done for homogeneous and non-homogeneous systems. Our results hold for any finite dimension and generalize various cases existing in the literature.

The notion of a semi-concurrent vector field (SCVF) is introduced and investigated. We show that some special Finsler manifolds admitting such a vector field turn out to be Riemannian. Different examples of non-Riemannian conic Finsler metrics admitting SCVF's are given. We finally conjecture that there is no regular Finsler metric admitting a SCVF.

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Cairo, Egypt. Ebtsam Taha
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Introduction

In this work we deal with the following topics:

(1) A study of conformal transformations in absolute parallelism geometry.

(2) An investigation of the notion of parallelizable distribution in connection with absolute parallelism and sub-Riemannian geometries.

(3) New solution of the Finsler metrizability problem for a two-dimensional non-flat spray.

(4) How to transform a non-conservative system (described by some Lagrangian) to a conservative one.

(5) How do the properties of a certain vector field defined on a Finsler space may govern the geometry of the space.

Finsler geometry [13, 22, 48, 63, 65, 88, 95] and absolute parallelism (AP-) geometry [83, 101, 102, 104, 113, 114] constitute the mathematical framework of the results obtained in this thesis. Many mathematical aspects of AP-geometry have received an extensive investigation in the last few years [104, 105, 113–117]. Both geometries have a rich mathematical content and are frequently used in many physical applications [9–11, 72, 73, 105]. They contain some canonical structures that lead to solutions of different physical and mathematical problems [40, 46, 62, 101, 102]. From the mathematical point of view, there is a crucial difference between AP and Finsler geometries. Unlike, AP-geometry, where the geometric objects live in the base manifold M, geometric objects in Finsler geometry are defined on the tangent bundle TM. Accordingly, their natural geometric framework is not the same. Both geometries have desirable properties for different reasons. Finsler geometry enjoys the richness and diversity of the tangent bundle while AP-geometry acquires a simple geometric structure induced by its parallelization. Both geometries share a common feature, however. Riemannian geometry is a special case of Finsler geometry while a Riemannain metric can be naturally defined in an AP-space. The work presented here involves the two types of geometries, namely, AP-geometry and Finsler geometry (and its generalized versions [4, 74]).

The thesis consists of two main parts which are divided into five chapters. The first two chapters constitute the first part and their framework is AP-geometry. The last three chapters constitute the second part and their framework is Finsler (or Lagrange) geometry. Each chapter is self contained and addresses some specified question and provides an answer in the appropriate framework. Some chapters are expressed locally while others are formulated in a global form.

Chapter 1 is devoted to the study of conformal changes in AP-geometry. We find out some new conformally invariant tensors fields expressed in terms of the Weitzenböck connection and the Levi-Civita connection of an AP-space. We show that these conformal invariants are the curvature tensors of constructed conformal connections. We believe that a physical interpretation of the geometric objects obtained is possible.
The main results of this chapter have been published, [116], in "International Journal of Geometric Methods in Modern Physics" Vol. 15 (2018) 1850012. DOI: 10.1142/S0219887818500123. arXiv: 1604.00474 [math.DG].

Chapter 2 is concerned with parallizable distributions. The notion of a parallelizable distribution has been introduced and investigated. A non-integrable parallelizable distribution carries a natural sub-Riemannian structure. The geometry of this structure has been studied from the bi-viewpoint of absolute parallelism geometry and sub-Riemannian geometry. Two remarkable linear connections have been constructed on a sub-Riemannian parallelizable distribution, namely, the Weitzenböck connection and the sub-Riemannian connection. The obtained results have been applied to two concrete examples: the spheres S^3 and S^7, where a classification of parallelizable distributions on S^3 and S^7 is provided.
The main results of this chapter have been published, [117], in "International Journal of Geometric Methods in Modern Physics" Vol. 14 (2017) 1750039. DOI: 10.1142/S0219887817500396. arXiv: 1603.06106 [math.DG].

Finsler geometry offers a natural geometric framework for variational calculus providing some version of Euler-Lagrange equations. These equations are associated with the notion of a spray [2, 49]. Every Finsler function determines uniquely a spray, called the geodesic spray. The inverse problem of the calculus of variation or the Finsler metrizability problem is the inverse of the last statement: Given a spray S, does there exist a Finsler function F such that the geodesic spray of F is S? In **chapter 3**, we give a new solution of the Finsler metrizability problem for a 2-dimensional non-flat spray S. For a 2-dimensional non-flat spray we associate a Berwald frame and a 3-dimensional distribution that we call the Berwald distribution. The Frobenius integrability of the Berwald distribution characterizes the Finsler metrizability of the given spray. In the integrable case, the sought after Finsler function is provided by a closed, homogeneous 1-form from the annihilator of the Berwald distribution. We discuss both the degenerate and non-degenerate cases using the fact that the regularity of the Finsler function is encoded into a regularity condition of a 2-form, canonically associated to the given spray. The integrability of the Berwald distribution and the regularity of the 2-form have simple and useful expressions in terms of the Berwald frame. The advantage of our approach consists in its constructive character, providing a simple algorithm to test the existence of the required Finsler structure and, in the affirmative case, to determine it. Finally, we end this chapter by some examples which show the utility of the results obtained. The main results of this chapter have been published, [20], in "Differential Geometry and its Applications" Vol. 56 (2018), 308-324. DOI: 10.1016/j.difgeo.2017.10.002. arXiv: 1610.03949 [math.DG].

In **Chapter 4**, we are interested in transforming a non-conservative system into a conservative one. This is achieved using the notion of differentiable deformation Φ. In more details, we study non-conservative SODEs admitting explicit Lagrangian descriptions. Such systems are equivalent to the system of Lagrange equations of some Lagrangian L, including a covariant force field which represents non-conservative forces. We find necessary and sufficient conditions for the existence of a differentiable function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that the initial system is equivalent to the system of Euler-Lagrange equations of the deformed Lagrangian $\Phi(L)$. This has been done for both the homogeneous and non-homogeneous cases. We give various examples of such deformations corresponding to different Lagrangian s.
The main results of this chapter have been submitted for publication, [31]. arXiv: 1712.01392 [math.DG].

In **Chapter 5**, we introduce and investigate the notion of a semi concurrent vector field on a Finsler manifold. We show that some special Finsler manifolds admitting such vector fields turn out to be Riemannian. We prove that Tachibana's

characterization of Finsler manifolds, admitting a concurrent vector field, leads to Riemannain metrics. We give an answer to the question raised in [42]: "Is any n-dimensional Finsler manifold (M, F), admitting a non-constant smooth function f on M such that $\frac{\partial f}{\partial x^i}\frac{\partial g^{ij}}{\partial x^k}$, a Riemannian manifold?". Various examples for conic Finsler and Riemannian spaces that admit semi-concurrent vector field are presented. Finally, we conjectured that there is no regular non-Riemannian Finsler metric that admits a semi-concurrent vector field. In other words, a Finsler metric admitting a semi-concurrent vector field is necessarily either Riemannian or conic Finslerian.
The main results of this chapter have been submitted for publication, [112]. arXiv: 1802.02405 [math.DG].

1

New Conformal Invariants in Absolute Parallelism Geometry

The notion of conformal transformations plays an important role in many branches of mathematics. In particular, conformal transformations have been intensively investigated in both Riemannian and Finsler geometries. Many authors [3, 43, 59, 79, 85, 87, 107] studied conformal transformations of Riemannian metrics. On the other hand, in the Finslerian domain, the conformal behavior of Finsler metrics were also examined [51, 55, 56, 66, 109, 110].

Conformal transformations have also proved to be effective in modern physical theories, see for example [62, 91]. Conformal field theories are constructed using the notion of a conformal map. Being conformally invariant, the physics of all these theories looks the same at all length scales. Conformal field theories care about angles and not distances. Such theories were successfully applied to string theory by Friedan, Martinec and Shenker [46].

Absolute Parallelism (AP-) geometry [17, 83, 101, 102, 113, 114] has a wide range of applications in the domain of physics. It has been successfully used in physical theories that are concerned with geometrization of physics [40, 78, 90, 91, 103, 105]. For these reasons (and others) we are motivated to discuss the notion of conformal changes in connection to AP-geometry. Because conformal changes are useful from both mathematical and physical points of view, an application of such a notion to the geometry of an AP-space should be fruitful and rewarding. In fact, the benefits of investigating conformal changes in AP-geometry is two-fold. Besides enriching the mathematical contents of the AP-geometry, the conformally invariant geometric objects constructed in the AP-context may also enlarge the scope of its physical applications [62, 91].

In this chapter we investigate conformal changes of the four built-in connections defined in an AP-space with their torsion and curvature associated tensors, the Wanas tensor and the contortion tensor. In addition, we introduce new conformal invariant connections and tensors. We study also some properties of those new invariants.

In the present chapter, we shall deal with the following items:

1.1. A brief account of AP-geometry

1.2. Conformal changes of geometric objects of AP-space

1.3. New conformal invariants

The main results of this chapter have been published in [116]: International Journal of Geometric Methods in Modern Physics, Vol. **15 (1)** (2018), 1850012 (10 pages). DOI: 10.1142/S0219887818500123. arXiv: 1604.00474 [math.DG].

1.1 A brief account of local AP-geometry

In this section, we give a short survey of the geometry of parallelizable manifolds or Absolute Parallelism geometry (AP-geometry). For more details, we refer, for example, to [102, 113, 114]. We use the notations and the results of [114].

Definition 1.1.1. *A parallelizable manifold is an n-dimensional smooth manifold M which admits n independent vector fields $\underset{i}{\lambda}$ $(i = 1, \ldots, n)$ defined globally on M.*

This space is also known in the literature as Absolute Parallelism space (*AP-space*) or teleparallel space. Let $\underset{i}{\lambda}^{\mu}$ $(\mu = 1, 2, \ldots, n)$ be the coordinate components of the *i-th* vector field $\underset{i}{\lambda}$. The Einstein summation convention is applied to both Latin (mesh) and Greek (world) indices, where all Latin indices are written beneath the symbols. The covariant components $\underset{i}{\lambda}_{\mu}$ of $\underset{i}{\lambda}$ are given via the relations

$$\underset{i}{\lambda}^{\mu} \underset{i}{\lambda}_{\nu} = \delta^{\mu}_{\nu}, \quad \underset{i}{\lambda}^{\mu} \underset{j}{\lambda}_{\mu} = \delta_{ij}.$$

The n^3 functions $\Gamma^{\alpha}_{\mu\nu}$ defined by

$$\Gamma^{\alpha}_{\mu\nu} := \underset{i}{\lambda}^{\alpha} \underset{i}{\lambda}_{\mu,\nu} \tag{1.1.1}$$

transform as the coefficients of a linear connection under a change of coordinates (where the comma here denotes partial differentiation with respect to the coordinate function x^{ν}). The connection $\Gamma^{\alpha}_{\mu\nu}$ is clearly non-symmetric and is referred to as the *Weitzenböck* or the *canonical connection* of the space.

As easily checked, we have

$$\underset{i}{\lambda}_{\mu|\nu} = 0, \quad \underset{i}{\lambda}^{\mu}{}_{|\nu} = 0, \tag{1.1.2}$$

where the stroke "|" denotes covariant differentiation with respect to the canonical connection $\Gamma^{\alpha}_{\mu\nu}$. The above relations are known in the literature as the condition of Absolute Parallelism or the *AP-condition*.

The torsion tensor of $\Gamma^{\alpha}_{\mu\nu}$ is given by

$$\Lambda^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}. \tag{1.1.3}$$

It is of particular importance to note that (1.1.2) together with the commutation formula

$$\underset{i}{\lambda}^{\alpha}{}_{|\mu\nu} - \underset{i}{\lambda}^{\alpha}{}_{|\nu\mu} = \underset{i}{\lambda}^{\epsilon}\, R^{\alpha}_{\epsilon\mu\nu} + \underset{i}{\lambda}^{\alpha}{}_{|\epsilon}\, \Lambda^{\epsilon}_{\nu\mu}$$

force the curvature tensor $R^{\alpha}_{\epsilon\mu\nu}$ of the canonical connection $\Gamma^{\alpha}_{\mu\nu}$ to vanish identically.

It is for this reason that many authors think that the AP-space is flat. This is by no means true. In fact, it is meaningless to speak of curvature without reference to a connection. All we can say is that the AP-space is flat with respect to the canonical connection, or that its canonical connection is flat. However, there are other three natural (built-in) connections in the AP-space which are non flat. Namely,
the *dual* connection

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\nu\mu}, \tag{1.1.4}$$

the *symmetric* connection

$$\widehat{\Gamma}^{\alpha}_{\mu\nu} := \frac{1}{2}(\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\mu}) = \Gamma^{\alpha}_{(\mu\nu)} \tag{1.1.5}$$

and the *Levi-Civita* connection

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} := \frac{1}{2} g^{\alpha\epsilon}(g_{\epsilon\nu,\mu} + g_{\epsilon\mu,\nu} - g_{\mu\nu,\epsilon}) \tag{1.1.6}$$

associated with the *metric structure* defined by

$$g_{\mu\nu} := \underset{i}{\lambda}_{\mu}\, \underset{i}{\lambda}_{\nu}$$

with inverse

$$g^{\mu\nu} := \underset{i}{\lambda}^{\mu}\, \underset{i}{\lambda}^{\nu}.$$

We have four types of covariant derivatives corresponding to the four connections mentioned above, namely,

$$A^{\mu}_{\sigma|\nu} = A^{\mu}_{\sigma,\nu} + A^{\epsilon}_{\sigma}\, \Gamma^{\mu}_{\epsilon\nu} - A^{\mu}_{\epsilon}\, \Gamma^{\epsilon}_{\sigma\nu},$$

$$A^{\mu}_{\sigma\widetilde{|}\nu} = A^{\mu}_{\sigma,\nu} + A^{\epsilon}_{\sigma}\,\widetilde{\Gamma}^{\mu}_{\epsilon\nu} - A^{\mu}_{\epsilon}\,\widetilde{\Gamma}^{\epsilon}_{\sigma\nu},$$

$$A^{\mu}_{\sigma\widehat{|}\nu} = A^{\mu}_{\sigma,\nu} + A^{\epsilon}_{\sigma}\,\widehat{\Gamma}^{\mu}_{\epsilon\nu} - A^{\mu}_{\epsilon}\,\widehat{\Gamma}^{\epsilon}_{\sigma\nu},$$

$$A^{\mu}_{\sigma\overset{\circ}{|}\nu} = A^{\mu}_{\sigma,\nu} + A^{\epsilon}_{\sigma}\,\overset{\circ}{\Gamma}{}^{\mu}_{\epsilon\nu} - A^{\mu}_{\epsilon}\,\overset{\circ}{\Gamma}{}^{\epsilon}_{\sigma\nu},$$

where A^{μ}_{σ} is an arbitrary $(1,1)$-tensor on M and $\widetilde{|}$, $\widehat{|}$ and $\overset{\circ}{|}$ are the covariant derivatives with respect to the dual, symmetric and Levi-Civita connections, respectively.

It is to be noted that the AP-condition (1.1.2) implies that the canonical connection $\Gamma^{\alpha}_{\mu\nu}$ is metric:

$$g_{\mu\nu|\sigma} = 0, \qquad g^{\mu\nu}{}_{|\sigma} = 0.$$

Consequently, the covariant differentiation with respect to the canonical connection commutes with contraction by the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. However, the covariant derivatives of the metric with respect to the dual and symmetric connections $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ and $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ are given respectively by:

$$g_{\alpha\beta\widetilde{|}\mu} = \Lambda_{\alpha\beta\mu} + \Lambda_{\beta\alpha\mu}, \quad g_{\alpha\beta\widehat{|}\mu} = \frac{1}{2}\,g_{\alpha\beta\widetilde{|}\mu},$$

where $\Lambda_{\alpha\beta\mu} := g_{\alpha\epsilon}\,\Lambda^{\epsilon}_{\beta\mu}$. Hence, $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ and $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ are non-metric.

The *contortion* tensor is defined by

$$\gamma^{\alpha}_{\mu\nu} := \underset{i}{\lambda}{}^{\alpha}\underset{i}{\lambda}{}_{\mu\overset{\circ}{|}\nu}.$$

Moreover, it can be shown that

$$\gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}. \tag{1.1.7}$$

Since $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ is symmetric, it follows that

$$\Lambda^{\alpha}_{\mu\nu} = \gamma^{\alpha}_{\mu\nu} - \gamma^{\alpha}_{\nu\mu}. \tag{1.1.8}$$

Moreover, the contortion tensor field is given in terms of the torsion tensor field by the relation [52]:

$$\gamma_{\mu\nu\sigma} = \frac{1}{2}(\Lambda_{\mu\nu\sigma} + \Lambda_{\sigma\nu\mu} + \Lambda_{\nu\sigma\mu}), \tag{1.1.9}$$

where $\gamma_{\mu\nu\sigma} := g_{\epsilon\mu}\, \gamma^{\epsilon}_{\nu\sigma}$. It is to be noted that $\Lambda_{\mu\nu\sigma}$ is skew-symmetric in the last pair of indices, whereas $\gamma_{\mu\nu\sigma}$ is skew symmetric in the first pair of indices. Moreover, in view of (1.1.8) and (1.1.9), the contortion tensor field vanishes if and only if the torsion tensor field vanishes.

Furthermore, the *basic form* C_μ is defined by

$$C_\mu := \Lambda^{\alpha}_{\alpha\mu} = \gamma^{\alpha}_{\alpha\mu}.$$

Theorem 1.1.2. [114] *The curvature tensor fields* $\widetilde{R}^{\alpha}_{\mu\nu\sigma}$, $\widehat{R}^{\alpha}_{\mu\nu\sigma}$ *and* $\overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma}$ *of the connections* $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$, $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$, *respectively, can be expressed in terms of the torsion tensor field* $\Lambda^{\alpha}_{\mu\nu}$ *of the canonical connection* $\Gamma^{\alpha}_{\mu\nu}$ *solely as follows:*

(a) $\widetilde{R}^{\alpha}_{\mu\nu\sigma} = \Lambda^{\alpha}_{\sigma\nu|\mu}$.

(b) $\widehat{R}^{\alpha}_{\mu\nu\sigma} = \frac{1}{2}\,(\Lambda^{\alpha}_{\mu\nu|\sigma} - \Lambda^{\alpha}_{\mu\sigma|\nu}) + \frac{1}{4}\,(\Lambda^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} - \Lambda^{\epsilon}_{\mu\sigma}\Lambda^{\alpha}_{\nu\epsilon}) + \frac{1}{2}\,(\Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu})$.

(c) $\overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma} = (\gamma^{\alpha}_{\mu\nu|\sigma} - \gamma^{\alpha}_{\mu\sigma|\nu}) + (\gamma^{\epsilon}_{\mu\sigma}\gamma^{\alpha}_{\epsilon\nu} - \gamma^{\epsilon}_{\mu\nu}\gamma^{\alpha}_{\epsilon\sigma}) + \gamma^{\alpha}_{\mu\epsilon}\Lambda^{\epsilon}_{\nu\sigma}$.

In conclusion, the AP-space has four curvature tensor fields $R^{\alpha}_{\mu\nu\sigma}$, $\widetilde{R}^{\alpha}_{\mu\nu\sigma}$, $\widehat{R}^{\alpha}_{\mu\nu\sigma}$ and $\overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma}$ corresponding to the four connections $\Gamma^{\alpha}_{\mu\nu}$, $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$, $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ and $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ respectively, with the curvature of the canonical connection vanishing identically. There is only one torsion (torsion of the canonical connection). The other surviving torsion is that of the dual connection which is the same as the torsion of the canonical connection up to a sign. The next table summarizes the geometry of the AP-space [114].

Table 1.1: Geometry of the AP-space

Connection	Coefficients	Covariant derivative	Torsion	Curvature	Metricity
Canonical	$\Gamma^{\alpha}_{\mu\nu}$	$\mid$	$\Lambda^{\alpha}_{\mu\,\nu}$	0	metric
Dual	$\widetilde{\Gamma}^{\alpha}_{\mu\nu}$	$\widetilde{\mid}$	$-\Lambda^{\alpha}_{\mu\,\nu}$	$\widetilde{R}^{\alpha}_{\mu\nu\sigma}$	non-metric
Symmetric	$\widehat{\Gamma}^{\alpha}_{\mu\nu}$	$\hat{\mid}$	0	$\widehat{R}^{\alpha}_{\mu\nu\sigma}$	non-metric
Levi-Civita	$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$	$\overset{o}{\mid}$	0	$\overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma}$	metric

The *Wanas tensor* [114] is one of the most important tensors in the context of AP-geometry. It has many physical applications related to the theory of general relativity and some unification theories, such as a remarkable geometric theory unifying gravity and electromagnetism [72, 73]. Thus, roughly speaking, the Wanas tensor expresses geometrically the interaction between curvature and torsion. On the other hand, as gravity is described in terms of curvature and electromagnetism is described in terms of torsion, we can roughly say that the Wanas tensor expresses physically the interaction between gravity and electromagnetism [72]. The Wanas tensor is given by the formula [114]

$$W^{\alpha}_{\mu\nu\sigma} = \widetilde{R}^{\alpha}_{\mu\nu\sigma} - \Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu} = \Lambda^{\alpha}_{\sigma\nu|\mu} - \Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu}.$$

1.2 Conformal change of geometric objects of AP-space

In this section, we shall investigate the conformal change of the natural connections, defined in an AP-space, together with their associated tensors. To this end, let $(M, \underset{i}{\lambda})$ be an n-dimensional AP-space. Let $\Gamma^{\alpha}_{\mu\nu}$, $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$, $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ and $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ be the Weitzenböck, dual, symmetric and Levi-Civita connections, respectively.

Definition 1.2.1. *Two AP-spaces $(M, \underset{i}{\lambda})$ and $(M, \underset{i}{\overline{\lambda}})$ are said to be conformal (or conformally related) if there exists a positive smooth function $\rho(x)$ such that*

$$\underset{i}{\overline{\lambda}}{}^{\mu} = e^{-\rho(x)}\ \underset{i}{\lambda}{}^{\mu} \quad (\text{or } \underset{i}{\overline{\lambda}}{}_{\mu} = e^{\rho(x)}\ \underset{i}{\lambda}{}_{\mu}), \tag{1.2.1}$$

or, equivalently,

$$\overline{g}_{\mu\nu} = e^{2\rho(x)} g_{\mu\nu}.$$

Now, we present the conformal change of the most important geometric objects associated with an AP-space. Let $\rho_{\mu} := \frac{\partial \rho}{\partial x^{\mu}}$ and $\rho^{\mu} := g^{\mu\nu}\rho_{\nu}$.

Proposition 1.2.2. *Under the conformal change* (1.2.1)*, we have:*

(a) *The Weitzenböck connections $\Gamma^{\alpha}_{\mu\nu}$ and $\overline{\Gamma}^{\alpha}_{\mu\nu}$ are related by*

$$\overline{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\mu}\rho_{\nu}.$$

(b) *The torsion tensors $\Lambda^{\alpha}_{\mu\nu}$ and $\overline{\Lambda}^{\alpha}_{\mu\nu}$ of $\Gamma^{\alpha}_{\mu\nu}$ and $\overline{\Gamma}^{\alpha}_{\mu\nu}$ are related by*

$$\overline{\Lambda}^{\alpha}_{\mu\nu} = \Lambda^{\alpha}_{\mu\nu} + \left(\delta^{\alpha}_{\mu}\rho_{\nu} - \delta^{\alpha}_{\nu}\rho_{\mu}\right).$$

Proof. The proof follows directly from (1.1.1) and (1.1.3). □

Remark 1.2.3. It can be shown easily that $\overline{\Gamma}^{\alpha}_{\mu\nu} = \underset{i}{\overline{\lambda}}{}^{\alpha}\, \underset{i}{\overline{\lambda}}{}_{\mu,\nu}$ and $\underset{i}{\overline{\lambda}}{}^{\alpha}{}_{\|\mu} = 0$ (the double stroke denotes covariant differentiation with respect to $\overline{\Gamma}^{\alpha}_{\mu\nu}$) so that the AP-condition (still) holds after a conformal change. Hence, the connection $\overline{\Gamma}^{\alpha}_{\mu\nu}$ of the space $(M, \underset{i}{\overline{\lambda}})$ behaves like the canonical connection of $(M, \underset{i}{\lambda})$. However, $\underset{i}{\lambda}{}^{\alpha}{}_{\|\mu} = \underset{i}{\lambda}{}^{\alpha}{}_{|\mu}\,\rho_{\mu}$ do not vanish unless the function ρ is a homothety.

Proposition 1.2.4. *Under the conformal change* (1.2.1)*, we have:*

(a) *The dual connections* $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widetilde{\Gamma}}}{}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\overline{\widetilde{\Gamma}}}{}^{\alpha}_{\mu\nu} = \widetilde{\Gamma}^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\nu}\rho_{\mu}.$$

(b) *The torsion tensors* $\widetilde{\Lambda}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widetilde{\Lambda}}}{}^{\alpha}_{\mu\nu}$ *of* $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widetilde{\Gamma}}}{}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\overline{\widetilde{\Lambda}}}{}^{\alpha}_{\mu\nu} = \widetilde{\Lambda}^{\alpha}_{\mu\nu} - \left(\delta^{\alpha}_{\mu}\rho_{\nu} - \delta^{\alpha}_{\nu}\rho_{\mu}\right).$$

(c) *The curvature tensors* $\widetilde{R}^{\alpha}_{\ \mu\nu\sigma}$ *and* $\overline{\overline{\widetilde{R}}}{}^{\alpha}_{\ \mu\nu\sigma}$ *of* $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widetilde{\Gamma}}}{}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\overline{\widetilde{R}}}{}^{\alpha}_{\ \mu\nu\sigma} = \widetilde{R}^{\alpha}_{\ \mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\sigma}\rho_{\mu\widetilde{|}\nu} + \delta^{\alpha}_{\nu}\,\rho_{\mu}\,\rho_{\sigma} + \frac{1}{2}\rho_{\mu}\Lambda^{\alpha}_{\nu\sigma}\right\},$$

where $\mathfrak{U}_{\mu\nu}\{A_{\mu\nu}\} := A_{\mu\nu} - A_{\nu\mu}$.

Proof. The proof follows directly from (1.1.4) and Theorem 1.1.2**(a)**. □

Proposition 1.2.5. *Under the conformal change* (1.2.1)*, we have:*

(a) *The symmetric connections* $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widehat{\Gamma}}}{}^{\alpha}_{\mu\nu}$ *are related by:*

$$\overline{\overline{\widehat{\Gamma}}}{}^{\alpha}_{\mu\nu} = \widehat{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}(\delta^{\alpha}_{\mu}\rho_{\nu} + \delta^{\alpha}_{\nu}\rho_{\mu}), \tag{1.2.2}$$

(b) *The curvature tensors* $\widehat{R}^{\alpha}_{\mu\nu\sigma}$ *and* $\overline{\overline{\widehat{R}}}{}^{\alpha}_{\mu\nu\sigma}$ *of* $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ *and* $\overline{\overline{\widehat{\Gamma}}}{}^{\alpha}_{\mu\nu}$ *are related by:*

$$\overline{\overline{\widehat{R}}}{}^{\alpha}_{\mu\nu\sigma} = \widehat{R}^{\alpha}_{\mu\nu\sigma} + \frac{1}{2}\mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\sigma}\rho_{\mu\widehat{|}\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho_{\sigma}\rho_{\mu}\right\}. \tag{1.2.3}$$

Proof. By some simple calculations, using (1.1.5) and Theorem 1.1.2**(b)**, we can prove (1.2.2) and (1.2.3). □

Remark 1.2.6. The tensor field $\widehat{r}_{\nu\sigma} := \widehat{R}^{\alpha}_{\alpha\nu\sigma}$ is conformally invariant. Indeed, from (1.2.3) we get

$$\overline{\overline{\widehat{r}}}_{\nu\sigma} := \overline{\overline{\widehat{R}}}{}^{\alpha}_{\alpha\nu\sigma} = \widehat{R}^{\alpha}_{\alpha\nu\sigma} + \frac{1}{2}\mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\sigma}\rho_{\alpha\widehat{|}\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho_{\sigma}\rho_{\alpha}\right\} = \widehat{R}^{\alpha}_{\alpha\nu\sigma} = \widehat{r}_{\nu\sigma}.$$

However, the Ricci-like tensor defined by $\widehat{R}_{\mu\nu} := \widehat{R}^{\alpha}_{\mu\nu\alpha}$ is not conformally invariant. Indeed, using (1.2.3) it follows that

$$\overline{\overline{\widehat{R}}}_{\mu\nu} := \overline{\overline{\widehat{R}}}{}^{\alpha}_{\mu\nu\alpha} = \widehat{R}^{\alpha}_{\mu\nu\alpha} + \frac{1}{2}\mathfrak{U}_{\nu\alpha}\left\{\delta^{\alpha}_{\alpha}\rho_{\mu\widehat{|}\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho_{\alpha}\rho_{\mu}\right\} = \widehat{R}_{\mu\nu} + \frac{(n-1)}{2}\left(\rho_{\mu\widehat{|}\nu} - \frac{1}{2}\rho_{\mu}\rho_{\nu}\right).$$

Proposition 1.2.7. *Under the conformal change* (1.2.1)*, we have:*

(a) *The Levi-Civita connections* $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ *and* $\overline{\overset{\circ}{\Gamma}}{}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\overset{\circ}{\Gamma}}{}^{\alpha}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} + \left(\delta^{\alpha}_{\mu}\rho_{\nu} + \delta^{\alpha}_{\nu}\rho_{\mu} - g_{\mu\nu}\rho^{\alpha}\right),$$

(b) *The curvature tensors* $\overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma}$ *and* $\overline{\overset{\circ}{R}}{}^{\alpha}_{\mu\nu\sigma}$ *of* $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ *and* $\overline{\overset{\circ}{\Gamma}}{}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\overset{\circ}{R}}{}^{\alpha}_{\mu\nu\sigma} = \overset{\circ}{R}{}^{\alpha}_{\mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\sigma}S_{\mu\nu} - g_{\mu\sigma}S^{\alpha}_{\nu}\right\},$$

where $S_{\mu\nu} := \rho_{\mu\overset{\circ}{|}\nu} - \rho_{\mu}\rho_{\nu} - \frac{1}{2}g_{\mu\nu}\rho^2$, $\rho^2 := \rho^{\epsilon}\rho_{\epsilon}$ *and* $S^{\alpha}_{\nu} := g^{\alpha\epsilon}S_{\epsilon\nu}$.

(c) *The contortion tensors* $\gamma^{\alpha}_{\mu\nu}$ *and* $\overline{\gamma}^{\alpha}_{\mu\nu}$ *are related by*

$$\overline{\gamma}^{\alpha}_{\mu\nu} = \gamma^{\alpha}_{\mu\nu} - \delta^{\alpha}_{\nu}\rho_{\mu} + g_{\mu\nu}\rho^{\alpha}.$$

Proof. The results can be obtained by some tedious though straightforward calculations taking into account (1.1.6), (1.1.7) and Theorem 1.1.2**(c)**. □

Remark 1.2.8. The Ricci tensor of $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ is transformed as

$$\overline{\overset{\circ}{R}}_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} + (n-2)S_{\mu\nu} + g_{\mu\nu}S^{\alpha}_{\alpha},$$

and the Ricci scalar is transformed as

$$\overline{\overset{\circ}{R}} = e^{-2\rho(x)}\left(\overset{\circ}{R} + 2(n-1)S^{\alpha}_{\alpha}\right).$$

As can be easily checked, the following transformation formula holds.

Proposition 1.2.9. *Under the conformal change* (1.2.1)*, the Wanas tensor* $W^{\alpha}_{\mu\nu\sigma}$ *is transformed as follows:*

$$\overline{W}^{\alpha}_{\mu\nu\sigma} = W^{\alpha}_{\mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\sigma}\rho_{\nu|\mu} - 2\delta^{\alpha}_{\sigma}\rho_{\nu}\rho_{\mu} - \frac{1}{2}\Lambda^{\alpha}_{\sigma\nu}\rho_{\mu} + \frac{1}{2}\delta^{\alpha}_{\mu}\Lambda^{\varepsilon}_{\sigma\nu}\rho_{\epsilon} - \Lambda^{\alpha}_{\sigma\mu}\rho_{\nu}\right\}.$$

1.3 New conformal invariants

In this section, we construct four conformally invariant tensors in the AP-context. Three of these tensors turn out to be the associated curvature tensors of three conformal connections to be constructed. The other one represents the torsion tensor of one of the conformal connections. It should be noted that our conformal curvature tensors are different from the (celebrated) Weyl tensor.

Definition 1.3.1. *Let $(M, \underset{i}{\lambda})$ be an AP-space. A linear connection $\Omega^{\alpha}_{\mu\nu}$ is said to be conformal if it is conformally invariant under the conformal change* (1.2.1)*:*

$$\overline{\Omega}^{\alpha}_{\mu\nu} = \Omega^{\alpha}_{\mu\nu}.$$

Theorem 1.3.2. *Let $(M, \underset{i}{\lambda})$ be an AP-space of dimension $n \geq 2$. Let $\Lambda^{\alpha}_{\mu\nu}$ and C_{μ} be respectively the torsion and the basic form associated with the Weitzanböck connection $\Gamma^{\alpha}_{\mu\nu}$. The tensors*

$$\begin{aligned} T^{\alpha}_{\mu\nu} &:= \Lambda^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu} C_{\nu} - \delta^{\alpha}_{\nu} C_{\mu}\right\}, \\ K^{\alpha}_{\mu\nu\sigma} &:= \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu} C_{\nu,\sigma} - \delta^{\alpha}_{\mu} C_{\sigma,\nu}\right\}, \end{aligned}$$

are conformally invariant. Moreover, the tensors $T^{\alpha}_{\mu\nu}$ and $K^{\alpha}_{\mu\nu\sigma}$ are precisely the torsion and curvature tensors of a conformal connection on M.

Proof. Under the conformal change (1.2.1), using Proposition 1.2.2**(b)**, we have

$$\overline{C}_{\nu} = C_{\nu} + (n-1)\rho_{\nu}. \tag{1.3.1}$$

This, combined with Proposition 1.2.2**(b)**, yields

$$\begin{aligned} \overline{T}^{\alpha}_{\mu\nu} &= \overline{\Lambda}^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu} \overline{C}_{\nu} - \delta^{\alpha}_{\nu} \overline{C}_{\mu}\right\} \\ &= \Lambda^{\alpha}_{\mu\nu} + (\delta^{\alpha}_{\mu}\rho_{\nu} - \delta^{\alpha}_{\nu}\rho_{\mu}) - \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu}(C_{\nu} + (n-1)\rho_{\nu}) - \delta^{\alpha}_{\nu}(C_{\mu} + (n-1)\rho_{\mu})\right\}, \\ &= \Lambda^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu} C_{\nu} - \delta^{\alpha}_{\nu} C_{\mu}\right\} = T^{\alpha}_{\mu\nu}. \end{aligned}$$

Similarly, from (1.3.1), we get

$$\begin{aligned}\overline{K}^{\,\alpha}_{\mu\nu\sigma} &= \frac{1}{(n-1)}\left\{\delta^\alpha_\mu \overline{C}_{\nu,\sigma} - \delta^\alpha_\mu \overline{C}_{\sigma,\nu}\right\} \\ &= \frac{1}{(n-1)}\left\{\delta^\alpha_\mu (C_\nu + (n-1)\rho_\nu)_{,\sigma} - \delta^\alpha_\mu (C_\sigma + (n-1)\rho_\sigma)_{,\nu}\right\} \\ &= \frac{1}{(n-1)}\left\{\delta^\alpha_\mu C_{\nu,\sigma} - \delta^\alpha_\mu C_{\sigma,\nu}\right\} = K^\alpha_{\mu\nu\sigma}.\end{aligned}$$

This means that $T^\alpha_{\mu\nu}$ and $K^{\,\alpha}_{\mu\nu\sigma}$ are conformally invariant.

Now, let us define the connection:

$$\mathbf{\Gamma}^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \frac{1}{(n-1)}\delta^\alpha_\mu C_\nu. \tag{1.3.2}$$

From (1.3.2) and (1.3.1), together with Proposition 1.2.2**(a)**, we conclude that

$$\begin{aligned}\overline{\mathbf{\Gamma}}^\alpha_{\mu\nu} &= \overline{\Gamma}^\alpha_{\mu\nu} - \frac{1}{(n-1)}\delta^\alpha_\mu \overline{C}_\nu \\ &= \Gamma^\alpha_{\mu\nu} + \delta^\alpha_\mu \rho_\nu - \frac{1}{(n-1)}\delta^\alpha_\mu \left(C_\nu + (n-1)\rho_\nu\right) \\ &= \Gamma^\alpha_{\mu\nu} - \frac{1}{(n-1)}\delta^\alpha_\mu C_\nu = \mathbf{\Gamma}^\alpha_{\mu\nu},\end{aligned}$$

which shows that the connection $\mathbf{\Gamma}^\alpha_{\mu\nu}$ is conformal. Next, we prove that the tensors $T^\alpha_{\mu\nu}$ and $K^\alpha_{\mu\nu\sigma}$ are respectively the torsion and the curvature tensors of the conformal connection $\mathbf{\Gamma}^\alpha_{\mu\nu}$:

$$\begin{aligned}\mathbf{\Gamma}^\alpha_{\mu\nu} - \mathbf{\Gamma}^\alpha_{\nu\mu} &= \left(\Gamma^\alpha_{\mu\nu} - \frac{1}{(n-1)}\delta^\alpha_\mu C_\nu\right) - \left(\Gamma^\alpha_{\nu\mu} - \frac{1}{(n-1)}\delta^\alpha_\nu C_\mu\right) \\ &= \Lambda^\alpha_{\mu\nu} - \frac{1}{(n-1)}\left\{\delta^\alpha_\mu C_\nu - \delta^\alpha_\nu C_\mu\right\} = T^\alpha_{\mu\nu}.\end{aligned}$$

$$\begin{aligned}\mathfrak{U}_{\nu\sigma}\left\{\mathbf{\Gamma}^\alpha_{\mu\sigma,\nu} + \mathbf{\Gamma}^\epsilon_{\mu\sigma}\mathbf{\Gamma}^\alpha_{\epsilon\nu}\right\} &= \left(\Gamma^\alpha_{\mu\sigma} - \frac{1}{(n-1)}\,\delta^\alpha_\mu C_\sigma\right)_{,\nu} - \left(\Gamma^\alpha_{\mu\nu} - \frac{1}{(n-1)}\,\delta^\alpha_\mu C_\nu\right)_{,\sigma} \\ &\quad + \left(\Gamma^\epsilon_{\mu\sigma} - \frac{1}{(n-1)}\,\delta^\epsilon_\mu C_\sigma\right)\left(\Gamma^\alpha_{\epsilon\nu} - \frac{1}{(n-1)}\,\delta^\alpha_\epsilon C_\nu\right) \\ &\quad - \left(\Gamma^\epsilon_{\mu\nu} - \frac{1}{(n-1)}\,\delta^\epsilon_\mu C_\nu\right)\left(\Gamma^\alpha_{\epsilon\sigma} - \frac{1}{(n-1)}\,\delta^\alpha_\epsilon C_\sigma\right)\end{aligned}$$

$$\begin{aligned}
&= \Gamma^{\alpha}_{\mu\sigma,\nu} - \Gamma^{\alpha}_{\mu\nu,\sigma} + \Gamma^{\epsilon}_{\mu\sigma}\Gamma^{\alpha}_{\epsilon\nu} - \Gamma^{\epsilon}_{\mu\nu}\Gamma^{\alpha}_{\epsilon\sigma} \\
&\quad -\frac{1}{(n-1)}\delta^{\alpha}_{\mu}C_{\sigma,\nu} + \frac{1}{(n-1)}\delta^{\alpha}_{\mu}C_{\nu,\sigma} \\
&\quad -\frac{1}{(n-1)}\Gamma^{\alpha}_{\mu\sigma}C_{\nu} - \frac{1}{(n-1)}\Gamma^{\alpha}_{\mu\nu}C_{\sigma} + \frac{1}{(n-1)^2}\delta^{\alpha}_{\mu}C_{\sigma}C_{\nu} \\
&\quad +\frac{1}{(n-1)}\Gamma^{\alpha}_{\mu\nu}C_{\sigma} + \frac{1}{(n-1)}\Gamma^{\alpha}_{\mu\sigma}C_{\nu} - \frac{1}{(n-1)^2}\delta^{\alpha}_{\mu}C_{\nu}C_{\sigma} \\
&= R^{\alpha}_{\mu\nu\sigma} - \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu}C_{\sigma,\nu} - \delta^{\alpha}_{\mu}C_{\nu,\sigma}\right\} \\
&= \frac{1}{(n-1)}\left\{\delta^{\alpha}_{\mu}C_{\nu,\sigma} - \delta^{\alpha}_{\mu}C_{\sigma,\nu}\right\} = K^{\alpha}_{\mu\nu\sigma},
\end{aligned}$$

since the curvature tensor of the Weitzenböck connection vanishes identically. This completes the proof. □

Theorem 1.3.3. *Let $(M, \underset{i}{\lambda})$ be an AP-space of dimension $n \geq 2$. The tensor*

$$\begin{aligned}
B^{\alpha}_{\mu\nu\sigma} &:= \frac{1}{4}\mathfrak{U}_{\nu\sigma}\left\{2\Lambda^{\alpha}_{\mu\nu|\sigma} + \Lambda^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} + \Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu}\right\} \\
&\quad -\frac{1}{2(n-1)}\mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\mu}C_{\sigma,\nu} + \delta^{\alpha}_{\sigma}C_{\mu\widehat{|}\nu} - \frac{1}{2(n-1)}\delta^{\alpha}_{\nu}C_{\mu}C_{\sigma}\right\}
\end{aligned}$$

is conformally invariant. Moreover, $B^{\alpha}_{\mu\nu\sigma}$ is precisely the curvature tensor of a conformal connection on M.

Proof. Let the covariant derivative with respect to $\overline{\widehat{\Gamma}}{}^{\alpha}_{\mu\nu}$ be denoted by $\widehat{||}$. Using Equations (1.3.1) and (1.2.2), one can show that

$$\begin{aligned}
\overline{C}_{\sigma,\nu} &= C_{\sigma,\nu} + (n-1)\rho_{\sigma,\nu} \\
\overline{C}_{\mu\widehat{||}\nu} &= C_{\mu\widehat{|}\nu} + (n-1)\rho_{\mu\widehat{|}\nu} - \frac{1}{2}(C_{\mu}\rho_{\nu} + C_{\nu}\rho_{\mu}) - (n-1)\rho_{\mu}\rho_{\nu}.
\end{aligned}$$

We show that the tensor $B^{\alpha}_{\mu\nu\sigma}$ is conformally invariant. From the above two relations, together with (1.3.1), (1.2.3) and Theorem 1.1.2**(b)**, we get, after some manipulations,

$$\begin{aligned}
\overline{B}{}^{\alpha}_{\mu\nu\sigma} &= \overline{\widehat{R}}{}^{\alpha}_{\mu\nu\sigma} - \frac{1}{2(n-1)}\Big\{\delta^{\alpha}_{\mu}\overline{C}_{\sigma,\nu} - \delta^{\alpha}_{\mu}\overline{C}_{\nu,\sigma} + \delta^{\alpha}_{\sigma}\overline{C}_{\mu\widehat{||}\nu} - \delta^{\alpha}_{\nu}\overline{C}_{\mu\widehat{||}\sigma} \\
&\quad -\frac{1}{2(n-1)}\delta^{\alpha}_{\nu}\overline{C}_{\mu}\overline{C}_{\sigma} + \frac{1}{2(n-1)}\delta^{\alpha}_{\sigma}\overline{C}_{\mu}\overline{C}_{\nu}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \widehat{R}^{\alpha}_{\mu\nu\sigma} + \frac{1}{2}\Big\{ \delta^{\alpha}_{\sigma}\rho_{\widehat{\mu|\nu}} - \delta^{\alpha}_{\nu}\rho_{\widehat{\mu|\sigma}} + \frac{1}{2}\,\delta^{\alpha}_{\nu}\rho_{\sigma}\rho_{\mu} - \frac{1}{2}\,\delta^{\alpha}_{\sigma}\rho_{\nu}\rho_{\mu}\Big\} \\
&\quad -\frac{1}{2(n-1)}\Big\{ \delta^{\alpha}_{\mu}\left(C_{\sigma,\nu} + (n-1)\rho_{\sigma,\nu}\right) - \delta^{\alpha}_{\mu}(C_{\nu,\sigma} + (n-1)\rho_{\nu,\sigma}) \\
&\quad +\delta^{\alpha}_{\sigma}\left(C_{\widehat{\mu|\nu}} + (n-1)\rho_{\widehat{\mu|\nu}} - \frac{1}{2}(C_{\mu}\rho_{\nu} + C_{\nu}\rho_{\mu}) - (n-1)\rho_{\mu}\rho_{\nu}\right) \\
&\quad -\delta^{\alpha}_{\nu}\left(C_{\widehat{\mu|\sigma}} + (n-1)\rho_{\widehat{\mu|\sigma}} - \frac{1}{2}(C_{\mu}\rho_{\sigma} + C_{\sigma}\rho_{\mu}) - (n-1)\rho_{\mu}\rho_{\sigma}\right) \\
&\quad -\frac{1}{2(n-1)}\,\delta^{\alpha}_{\nu}(C_{\mu} + (n-1)\rho_{\mu})(C_{\sigma} + (n-1)\rho_{\sigma}) \\
&\quad +\frac{1}{2(n-1)}\,\delta^{\alpha}_{\sigma}(C_{\mu} + (n-1)\rho_{\mu})(C_{\nu} + (n-1)\rho_{\nu})\Big\} \\
&= \widehat{R}^{\alpha}_{\mu\nu\sigma} - \frac{1}{2(n-1)}\Big\{ \delta^{\alpha}_{\mu}C_{\sigma,\nu} - \delta^{\alpha}_{\mu}C_{\nu,\sigma} + \delta^{\alpha}_{\sigma}C_{\widehat{\mu|\nu}} - \delta^{\alpha}_{\nu}C_{\widehat{\mu|\sigma}} \\
&\quad -\frac{1}{2(n-1)}\,\delta^{\alpha}_{\nu}C_{\mu}C_{\sigma} + \frac{1}{2(n-1)}\,\delta^{\alpha}_{\sigma}C_{\mu}C_{\nu}\Big\} = B^{\alpha}_{\mu\nu\sigma}.
\end{aligned}
$$

Now, let us define the connection:

$$
\widehat{\mathbf{\Gamma}}^{\alpha}_{\mu\nu} := \widehat{\Gamma}^{\alpha}_{\mu\nu} - \frac{1}{2(n-1)}(\delta^{\alpha}_{\mu}C_{\nu} + \delta^{\alpha}_{\nu}C_{\mu}). \tag{1.3.3}
$$

One can easily show that this connection is conformal. We now prove that the tensor $B^{\alpha}_{\mu\nu\sigma}$ is the curvature tensor of the conformal connection $\widehat{\mathbf{\Gamma}}^{\alpha}_{\mu\nu}$:

$$
\begin{aligned}
&\mathfrak{A}_{\nu\sigma}\left\{\widehat{\mathbf{\Gamma}}^{\alpha}_{\mu\sigma,\nu} + \widehat{\mathbf{\Gamma}}^{\epsilon}_{\mu\sigma}\widehat{\mathbf{\Gamma}}^{\alpha}_{\epsilon\nu}\right\} \\
&= \left(\widehat{\Gamma}^{\alpha}_{\mu\sigma} - \frac{1}{2(n-1)}\left(\delta^{\alpha}_{\mu}C_{\sigma} + \delta^{\alpha}_{\sigma}C_{\mu}\right)\right)_{,\nu} - \left(\widehat{\Gamma}^{\alpha}_{\mu\nu} - \frac{1}{2(n-1)}\left(\delta^{\alpha}_{\mu}C_{\nu} + \delta^{\alpha}_{\nu}C_{\mu}\right)\right)_{,\sigma} \\
&\quad + \left(\widehat{\Gamma}^{\epsilon}_{\mu\sigma} - \frac{1}{2(n-1)}\left(\delta^{\epsilon}_{\mu}C_{\sigma} + \delta^{\epsilon}_{\sigma}C_{\mu}\right)\right)\left(\widehat{\Gamma}^{\alpha}_{\epsilon\nu} - \frac{1}{2(n-1)}\left(\delta^{\alpha}_{\epsilon}C_{\nu} + \delta^{\alpha}_{\nu}C_{\epsilon}\right)\right) \\
&\quad - \left(\widehat{\Gamma}^{\epsilon}_{\mu\nu} - \frac{1}{2(n-1)}\left(\delta^{\epsilon}_{\mu}C_{\nu} + \delta^{\epsilon}_{\nu}C_{\mu}\right)\right)\left(\widehat{\Gamma}^{\alpha}_{\epsilon\sigma} - \frac{1}{2(n-1)}\left(\delta^{\alpha}_{\epsilon}C_{\sigma} + \delta^{\alpha}_{\sigma}C_{\epsilon}\right)\right) \\
&= \widehat{\Gamma}^{\alpha}_{\mu\sigma,\nu} - \widehat{\Gamma}^{\alpha}_{\mu\nu,\sigma} + \widehat{\Gamma}^{\epsilon}_{\mu\sigma}\widehat{\Gamma}^{\alpha}_{\epsilon\nu} - \widehat{\Gamma}^{\epsilon}_{\mu\nu}\widehat{\Gamma}^{\alpha}_{\epsilon\sigma} \\
&\quad -\frac{1}{2(n-1)}\left(\delta^{\alpha}_{\mu}C_{\sigma,\nu} + \delta^{\alpha}_{\sigma}C_{\mu,\nu}\right) + \frac{1}{2(n-1)}\left(\delta^{\alpha}_{\mu}C_{\nu,\sigma} + \delta^{\alpha}_{\nu}C_{\mu,\sigma}\right) \\
&\quad -\frac{1}{2(n-1)}\left(\widehat{\Gamma}^{\alpha}_{\mu\sigma}C_{\nu} + \delta^{\alpha}_{\nu}\widehat{\Gamma}^{\epsilon}_{\mu\sigma}C_{\epsilon}\right) - \frac{1}{2(n-1)}\left(\widehat{\Gamma}^{\alpha}_{\mu\nu}C_{\sigma} + \widehat{\Gamma}^{\alpha}_{\sigma\nu}C_{\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4(n-1)^2}\left(\delta^\alpha_\mu C_\nu C_\sigma+\delta^\alpha_\nu C_\mu C_\sigma+\delta^\alpha_\sigma C_\nu C_\mu+\delta^\alpha_\nu C_\sigma C_\mu\right)\\
&+\frac{1}{2(n-1)}\left(\widehat{\Gamma}^\alpha_{\mu\nu}C_\sigma+\delta^\alpha_\sigma\widehat{\Gamma}^\epsilon_{\mu\nu}C_\epsilon\right)+\frac{1}{2(n-1)}\left(\widehat{\Gamma}^\alpha_{\mu\sigma}C_\nu+\widehat{\Gamma}^\alpha_{\nu\sigma}C_\mu\right)\\
&-\frac{1}{4(n-1)^2}\left(\delta^\alpha_\mu C_\sigma C_\nu+\delta^\alpha_\sigma C_\mu C_\nu+\delta^\alpha_\nu C_\sigma C_\mu+\delta^\alpha_\sigma C_\nu C_\mu\right)\\
= \quad &\widehat{R}^\alpha_{\mu\nu\sigma}-\frac{1}{2(n-1)}\Bigg\{\delta^\alpha_\mu C_{\sigma,\nu}-\delta^\alpha_\mu C_{\nu,\sigma}+\delta^\alpha_\sigma C_{\mu\widehat{|}\nu}-\delta^\alpha_\nu C_{\mu\widehat{|}\sigma}\\
&-\frac{1}{2(n-1)}\,\delta^\alpha_\nu C_\mu C_\sigma+\frac{1}{2(n-1)}\,\delta^\alpha_\sigma C_\mu C_\nu\Bigg\}=B^\alpha_{\mu\nu\sigma}.
\end{aligned}
$$

This completes the proof. □

The following lemma will be needed for the proof of the next theorem below.

Lemma 1.3.4. *Under the conformal change* (1.2.1), *we have*

(a) $\overline{C}_\sigma = C_\sigma + (n-1)\rho_\sigma\,,\ \overline{C}^\sigma = e^{-2\rho(x)}\left(C^\sigma + (n-1)\rho^\sigma\right)$

(b) $\overline{C}^2 = e^{-2\rho(x)}\left(C^2 + 2(n-1)C_\epsilon\rho^\epsilon + (n-1)^2\rho^2\right)$

(c) $\overline{C}_{\mu \overset{\circ}{\|} \nu} = C_{\mu\overset{\circ}{|}\nu} + (n-1)\rho_{\mu\overset{\circ}{|}\nu} - (C_\mu\rho_\nu + C_\nu\rho_\mu - g_{\mu\nu}C_\epsilon\rho^\epsilon) - (n-1)\left(2\rho_\mu\rho_\nu - g_{\mu\nu}\rho^2\right)$

(d) $\overline{C}^\alpha_{\ \overset{\circ}{\|}\nu} = e^{-2\rho(x)}\{C^\alpha_{\ \overset{\circ}{|}\nu} + (n-1)\rho^\alpha_{\ \overset{\circ}{|}\nu} + (\delta^\alpha_\nu C^\epsilon\rho_\epsilon - C^\alpha\rho_\nu - C_\nu\rho^\alpha) + (n-1)\left(\delta^\alpha_\nu\rho^2 - 2\rho^\alpha\rho_\nu\right)\}$,

where $C^\alpha := g^{\alpha\epsilon}C_\epsilon$, $\overline{C}^2 := \overline{C}_\epsilon\overline{C}^\epsilon$. *Here* $\overset{o}{\|}$ *denotes the covariant derivative with respect to the connection* $\overline{\overset{\circ}{\Gamma}}{}^\alpha_{\mu\nu}$.

Theorem 1.3.5. *Let* $(M, \underset{i}{\lambda})$ *be an AP-space of dimension* $n \geq 2$. *The tensor*

$$
\begin{aligned}
Q^\alpha_{\mu\nu\sigma} \;:=\; &\mathfrak{U}_{\nu\sigma}\left\{\gamma^\alpha_{\mu\nu|\sigma}+\gamma^\epsilon_{\mu\sigma}\gamma^\alpha_{\epsilon\nu}+\frac{1}{2}\gamma^\alpha_{\mu\epsilon}\Lambda^\epsilon_{\nu\sigma}\right\}-\frac{1}{(n-1)}\mathfrak{U}_{\nu\sigma}\Bigg\{\delta^\alpha_\mu C_{\sigma,\nu}+\delta^\alpha_\sigma C_{\mu\overset{\circ}{|}\nu}+g_{\mu\sigma}C^\alpha_{\ \overset{\circ}{|}\nu}\\
&-\frac{1}{(n-1)}(\delta^\alpha_\nu C_\mu C_\sigma-\delta^\alpha_\nu g_{\mu\sigma}C^2+g_{\mu\sigma}C_\nu C^\alpha)\Bigg\}
\end{aligned}
$$

is conformally invariant. Moreover, $Q^\alpha_{\mu\nu\sigma}$ *is precisely the curvature tensor of a conformal connection on* M.

Proof. From Lemma 1.3.4 and Theorem 1.1.2**(c)**, after some calculations, we have

$$
\begin{aligned}
\overline{Q}^{\alpha}_{\ \mu\nu\sigma} &= \overline{\overset{o}{R}}{}^{\alpha}_{\ \mu\nu\sigma} - \frac{1}{(n-1)}\Big\{ \delta^{\alpha}_{\mu}\overline{C}_{\sigma,\nu} - \delta^{\alpha}_{\mu}\overline{C}_{\nu,\sigma} + \delta^{\alpha}_{\sigma}\overline{C}_{\underset{\mu\|\nu}{o}} - \delta^{\alpha}_{\nu}\overline{C}_{\underset{\mu\|\sigma}{o}} + \overline{g}_{\mu\sigma}\overline{C}^{\alpha}_{\underset{\|\nu}{o}} - \overline{g}_{\mu\nu}\overline{C}^{\alpha}_{\underset{\|\sigma}{o}} \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu}\overline{C}_{\mu}\overline{C}_{\sigma} + \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma}\overline{C}_{\mu}\overline{C}_{\nu} + \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu}\overline{g}_{\mu\sigma}\overline{C}^{2} \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma}\overline{g}_{\mu\nu}\overline{C}^{2} - \frac{1}{(n-1)}\overline{g}_{\mu\sigma}\overline{C}^{\alpha}\overline{C}_{\nu} + \frac{1}{(n-1)}\overline{g}_{\mu\nu}\overline{C}^{\alpha}\overline{C}_{\sigma}\Big\} \\
&= \overset{o}{R}{}^{\alpha}_{\ \mu\nu\sigma} + \delta^{\alpha}_{\sigma}\left(\rho_{\underset{\mu|\nu}{o}} - \rho_{\mu}\rho_{\nu} + \frac{1}{2}g_{\mu\nu}\rho^{2}\right) - \delta^{\alpha}_{\nu}\left(\rho_{\underset{\mu|\sigma}{o}} - \rho_{\mu}\rho_{\sigma} + \frac{1}{2}g_{\mu\sigma}\rho^{2}\right) \\
&\quad - g_{\mu\sigma}\left(\rho^{\alpha}_{\underset{|\nu}{o}} - \rho^{\alpha}\rho_{\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho^{2}\right) + g_{\mu\nu}\left(\rho^{\alpha}_{\underset{|\sigma}{o}} - \rho^{\alpha}\rho_{\sigma} + \frac{1}{2}\delta^{\alpha}_{\sigma}\rho^{2}\right) \\
&\quad - \frac{1}{(n-1)}\Big\{ \delta^{\alpha}_{\mu}(C_{\sigma,\nu} + (n-1)\rho_{\sigma,\nu}) - \delta^{\alpha}_{\mu}(C_{\nu,\sigma} + (n-1)\rho_{\nu,\sigma}) \\
&\quad + \delta^{\alpha}_{\sigma}\left[C_{\underset{\mu|\nu}{o}} - (C_{\mu}\rho_{\nu} + C_{\nu}\rho_{\mu} - g_{\mu\nu}C_{\epsilon}\rho^{\epsilon}) - (n-1)\left(2\rho_{\mu}\rho_{\nu} - g_{\mu\nu}\rho^{2} - \rho_{\underset{\mu|\nu}{o}}\right)\right] \\
&\quad - \delta^{\alpha}_{\nu}\left[C_{\underset{\mu|\sigma}{o}} - (C_{\mu}\rho_{\sigma} + C_{\sigma}\rho_{\mu} - g_{\mu\sigma}C_{\epsilon}\rho^{\epsilon}) - (n-1)\left(2\rho_{\mu}\rho_{\sigma} - g_{\mu\sigma}\rho^{2} - \rho_{\underset{\mu|\sigma}{o}}\right)\right] \\
&\quad + g_{\mu\sigma}\left[C^{\alpha}_{\underset{|\nu}{o}} + (\delta^{\alpha}_{\nu}C^{\epsilon}\rho_{\epsilon} - C^{\alpha}\rho_{\nu} - C_{\nu}\rho^{\alpha}) + (n-1)\left(\delta^{\alpha}_{\nu}\rho^{2} - 2\rho^{\alpha}\rho_{\nu} + \rho^{\alpha}_{\underset{|\nu}{o}}\right)\right] \\
&\quad - g_{\mu\nu}\left[C^{\alpha}_{\underset{|\sigma}{o}} + (\delta^{\alpha}_{\sigma}C^{\epsilon}\rho_{\epsilon} - C^{\alpha}\rho_{\sigma} - C_{\sigma}\rho^{\alpha}) + (n-1)\left(\delta^{\alpha}_{\sigma}\rho^{2} - 2\rho^{\alpha}\rho_{\sigma} + \rho^{\alpha}_{\underset{|\sigma}{o}}\right)\right] \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu}(C_{\mu} + (n-1)\rho_{\mu})(C_{\sigma} + (n-1)\rho_{\sigma}) \\
&\quad + \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma}(C_{\mu} + (n-1)\rho_{\mu})(C_{\nu} + (n-1)\rho_{\nu}) \\
&\quad + \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu}g_{\mu\sigma}\left(C^{2} + 2(n-1)C_{\epsilon}\rho^{\epsilon} + (n-1)^{2}\rho^{2}\right) \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma}g_{\mu\nu}\left(C^{2} + 2(n-1)C_{\epsilon}\rho^{\epsilon} + (n-1)^{2}\rho^{2}\right) \\
&\quad - \frac{1}{(n-1)}g_{\mu\sigma}(C^{\alpha} + (n-1)\rho^{\alpha})(C_{\nu} + (n-1)\rho_{\nu}) \\
&\quad + \frac{1}{(n-1)}g_{\mu\nu}(C^{\alpha} + (n-1)\rho^{\alpha})(C_{\sigma} + (n-1)\rho_{\sigma})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \overset{o}{R}{}^{\alpha}_{\ \mu\nu\sigma} - \frac{1}{(n-1)}\Big\{ \delta^{\alpha}_{\mu} C_{\sigma,\nu} - \delta^{\alpha}_{\mu} C_{\nu,\sigma} + \delta^{\alpha}_{\sigma} C_{\underset{\mu|\nu}{\circ}} - \delta^{\alpha}_{\nu} C_{\underset{\mu|\sigma}{\circ}} + g_{\mu\sigma} C^{\alpha}_{\underset{|\nu}{\circ}} - g_{\mu\nu} C^{\alpha}_{\underset{|\sigma}{\circ}} \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu} C_{\mu} C_{\sigma} + \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma} C_{\mu} C_{\nu} + \frac{1}{(n-1)}\,\delta^{\alpha}_{\nu} g_{\mu\sigma} C^{2} \\
&\quad - \frac{1}{(n-1)}\,\delta^{\alpha}_{\sigma} g_{\mu\nu} C^{2} - \frac{1}{(n-1)} g_{\mu\sigma} C^{\alpha} C_{\nu} + \frac{1}{(n-1)} g_{\mu\nu} C^{\alpha} C_{\sigma} \Big\} = Q^{\ \alpha}_{\mu\nu\sigma}.
\end{aligned}
$$

Hence, $Q^{\ \alpha}_{\mu\nu\sigma}$ is a conformally invariant tensor.

Now, we define the following connection

$$
\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \mu\nu} := \overset{o}{\Gamma}{}^{\alpha}_{\ \mu\nu} - \frac{1}{(n-1)}(\delta^{\alpha}_{\mu} C_{\nu} + \delta^{\alpha}_{\nu} C_{\mu} - g_{\mu\nu} C^{\alpha}). \tag{1.3.4}
$$

One can easily show that this connection is a conformal connection. Now, we show that the given tensor $Q^{\ \alpha}_{\mu\nu\sigma}$ is precisely the curvature tensor of the conformal connection $\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \mu\nu}$:

$$
\begin{aligned}
&\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \mu\sigma,\nu} - \overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \mu\nu,\sigma} + \overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\epsilon}_{\ \mu\sigma}\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \epsilon\nu} - \overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\epsilon}_{\ \mu\nu}\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\ \epsilon\sigma} \\
&= \left(\overset{o}{\Gamma}{}^{\alpha}_{\ \mu\sigma} - \frac{1}{(n-1)}(\delta^{\alpha}_{\mu} C_{\sigma} + \delta^{\alpha}_{\sigma} C_{\mu} - g_{\mu\sigma} C^{\alpha}) \right)_{,\nu} \\
&\quad - \left(\overset{o}{\Gamma}{}^{\alpha}_{\ \mu\nu} - \frac{1}{(n-1)}(\delta^{\alpha}_{\mu} C_{\nu} + \delta^{\alpha}_{\nu} C_{\mu} - g_{\mu\nu} C^{\alpha}) \right)_{,\sigma} \\
&\quad + \left(\overset{o}{\Gamma}{}^{\epsilon}_{\ \mu\sigma} - \frac{1}{(n-1)}(\delta^{\epsilon}_{\mu} C_{\sigma} + \delta^{\epsilon}_{\sigma} C_{\mu} - g_{\mu\sigma} C^{\epsilon}) \right) \left(\overset{o}{\Gamma}{}^{\alpha}_{\ \epsilon\nu} - \frac{1}{(n-1)}(\delta^{\alpha}_{\epsilon} C_{\nu} + \delta^{\alpha}_{\nu} C_{\epsilon} - g_{\epsilon\nu} C^{\alpha}) \right) \\
&\quad - \left(\overset{o}{\Gamma}{}^{\epsilon}_{\ \mu\nu} - \frac{1}{(n-1)}(\delta^{\epsilon}_{\mu} C_{\nu} + \delta^{\epsilon}_{\nu} C_{\mu} - g_{\mu\nu} C^{\epsilon}) \right) \left(\overset{o}{\Gamma}{}^{\alpha}_{\ \epsilon\sigma} - \frac{1}{(n-1)}(\delta^{\alpha}_{\epsilon} C_{\sigma} + \delta^{\alpha}_{\sigma} C_{\epsilon} - g_{\epsilon\sigma} C^{\alpha}) \right) \\
&= \overset{o}{\Gamma}{}^{\alpha}_{\ \mu\sigma,\nu} - \overset{o}{\Gamma}{}^{\alpha}_{\ \mu\nu,\sigma} + \overset{o}{\Gamma}{}^{\epsilon}_{\ \mu\sigma}\overset{o}{\Gamma}{}^{\alpha}_{\ \epsilon\nu} - \overset{o}{\Gamma}{}^{\epsilon}_{\ \mu\nu}\overset{o}{\Gamma}{}^{\alpha}_{\ \epsilon\sigma} \\
&\quad - \frac{1}{(n-1)}(\delta^{\alpha}_{\mu} C_{\sigma,\nu} + \delta^{\alpha}_{\sigma} C_{\mu,\nu} - g_{\mu\sigma,\nu} C^{\alpha} - g_{\mu\sigma} C^{\alpha}_{,\nu}) \\
&\quad + \frac{1}{(n-1)}(\delta^{\alpha}_{\mu} C_{\nu,\sigma} + \delta^{\alpha}_{\nu} C_{\mu,\sigma} - g_{\mu\nu,\sigma} C^{\alpha} - g_{\mu\nu} C^{\alpha}_{,\sigma}) \\
&\quad - \frac{1}{(n-1)}(\Gamma^{\alpha}_{\mu\sigma} C_{\nu} + \delta^{\alpha}_{\nu} \Gamma^{\epsilon}_{\mu\sigma} C_{\epsilon} - g_{\epsilon\nu} \Gamma^{\epsilon}_{\mu\sigma} C^{\alpha}) - \frac{1}{(n-1)}(\Gamma^{\alpha}_{\mu\nu} C_{\sigma} + \Gamma^{\alpha}_{\sigma\nu} C_{\mu} - g_{\mu\sigma} \Gamma^{\alpha}_{\epsilon\nu} C^{\epsilon}) \\
&\quad + \frac{1}{(n-1)^2}(\delta^{\alpha}_{\mu} C_{\nu} C_{\sigma} + \delta^{\alpha}_{\nu} C_{\mu} C_{\sigma} - g_{\mu\nu} C_{\sigma} C^{\alpha} + \delta^{\alpha}_{\sigma} C_{\nu} C_{\mu} + \delta^{\alpha}_{\nu} C_{\sigma} C_{\mu} - g_{\sigma\nu} C_{\mu} C^{\alpha} \\
&\quad - g_{\mu\sigma} C^{\alpha} C_{\nu} - \delta^{\alpha}_{\nu} g_{\mu\sigma} C^{2} + g_{\mu\sigma} C^{\alpha} C_{\nu}) \\
&\quad + \frac{1}{(n-1)}(\Gamma^{\alpha}_{\mu\nu} C_{\sigma} + \delta^{\alpha}_{\sigma} \Gamma^{\epsilon}_{\mu\nu} C_{\epsilon} - g_{\epsilon\sigma} \Gamma^{\epsilon}_{\mu\nu} C^{\alpha}) + \frac{1}{(n-1)}(\Gamma^{\alpha}_{\mu\sigma} C_{\nu} + \Gamma^{\alpha}_{\nu\sigma} C_{\mu} - g_{\mu\nu} \Gamma^{\alpha}_{\epsilon\sigma} C^{\epsilon})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{(n-1)^2}(\delta^\alpha_\mu C_\sigma C_\nu + \delta^\alpha_\sigma C_\mu C_\nu - g_{\mu\sigma} C_\nu C^\alpha + \delta^\alpha_\nu C_\sigma C_\mu + \delta^\alpha_\sigma C_\nu C_\mu - g_{\nu\sigma} C_\mu C^\alpha \\
&-g_{\mu\nu} C^\alpha C_\sigma - \delta^\alpha_\sigma g_{\mu\nu} C^2 + g_{\mu\nu} C^\alpha C_\sigma) \\
= {}& \overset{o}{R}{}^{\ \alpha}_{\mu\nu\sigma} - \frac{1}{(n-1)}\Big\{ \delta^\alpha_\mu C_{\sigma,\nu} - \delta^\alpha_\mu C_{\nu,\sigma} + \delta^\alpha_\sigma C_{\underset{\mu|\nu}{\circ}} - \delta^\alpha_\nu C_{\underset{\mu|\sigma}{\circ}} + g_{\mu\sigma} C^\alpha_{\underset{|\nu}{\circ}} - g_{\mu\nu} C^\alpha_{\underset{|\sigma}{\circ}} \\
&-\frac{1}{(n-1)}\,\delta^\alpha_\nu C_\mu C_\sigma + \frac{1}{(n-1)}\,\delta^\alpha_\sigma C_\mu C_\nu + \frac{1}{(n-1)}\,\delta^\alpha_\nu g_{\mu\sigma} C^2 \\
&-\frac{1}{(n-1)}\,\delta^\alpha_\sigma g_{\mu\nu} C^2 - \frac{1}{(n-1)} g_{\mu\sigma} C^\alpha C_\nu + \frac{1}{(n-1)} g_{\mu\nu} C^\alpha C_\sigma \Big\} = Q^{\ \alpha}_{\mu\nu\sigma}.
\end{aligned}
$$

This completes the proof. □

Remark 1.3.6. Some of the properties enjoyed by the new invariant connections are the following:

(a) The invariant connection $\mathbf{\Gamma}^\alpha_{\mu\nu}$ defined by (1.3.2) is non-metric and non-symmetric. In more detail, we have

$$
g_{\mu\nu||\sigma} = \frac{2}{n-1}\, g_{\mu\nu}\, C_\sigma,
$$

i.e. the metric is a recurrent metric (with respect to $\mathbf{\Gamma}^\alpha_{\mu\nu}$) with recurrence form $\frac{2}{n-1}\, C_\sigma$.

(b) For the invariant connection $\widehat{\mathbf{\Gamma}}^\alpha_{\mu\nu}$ (1.3.3), we have:

$$
g_{\mu\nu\widehat{||}\sigma} = g_{\mu\nu\widehat{|}\sigma} + \frac{1}{n-1}\, g_{\mu\nu}\, C_\sigma + \frac{1}{2(n-1)}\, (g_{\mu\sigma}\, C_\nu + g_{\nu\sigma}\, C_\mu)\,.
$$

(c) The invariant connection $\overset{\mathbf{o}}{\mathbf{\Gamma}}{}^{\alpha}_{\mu\nu}$ defined by (1.3.4) is non-metric, symmetric and recurrent metric with recurrence form $\frac{2}{n-1}\, C_\sigma$. That is,

$$
g_{\underset{\mu\nu\|\sigma}{\circ}} = \frac{2}{n-1}\, g_{\mu\nu}\, C_\sigma.
$$

We end this chapter with the following remarks.

Concluding Remarks

In this chapter we have studied conformally invariant geometric objects in AP-space from a purely mathematical point of view. The physical meaning of these entities was however not discussed. Because both AP-geometry and conformal transformations have proved to be useful in connection to general relativity, it is possible and even likely that our conformal geometric objects should acquire some physical significance beyond their mathematical aspects.

This work is not an end in itself. A natural continuation of this work is the study of the the Ricci-tensors and the first and second Bianichi identities of the new invariant connections. Another point worth investigating is the consequences of the connections (1.3.2) and (1.3.4) being recurrent metric. A global approach to the notion of conformal changes in an AP-space would be also a natural extension of the present work (with the aid of [113]). We intend to deal with these issues in the near future.

2

Connections in Sub-Riemannian Geometry of Parallelizable Distributions

Sub-Riemannian Geometry [1, 25, 61, 92, 93] has many applications such as diffusion, mechanics, gauge theories and control theory [12,45,82], among other fields. Absolute parallelism geometry or the geometry of parallelizable manifolds [17, 102, 113, 114] is frequently used for applications in physics, especially in the geometrization of physical theories such as general relativity [40, 78, 90, 103, 105].

Several attempts [14, 53, 54] have been made to construct a connection theory in sub-Riemannian geometry. Our approach is different. We define a parallelizable distribution (PD) on a finite dimensional manifold M. A non-integrable PD on M carries simultaneously two structures: an absolute parallelism structure and a sub-Riemannian structure. We make use of both structures to build up some new connections in a sub-Riemannian setting.

The wide spectrum of applications of both sub-Riemannian geometry and absolute parallelism geometry makes our approach, which enjoys the advantages of both geometries, a potential candidate for many applications in different fields of mathematical physics.

The aim of this chapter is to use the results of [113] to give a sub-Riemannian formulation of AP-geometry. As should be expected, the passage from the Riemannain to sub-Riemannian framework complicates matters. First of all, some notions had to be redefined in order to be compatible with the new sub-Riemannian language. Secondly, some results are no longer true or require different proofs. Finally and more importantly, new relations emerged that have no analogue in [113]. Obviously, our results reduce to these obtained in [113] when the frame is defined on all of M, that is when the distribution coincides with TM.

In the present chapter, we shall deal with the following items:

2.1. A short note on sub-Riemannian geometry

2.2. Parallelizable distribution and Sub-Riemannian structure

2.3. Linear connections on parallelizable distributions

2.4. Examples of sub-Riemannian parallelizable spaces

The main results of this chapter have been published [117] in: International Journal of Geometric Methods in Modern Physics, **14** (2017), 1750039 (11 pages). DOI: 10.1142/S0219887817500396. arXiv: 1603.06106 [math.DG].

2.1 A short note on sub-Riemannian geometry

Sub-Riemannian geometry is roughly the study of the geometry of a smooth manifold M equipped with a smoothly varying positive definite metric defined on a sub-bundle D of the tangent bundle TM. Here D is assumed to be bracket generating.

We first give some fundamental definitions and important theorem concerning sub-Riemannian geometry. For a more detailed exposition of sub-Riemannian geometry, we refer to $[1, 25, 61, 92, 93]$. By a manifold M, we shall mean an n-dimensional smooth manifold.

Definition 2.1.1. *A distribution of rank k on a manifold M is a map D that assigns to each point $p \in M$ a k-dimensional subspace D_p of T_pM. A distribution D may be regarded as a vector sub-bundle $(\bigcup_{p\in M} D_p) \longrightarrow M$ of the tangent bundle $TM \longrightarrow M$. A distribution D of rank k is differentiable if every point $p \in M$ has a neighborhood U and smooth k-vector fields $Y_1, \ldots, Y_k$ on U such that $Y_1(q), \ldots, Y_k(q)$ form a basis of D_q for all $q \in U$, i.e., $D = \mathrm{Span}\{Y_1, \ldots, Y_k\}$ on U.*

We shall always deal with differentiable distributions.

Definition 2.1.2. *A distribution D on M is integrable if it admits a unique maximal integral manifold through each point of M. A distribution D is involutive if $[X, Y] \in D$ for all $X, Y \in D$.*

According to Frobenius theorem, a distribution D is integrable if and only if it is involutive.

Definition 2.1.3. *A distribution D on M is bracket generating (BG) if there exists an integer $r \geq 1$ such that $D_p^r = T_pM$ for all $p \in M$, where*

$$\begin{aligned} D_p^1 &= D_p, \\ D_p^{s+1} &= D_p^s + [D_p, D_p^s], \text{ for } s \geq 1, \end{aligned}$$

and $[D_p, D_p^s] = \{[X, Y] : X \in D_p, Y \in D_p^s\}$.
The smallest integer r such that $D_p^r = T_pM$ is called the step of the distribution D. If r does not exist, we say that the distribution is of infinite step.

The above definition means that the tangent spaces T_pM; $p \in M$, are generated by a finite number of brackets, of which the first-order brackets are taken from D_p. Roughly speaking, the BG property of D is a necessary condition to recover the ambient space. As an illustration of the importance of this property, we mention the fact that for the case of BG distributions, any two points can be joined by piecewise horizontal curve.

Remark 2.1.4. **(a)** It should be noted that $\mathrm{rank}(D_p^r) = \dim(M)$.

(b) A distribution D on M that satisfies the bracket generating condition is also said to satisfy *Hörmander's condition* or *Chow's condition.*

(c) The step is a property of the distribution D and does not depend on the sub-Riemannian metric (see Def. 2.1.5).

(d) There are distributions where the step is the same for all points. They are called *constant-step distribution.*

Definition 2.1.5. *A sub-Riemannian metric on a distribution D is a map g that assigns to each point $p \in M$ a positive definite inner product $g_p : D_p \times D_p \longrightarrow \mathbb{R}$ and g_p varies smoothly.*

Definition 2.1.6. *A sub-Riemannian structure on M is a pair (D, g) where D is a non-integrable (bracket generating) distribution on M and g is a smooth sub-Riemannian metric on D. In this case, M is said to be a sub-Riemannian manifold and the distribution D is called the horizontal distribution.*

Definition 2.1.7. *A tangent vector v at $p \in M$ is a horizontal vector if $v \in D_p$. A horizontal vector field X is a section of the horizontal distribution; i.e.,*

$$X : M \ni p \longrightarrow X_p \in D_p.$$

The set of the horizontal vector fields on M will be denoted by $\Gamma(D)$.

We now recall the definitions of a horizontal curve, its length and the sub-Riemannian distance between two points on M.

Definition 2.1.8. (a) *A curve γ on M, $\gamma : [0, \tau] \to M$, is a* horizontal curve *if*

$$\dot\gamma(s) \in D_{\gamma(s)} \quad \forall\, s \in [0, \tau].$$

(b) *The length of a horizontal curve γ on M is defined by:*

$$\ell(\gamma) = \int_0^\tau g(\dot\gamma(s), \dot\gamma(s))^{1/2} ds.$$

(c) *The sub-Riemannian distance between p and q is defined by*

$$d_\gamma(p, q) = inf\{\ell(\gamma);\, \gamma\, is\ a\ horizontal\ curve\ joining\ p\ and\ q\}.$$

This distance is called Carnot-Carathéodory distance and denoted by d_{CC}.

In Riemannian geometry, "Whitehead Theorem" states that any two neighboring points of M can be joined by a unique geodesic of shortest length. This theorem does not necessarily hold in sub-Riemannian geometry. The corresponding theorem in the sub-Riemannian context is due to Chow.

Theorem 2.1.9. (*Chow's Theorems*)

(a) *If a smooth distribution satisfies Chow's condition at some point p, then any point q which is sufficiently close to p can be joined to p by a horizontal curve.*

(b) *For a BG-distribution, any two points can be joined by a piecewise horizontal curve.*

(c) *If M is connected and Chow's condition holds at any point, then the Carnot-Carathéodory distance d_{CC} is finite.*

2.2 Parallelizable distribution and Sub-Riemannian structure

In the following we shall deal with a certain type of differentiable distributions, namely, parallelizable distributions. Throughout this work, M will denote an n-dimensional smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M, $\mathfrak{X}(M)$ the $C^\infty(M)$-module of smooth vector fields on M and $\Gamma(D)$ the $C^\infty(M)$-module of smooth sections of a distribution D. Einstein summation convention will be applied to all pairs of repeated indices even when they are both down. We shall follow the notations and use the results of [113].

Definition 2.2.1. *A distribution HM on M of rank $k < n$ is said to be parallelizable if it admits k independent global smooth sections $\underset{1}{X}, \dots, \underset{k}{X} \in \Gamma(HM)$, called paralellization sections. Consequently, $HM = span\{\underset{i}{X} \mid i = 1, \dots, k\}$. A parallelizable distribution (PD) generated by $\underset{i}{X}$ will be denoted by $(M, HM, \underset{i}{X})$ or, simply, by $(HM, \underset{i}{X})$.*

Proposition 2.2.2. *Every parallelizable distribution admits a sub-Riemannian metric.*

Proof. Let $(HM, \underset{i}{X})$ be a parallelizable distribution of rank $k < n$ on M. Define the k differential 1-forms $\underset{i}{\Omega} : \Gamma(HM) \longrightarrow C^\infty(M)$ by

$$\underset{i}{\Omega}(\underset{j}{X}) = \delta_{ij}.$$

We call $\underset{i}{\Omega}$ the parallelization forms. Clearly, if $Y = Y^i \underset{i}{X} \in \Gamma(HM)$, then

$$\underset{i}{\Omega}(Y) = Y^i, \qquad \underset{i}{\Omega}(Y)\underset{i}{X} = Y. \tag{2.2.1}$$

The parallelization forms $\underset{i}{\Omega}$ are independent in the $C^\infty(M)$-module $\Gamma^*(HM)$. It is then easy to show that

$$g := \underset{i}{\Omega} \otimes \underset{i}{\Omega} \tag{2.2.2}$$

defines a sub-Riemannian metric on HM. Indeed, g is a symmetric tensor of type $(0,2)$ on HM. Moreover, for all $Y \in \Gamma(HM)$, we have

$$g(Y, Y) = (\underset{i}{\Omega} \otimes \underset{i}{\Omega})(Y, Y) = \sum_{i=1}^{k} \big(\underset{i}{\Omega}(Y)\big)^2 \geq 0,$$

$$g(Y, Y) = 0 \implies \sum_{i=1}^{k} \big(\underset{i}{\Omega}(Y)\big)^2 = 0 \implies \underset{i}{\Omega}(Y) = 0 \ \forall i \implies \underset{i}{\Omega}(Y)\underset{i}{X} = 0 \overset{(2.2.1)}{\implies} Y = 0.$$

Hence, g is a sub-Riemannian metric on HM. □

Corollary 2.2.3. *A non-integrable* (*bracket generating*) *parallelizable distribution* $(HM, \underset{i}{X})$ *defines a sub-Riemannian structure on* M *with the sub-Riemannian metric* g *defined by* (2.2.2).

It should be noted that the parallelization sections $\underset{i}{X}$ are g-orthonormal: $g(\underset{i}{X}, \underset{j}{X}) = \delta_{ij}$. Moreover, we have the following duality relation $g(\underset{i}{X}, Y) = \underset{i}{\Omega}(Y)$ for all $Y \in \Gamma(HM)$.

According to [92, 93], there exists a metric extension G for g that makes the split $TM = HM \oplus VM$ G-orthogonal, where $VM := (HM)^\perp$. This decomposition of TM induces two projectors $h : TM \to HM$ and $v : TM \to VM$, called the horizontal and vertical projectors, respectively. The projectors h and v are $C^\infty(M)$-linear with the properties $h^2 = h$, $v^2 = v$, $h \circ v = v \circ h = 0$ and $h + v = id_{TM}$. Throughout, the projector h will be used to transform vector fields on M to sections of D; in fact, for every $X \in \mathfrak{X}(M)$, we have $hX \in D$.

2.3 Linear connections on parallelizable distributions

In this section, we explore the natural sub-Riemannian structure associated with a parallelizable distribution $(HM, \underset{i}{X})$. We introduce and investigate two remarkable connections on HM, namely, the Weitzenböck connection and the sub-Riemannian connection. The later generalizes the Levi-Civita connection to the sub-Riemannian case. Moreover, two useful connections related to the Weitzenböck connection, namely, the symmetric and dual connections, are also introduced. We shall continue to follow the notations and use the results of [113].

Theorem 2.3.1. *Let $(HM, \underset{i}{X})$ be a parallelizable distribution of rank k on M. Then, there exists a unique linear connection ∇ on HM for which the parallelization sections $\underset{i}{X}$ are parallel:*

$$\nabla_Y \underset{i}{X} = 0 \quad \forall\, Y \in \mathfrak{X}(M). \tag{2.3.1}$$

Proof. To prove the uniqueness, assume that ∇ is a linear connection satisfying the condition $\nabla \underset{i}{X} = 0$. For all $Y \in \mathfrak{X}(M)$, $Z \in \Gamma(HM)$ we have, by (2.2.1) and (2.3.1),

$$\nabla_Y Z = \nabla_Y \big(\underset{i}{\Omega}(Z)\underset{i}{X}\big) = \underset{i}{\Omega}(Z)\nabla_Y \underset{i}{X} + \big(Y \cdot \underset{i}{\Omega}(Z)\big)\underset{i}{X} = \big(Y \cdot \underset{i}{\Omega}(Z)\big)\underset{i}{X}.$$

Hence, the connection ∇ is uniquely determined by the relation

$$\nabla_Y Z = \big(Y \cdot \underset{i}{\Omega}(Z)\big)\underset{i}{X}. \tag{2.3.2}$$

To prove the existence, let $\nabla : \mathfrak{X}(M) \times \Gamma(HM) \longrightarrow \Gamma(HM)$ be defined by (2.3.2). It is easy to show that ∇ is a linear connection on HM with the required properties. □

Definition 2.3.2. *The unique linear connection ∇ on HM defined by* (2.3.2) *will be called the Weitzenböck or canonical connection of $(HM, \underset{i}{X})$.*

We have the following immediate result:

Corollary 2.3.3. *The connection ∇ defined by* (2.3.2) *is metric: $\nabla g = 0$.*

Definition 2.3.4. *Let $(HM, \underset{i}{X})$ be a parallelizable distribution. Let ∇ be the Weitzenböck connection on $(HM, \underset{i}{X})$.*

(a) *The torsion tensor of the Weitzenböck connection is defined by:*

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Gamma(HM);$$
$$T(Y, Z) = \nabla_Y hZ - \nabla_Z hY - h[Y, Z].$$

(b) *The curvature tensor of the Weitzenböck connection is defined by:*

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(HM) \longrightarrow \Gamma(HM);$$
$$R(Y, Z)W := \nabla_Y \nabla_Z W - \nabla_Z \nabla_Y W - \nabla_{[Y, Z]} W.$$

The torsion and curvature tensors of an arbitrary linear connection on $(HM, \underset{i}{X})$ can be defined similarly.

The torsion tensor T of ∇ has the following properties:

$$T(\underset{i}{X}, \underset{j}{X}) = -h[\underset{i}{X}, \underset{j}{X}], \quad T(vY, \underset{j}{X}) = -h[vY, \underset{j}{X}], \quad T(vY, vZ) = -h[vY, vZ].$$

Because of the property (2.3.1), the curvature tensor of the Weitzenböck connection vanishes identically.

Definition 2.3.5. *Given a vector field $W \in \mathfrak{X}(M)$, the horizontal Lie derivative with respect to W of the metric tensor g is defined, for all $Y, Z \in \Gamma(HM)$, by:*

$$(\mathfrak{L}_W g)(Y, Z) := W.g(Y, Z) + g(h[Y, W], Z) + g(h[Z, W], Y).$$

It is clear that

$$(\mathfrak{L}_{fW} g)(Y, Z) = f(\mathfrak{L}_W g)(Y, Z) + (Y.f)\, g(hW, Z) + (Z.f)\, g(hW, Y).$$

As far as we know, there is no complete well-established theory of linear connections in sub-Riemannian geometry. There are some successful trials to construct a theorem similar to the fundamental theorem of Riemannian geometry.

In the next theorem, we provide a sub-Riemannian analogue of the Riemannian or the Levi-Civita connection. More precisely, we have

Theorem 2.3.6. *On any parallelizable distribution* $(M,\, HM, \underset{k}{X})$, *there exists a unique linear connection* $\overset{\circ}{\nabla}$, *called the sub-Riemannian connection (sR-connection), such that*

(a) $\overset{\circ}{\nabla}$ *is metric:* $(\overset{\circ}{\nabla}_W g)(Y,\, Z) = 0,\ \forall\, Y,\, Z \in \Gamma(HM),\, W \in \mathfrak{X}(M)$.

(b) $\overset{\circ}{T}(HM,\, HM) = 0$.

(c) $g(\overset{\circ}{T}(V,\, Y), Z) = g(\overset{\circ}{T}(V,\, Z), Y),\ \forall\, Y,\, Z \in \Gamma(HM),\, V \in \Gamma(VM)$.

Proof. Let $W \in \mathfrak{X}(M)$ and $Z \in \Gamma(HM)$. Set

$$\overset{\circ}{\nabla}_W Z = \widehat{\nabla}_W Z - \frac{1}{2}(\mathfrak{L}_{\underset{i}{X}} g)(hW, Z)\,\underset{i}{X} - \frac{1}{2} g(Z,\, h[vW,\, \underset{k}{X}])\underset{k}{X}, \tag{2.3.3}$$

where $\widehat{\nabla}_Y Z := \frac{1}{2}(\nabla_Y Z + \nabla_Z hY + h[Y, Z])$.

It is clear that $\overset{\circ}{\nabla}$ is a linear connection with the desired properties. For example, let us prove the property (c):

$$\begin{aligned}
\overset{\circ}{T}(V,\, \underset{k}{X}) &= \overset{\circ}{\nabla}_V \underset{k}{X} - h[V,\, \underset{k}{X}] = \widehat{\nabla}_V \underset{k}{X} - \frac{1}{2} g(\underset{k}{X},\, h[V, \underset{i}{X}])\underset{i}{X} - h[V,\, \underset{k}{X}] \\
&= \frac{1}{2}\Big\{\nabla_V \underset{k}{X} + \nabla_{\underset{k}{X}} hV + h[V,\, \underset{k}{X}] - g(\underset{k}{X},\, h[V, \underset{i}{X}])\underset{i}{X} - 2h[V,\, \underset{k}{X}]\Big\} \\
&= -\frac{1}{2}\Big\{g(\underset{k}{X},\, h[V, \underset{i}{X}])\underset{i}{X} + h[V,\, \underset{k}{X}]\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
g(\overset{\circ}{T}(V,\, \underset{k}{X}),\, \underset{j}{X}) &= -\frac{1}{2}\Big\{g(\underset{k}{X},\, h[V, \underset{i}{X}])g(\underset{i}{X},\, \underset{j}{X}) + g(h[V,\, \underset{k}{X}],\, \underset{j}{X})\Big\} \\
&= -\frac{1}{2}\Big\{g(\underset{k}{X},\, h[V, \underset{j}{X}]) + g(h[V,\, \underset{k}{X}],\, \underset{j}{X})\Big\} \\
&= g(\overset{\circ}{T}(V,\, \underset{j}{X}),\, \underset{k}{X}).
\end{aligned}$$

For the uniqueness, assume that $\overline{\nabla}$ is another linear connection satisfying (a), (b) and (c). Define the tensor $\psi(W,\, Z) := \overset{\circ}{\nabla}_W Z - \overline{\nabla}_W Z$. Then for $Y,\, Z \in \Gamma(HM)$, $V \in \Gamma(VM)$,

$$g(\psi(V,\, Z),\, Y) \overset{(a)}{=} -g(Z,\, \psi(V,\, Y)) = -g(Z,\, \overset{\circ}{T}(V,\, Y) - \overline{T}(V,\, Y))$$

$$\begin{aligned} &\overset{(c)}{=} -g(Y, \overset{\circ}{T}(V, Z) - \overline{T}(V, Z)) \\ &= -g(Y, \psi(V, Z)). \end{aligned}$$

On the other hand, for $X, Y, Z \in \Gamma(HM)$,

$$\begin{aligned} g(\psi(X, Z), Y) &\overset{(a)}{=} -g(Z, \psi(X, Y)) \overset{(b)}{=} -g(Z, \psi(Y, X)) \overset{(a)}{=} g(X, \psi(Y, Z)) \\ &\overset{(b)}{=} g(X, \psi(Z, Y)) \overset{(a)}{=} -g(Y, \psi(Z, X)) \overset{(b)}{=} -g(Y, \psi(X, Z)). \end{aligned}$$

Hence, $\psi(W, Z) := 0 \ \forall W \in \mathfrak{X}(M),\ Z \in \Gamma(HM)$, which completes the proof. □

Remark 2.3.7. **(a)** The non vanishing counterparts of the torsion tensor $\overset{\circ}{T}$ of $\overset{\circ}{\nabla}$ are given, for all $V, U \in \Gamma(VM)$, by

$$\begin{aligned} \overset{\circ}{T}(\underset{k}{X}, V) &= \frac{1}{2}\Big\{T(\underset{k}{X}, V) - g(h[\underset{k}{X}, V], \underset{i}{X})\underset{i}{X}\Big\}, \\ \overset{\circ}{T}(V, U) &= -h[V, U]. \end{aligned}$$

On the other hand, $\overset{\circ}{T}$ vanishes on the horizontal distribution. The connection $\overset{\circ}{\nabla}$ is thus a generalization of the Levi-Civita connection to the sub-Riemannian case. The advantage of formula (2.3.3) is that it gives the connection $\overset{\circ}{\nabla}$ an *explicit* form, contrary to the Levi-Civita connection.

(b) If, in particular, M is parallelizable ($k = n$), then the sR-connection is just the well known Levi-Civita connection of the parallelizable manifold M [113, 114].

Proposition 2.3.8. *For a parallelizable distribution* $(HM, \underset{i}{X})$ *of rank* k *on* M *there exist two other linear connections on* HM *associated with the Weitzenböck connection* ∇*:*

(a) *The dual connection* $\widetilde{\nabla}$ *given by*

$$\widetilde{\nabla}_Y Z := \nabla_Z hY + h[Y, Z]. \tag{2.3.4}$$

(b) *The symmetric connection* $\widehat{\nabla}$ *given by*

$$\widehat{\nabla}_Y Z := \frac{1}{2}(\nabla_Y Z + \nabla_Z hY + h[Y, Z]), \tag{2.3.5}$$

for all $Y \in \mathfrak{X}(M)$, $Z \in \Gamma(HM)$.

Some tedious but straightforward calculations give the following result.

Proposition 2.3.9. *Let* $\widetilde{\widetilde{\nabla}}$ *and* $\widehat{\nabla}$ *be the dual and symmetric connections, respectively.*

(a) *The torsion and curvature tensors of* $\widetilde{\widetilde{\nabla}}$ *are given respectively by:*

$$\widetilde{T}(Y,\,Z) = T(hY,\,hZ) - h[vY,\,vZ],$$

$$\begin{aligned}\widetilde{R}(Y,\,Z)\underset{i}{X} &= \underset{Y,Z,\underset{i}{X}}{\mathfrak{S}}\left\{T\big(Y,\,h[\underset{i}{X},\,Z]\big)\right\} + T(v[Y,\,Z],\,\underset{i}{X}) - T(Y,\,\nabla_{\underset{i}{X}}hZ) + T(Z,\,\nabla_{\underset{i}{X}}hY)\\ &\quad + \nabla_Y h[\underset{i}{X},\,Z] - \nabla_Z h[\underset{i}{X},\,Y] + \nabla_Y \nabla_{\underset{i}{X}} hZ - \nabla_Z \nabla_{\underset{i}{X}} hY.\end{aligned}$$

(b) *The torsion and curvature tensors of* $\widehat{\nabla}$ *are given respectively by:*

$$\widehat{T}(Y,\,Z) = \frac{1}{2}(T(Y,\,vZ) + T(vY,\,Z)),$$

$$\widehat{R}(Y,\,Z)\underset{i}{X} = \tfrac{1}{4}\Big\{\widetilde{R}(Y,\,Z)\underset{i}{X} + T([Y,\,Z],\,\underset{i}{X}) - \nabla_Y T(Z,\,\underset{i}{X}) - \nabla_Z T(Y,\,\underset{i}{X})\Big\}.$$

Corollary 2.3.10. *If* $[\underset{i}{X},\,\underset{j}{X}] \in VM,\ \ \forall i,j \in \{1,\ldots,k\}$*, then*

(a) $T(HM,\,HM) = 0$.

(b) $T(VM,\,VM) = \widetilde{T}(VM,\,VM) = \widehat{T}(VM,\,VM)$.

(c) $T(HM,\,VM) = 2\,\widehat{T}(HM,\,VM),\ \ \widetilde{T}(HM,\,VM) = 0$.

(d) $\widetilde{R}(\underset{i}{X},\,\underset{j}{X})\,\underset{k}{X} = T([\underset{i}{X},\,\underset{j}{X}],\,\underset{k}{X})$.

(e) $\widehat{R}(\underset{i}{X},\,\underset{j}{X})\,\underset{k}{X} = \tfrac{1}{2}\,T([\underset{i}{X},\,\underset{j}{X}],\,\underset{k}{X})$.

The next table summarizes the geometry of sub-Riemannian parallelizable distribution.

Table 2.1: Geometry of sub-Riemannian parallelizable distribution

Connection	$\{\underset{i}{X}\}_{i=1}^{k}$ parallel	Torsion	Curvature	Metricity
Weitzenböck ∇	Yes	$T(\underset{i}{X}, \underset{j}{X}) = -h[\underset{i}{X}, \underset{j}{X}]$ $T(\underset{i}{X}, vY) = h[\underset{i}{X}, vY]$ $T(vY, vZ) = -h[vY, vZ]$	0	metric
Dual $\widetilde{\nabla}$	No	$\widetilde{T}(\underset{i}{X}, \underset{j}{X}) = T(\underset{i}{X}, \underset{j}{X})$ $\widetilde{T}(\underset{i}{X}, vY) = 0$ $\widetilde{T}(vY, vZ) = T(vY, vZ)$	$\widetilde{R}$	non-metric
Symmetric $\widehat{\nabla}$	No	$\widehat{T}(\underset{i}{X}, \underset{j}{X}) = 0$ $\widehat{T}(\underset{i}{X}, vY) = \frac{1}{2}T(\underset{i}{X}, vY)$ $\widehat{T}(vY, vZ) = T(vY, vZ)$	$\widehat{R}$	non-metric
Sub-Riemannian $\overset{\circ}{\nabla}$	No	$\overset{\circ}{T}(\underset{i}{X}, \underset{j}{X})=0$ $\overset{\circ}{T}(\underset{i}{X}, vY) = \frac{1}{2}T(\underset{i}{X}, vY)$ $-\frac{1}{2}g(h[\underset{k}{X}, V], \underset{i}{X})\underset{i}{X}$ $\overset{\circ}{T}(vY, vZ) = T(vY, vZ)$	$\overset{\circ}{R}$	metric

2.4 Examples

In this section we shall apply our results to the two famous parallelizable spheres S^3 and S^7. The parallelizations are given in [75].

2.4.1 The sphere S^3

Consider the 3-sphere S^3 and let (y_0, y_1, y_2, y_3) be the coordinates on S^3. Consider the parallelization vector fields on S^3 given by [75]:

$$\begin{aligned}
\underset{1}{X} &= -y_2\,\partial_{y_0} + y_3\,\partial_{y_1} + y_0\,\partial_{y_2} - y_1\,\partial_{y_3},\\
\underset{2}{X} &= -y_3\,\partial_{y_0} - y_2\,\partial_{y_1} + y_1\,\partial_{y_2} + y_0\,\partial_{y_3},\\
\underset{3}{X} &= -y_1\,\partial_{y_0} + y_0\,\partial_{y_1} - y_3\,\partial_{y_2} + y_2\,\partial_{y_3} = (1/2)[\underset{1}{X},\ \underset{2}{X}].
\end{aligned}$$

Let $HM = \text{span}\{\underset{1}{X},\ \underset{2}{X}\}$ and $VM = \text{span}\{\underset{3}{X}\}$. The distribution HM is non-integrable and bracket generating of step 2. The parallelization forms associated with $\underset{1}{X},\ \underset{2}{X}$ are given by:

$$\underset{1}{\Omega} = -y_2\,dy_0 + y_3\,dy_1 + y_0\,dy_2 - y_1\,dy_3,$$

$$\underset{2}{\Omega} = -y_3\,dy_0 - y_2\,dy_1 + y_1\,dy_2 + y_0\,dy_3.$$

The sub-Riemannian metric of HM, defined by (2.2.2), is given by

$$\begin{aligned}
g \;=\;& (y_2^2 + y_3^2)\,({dy_0}^2 + {dy_1}^2) + (y_1^2 + y_0^2)\,({dy_2}^2 + {dy_3}^2)\\
&+2(y_0\,y_3 - y_1\,y_2)(dy_0\,dy_3 - dy_1\,dy_2) - 2(y_1\,y_3 - y_0\,y_2)(dy_0\,dy_2 - dy_1\,dy_3).
\end{aligned}$$

We have $g(HM, HM) = \langle HM, HM\rangle$ and $\langle HM, VM\rangle = 0$, where $\langle .\,,\,.\rangle$ is the usual inner product of $\mathbb{R}^4$.

- The Weitzenböck connection ∇ defined by (2.3.2) has coefficients

$$\nabla_Y \underset{1}{X} = \nabla_Y \underset{2}{X} = 0, \text{ where } \ Y = \underset{1}{X}, \underset{2}{X} \text{ or } \underset{3}{X}.$$

The torsion tensor T of ∇ is given by

$$T(\underset{1}{X}, \underset{2}{X}) = 0,\ \ T(\underset{1}{X}, \underset{3}{X}) = -2\,\underset{2}{X},\ \ T(\underset{2}{X}, \underset{3}{X}) = 2\,\underset{1}{X}$$

and the curvature tensor of ∇ vanish identically.

- The sR-connection defined by (2.3.3) has coefficients

$$\overset{\circ}{\nabla}_{\underset{i}{X}} \underset{j}{X} = 0, \;\; i,j \in \{1,2\}, \quad \overset{\circ}{\nabla}_{\underset{3}{X}} \underset{1}{X} = -3 \underset{2}{X}, \quad \overset{\circ}{\nabla}_{\underset{3}{X}} \underset{2}{X} = 3 \underset{1}{X}.$$

The non vanishing components of the torsion tensor $\overset{\circ}{T}$ of $\overset{\circ}{\nabla}$ are

$$\overset{\circ}{T}(\underset{1}{X}, \underset{3}{X}) = \underset{2}{X}, \quad \overset{\circ}{T}(\underset{2}{X}, \underset{3}{X}) = -\underset{1}{X}$$

and the non vanishing components of the curvature tensor $\overset{\circ}{R}$ of $\overset{\circ}{\nabla}$ are

$$\overset{\circ}{R}(\underset{1}{X}, \underset{2}{X}) \underset{1}{X} = 6 \underset{2}{X}, \quad \overset{\circ}{R}(\underset{1}{X}, \underset{2}{X}) \underset{2}{X} = -6 \underset{1}{X}.$$

- The dual connection $\widetilde{\nabla}$ defined by (2.3.4) has coefficients

$$\widetilde{\nabla}_{\underset{1}{X}} \underset{2}{X} = \widetilde{\nabla}_{\underset{2}{X}} \underset{1}{X} = 0, \quad \widetilde{\nabla}_{\underset{3}{X}} \underset{1}{X} = -2\underset{2}{X}, \quad \widetilde{\nabla}_{\underset{3}{X}} \underset{2}{X} = 2\underset{1}{X}.$$

The torsion tensor of $\widetilde{\nabla}$ vanishes identically. Moreover, the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{R}(\underset{1}{X}, \underset{2}{X})\underset{1}{X} = 4\underset{2}{X}, \;\; \widetilde{R}(\underset{1}{X}, \underset{2}{X})\underset{2}{X} = -4\underset{1}{X},$$
$$\widetilde{R}(\underset{3}{X}, \underset{i}{X})\underset{j}{X} = 0, \;\; i,j \in \{1,2\}.$$

- The symmetric connection $\widehat{\nabla}$ defined by (2.3.5) has coefficients

$$\widehat{\nabla}_{\underset{1}{X}} \underset{2}{X} = \widehat{\nabla}_{\underset{2}{X}} \underset{1}{X} = 0, \quad \widehat{\nabla}_{\underset{3}{X}} \underset{1}{X} = -\underset{2}{X}, \quad \widehat{\nabla}_{\underset{3}{X}} \underset{2}{X} = \underset{1}{X}.$$

The torsion tensor $\widehat{T}$ of $\widehat{\nabla}$ is given by

$$\widehat{T}(\underset{1}{X}, \underset{2}{X}) = 0, \quad \widehat{T}(\underset{1}{X}, \underset{3}{X}) = -\underset{2}{X}, \quad \widehat{T}(\underset{2}{X}, \underset{3}{X}) = \underset{1}{X}$$

and the non vanishing components of the curvature tensor $\widehat{R}$ of $\widehat{\nabla}$ are

$$\widehat{R}(\underset{1}{X}, \underset{2}{X}) \underset{1}{X} = 2 \underset{2}{X}, \quad \widehat{R}(\underset{1}{X}, \underset{2}{X}) \underset{2}{X} = -2 \underset{1}{X}.$$

The last computations can be summarized in the following table:

Table 2.2: Connections on the sphere S^3

Connection	Connection coefficients	Non vanishing torsions	Non vanishing curvatures	Metricity
Weitzenböck ∇ $\nabla_Y Z = (Y \cdot \Omega_i(Z)) X_i$	$\nabla_{X_i} X_j = 0$ $\nabla_V X_i = 0$	$T(V, X_1) = 2X_2$ $T(V, X_2) = -2X_1$	—	Yes
Dual $\widetilde{\nabla}$ $\widetilde{\nabla}_Y Z = \nabla_Z hY + h[Y, Z]$	$\widetilde{\nabla}_{X_i} X_j = 0$ $\widetilde{\nabla}_V X_1 = -2X_2$ $\widetilde{\nabla}_V X_2 = 2X_1$	—	$\widetilde{R}(X_1, X_2)X_1 = 4X_2$ $\widetilde{R}(X_1, X_2)X_2 = -4X_1$	No
Symmetric $\widehat{\nabla}$ $\widehat{\nabla}_Y Z = \frac{1}{2}(\nabla_Y Z + \widetilde{\nabla}_Y Z)$	$\widehat{\nabla}_{X_i} X_j = 0$ $\widehat{\nabla}_V X_1 = -X_2$ $\widehat{\nabla}_V X_2 = X_1$	$\widehat{T}(V, X_1) = X_2$ $\widehat{T}(V, X_2) = -X_1$	$\widehat{R}(X_1, X_2)X_1 = 2X_2$ $\widehat{R}(X_1, X_2)X_2 = -2X_1$	No
SR-Connection $\mathring{\nabla}$ $\mathring{\nabla}_Y Z = \widehat{\nabla}_Y Z - \frac{1}{2}(\mathfrak{L}_{X_i} g)(hY, Z)\, X_i$ $-\frac{1}{2} g(Z,\, h[vY,\, X_k])X_k$	$\mathring{\nabla}_{X_i} X_j = 0$ $\mathring{\nabla}_V X_1 = -3X_2$ $\mathring{\nabla}_V X_2 = 3X_1$	$\mathring{T}(V, X_1) = -X_2$ $\mathring{T}(V, X_2) = X_1$	$\mathring{R}(X_1, X_2)X_1 = 6X_2$ $\mathring{R}(X_1, X_2)X_2 = -6X_1$	Yes

Note that we have used the notation $V := X_3$.

Remark 2.4.1. From the above table, we deduce the following:

(a) $\widehat{\nabla} = \frac{1}{3}\mathring{\nabla} = \frac{1}{2}\widetilde{\nabla}, \quad \widehat{T} = -\mathring{T} = \frac{1}{2}T, \quad \widehat{R} = \frac{1}{3}\mathring{R} = \frac{1}{2}\widetilde{R}.$

(b) The restriction of the torsion tensor of ∇ on the horizontal distribution vanishes identically, while the restriction of the curvature tensor on the horizontal distribution gives the non-vanishing components. Both these properties occur for the four connections.

(c) It should be noted that $X_2 = \frac{1}{2}[X_1, X_3]$, $X_1 = \frac{1}{2}[X_2, X_3]$. This implies that the distributions $\text{span}\{X_1, X_3\}$ and $\text{span}\{X_2, X_3\}$ are also non-integrable bracket generating of step 2 and we can perform the same calculation as above for each of them. Consequently, there are exactly three distinct sub-Riemannian parallelizable structures on S^3.

2.4.2 The sphere S^7

Consider the 7-sphere S^7 and let $(y_0, y_1, \dots, y_7)$ be the coordinates on S^7. For each point $p \in S^7$, the vector fields $\underset{1}{X}, \dots, \underset{7}{X}$ form an orthonormal frame of T_pS^7, where [75]

$$\underset{1}{X} = -y_2\partial_{y_0} + y_3\partial_{y_1} + y_0\partial_{y_2} - y_1\partial_{y_3} - y_6\partial_{y_4} + y_7\partial_{y_5} + y_4\partial_{y_6} - y_5\partial_{y_7}$$

$$\underset{2}{X} = -y_3\partial_{y_0} - y_2\partial_{y_1} + y_1\partial_{y_2} + y_0\partial_{y_3} + y_7\partial_{y_4} + y_6\partial_{y_5} - y_5\partial_{y_6} - y_4\partial_{y_7}$$

$$\underset{3}{X} = -y_4\partial_{y_0} + y_5\partial_{y_1} + y_6\partial_{y_2} - y_7\partial_{y_3} + y_0\partial_{y_4} - y_1\partial_{y_5} - y_2\partial_{y_6} + y_3\partial_{y_7}$$

$$\underset{4}{X} = -y_5\partial_{y_0} - y_4\partial_{y_1} - y_7\partial_{y_2} - y_6\partial_{y_3} + y_1\partial_{y_4} + y_0\partial_{y_5} + y_3\partial_{y_6} + y_2\partial_{y_7}$$

$$\underset{5}{X} = -y_6\partial_{y_0} + y_7\partial_{y_1} - y_4\partial_{y_2} + y_5\partial_{y_3} + y_2\partial_{y_4} - y_3\partial_{y_5} + y_0\partial_{y_6} - y_1\partial_{y_7}$$

$$\underset{6}{X} = -y_7\partial_{y_0} - y_6\partial_{y_1} + y_5\partial_{y_2} + y_4\partial_{y_3} - y_3\partial_{y_4} - y_2\partial_{y_5} + y_1\partial_{y_6} + y_0\partial_{y_7}$$

$$\underset{7}{X} = -y_1\partial_{y_0} + y_0\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} - y_5\partial_{y_4} + y_4\partial_{y_5} - y_7\partial_{y_6} + y_6\partial_{y_7}$$

Let $HM = \text{Span}\{\underset{1}{X}, \dots, \underset{6}{X}\}$ and $VM = \text{Span}\{\underset{7}{X}\}$. Clearly, the distribution HM is non-integrable and bracket generating of step 2. The parallelization forms associated with $\{\underset{1}{X}, \dots, \underset{6}{X}\}$ are given by

$$\underset{1}{\Omega} = -y_2\,dy_0 + y_3\,dy_1 + y_0\,dy_2 - y_1\,dy_3 - y_6\,dy_4 + y_7\,dy_5 + y_4\,dy_6 - y_5\,dy_7$$

$$\underset{2}{\Omega} = -y_3\,dy_0 - y_2\,dy_1 + y_1\,dy_2 + y_0\,dy_3 + y_7\,dy_4 + y_6\,dy_5 - y_5\,dy_6 - y_4\,dy_7$$

$$\underset{3}{\Omega} = -y_4\,dy_0 + y_5\,dy_1 + y_6\,dy_2 - y_7\,dy_3 + y_0\,dy_4 - y_1\,dy_5 - y_2\,dy_6 + y_3\,dy_7$$

$$\underset{4}{\Omega} = -y_5\,dy_0 - y_4\,dy_1 - y_7\,dy_2 - y_6\,dy_3 + y_1\,dy_4 + y_0\,dy_5 + y_3\,dy_6 + y_2\,dy_7$$

$$\underset{5}{\Omega} = -y_6\,dy_0 + y_7\,dy_1 - y_4\,dy_2 + y_5\,dy_3 + y_2\,dy_4 - y_3\,dy_5 + y_0\,dy_6 - y_1\,dy_7$$

$$\underset{6}{\Omega} = -y_7\,dy_0 - y_6\,dy_1 + y_5\,dy_2 + y_4\,dy_3 - y_3\,dy_4 - y_2\,dy_5 + y_1\,dy_6 + y_0\,dy_7.$$

The metric g of HM, defined by (2.2.2), is given by

$$\begin{aligned} g \;=\; & (1-{y_0}^2-{y_1}^2)({dy_0}^2+{dy_1}^2)+(1-{y_2}^2-{y_3}^2)({dy_2}^2+{dy_3}^2)\\ & +(1-{y_4}^2-{y_5}^2)({dy_4}^2+{dy_5}^2)+(1-{y_6}^2-{y_7}^2)({dy_6}^2+{dy_7}^2)\\ & +2(-y_1\,y_6+y_7\,y_0)(dy_6\,dy_1-dy_7\,dy_0)+2(-y_3\,y_0+y_2\,y_1)(dy_3\,dy_0-dy_1\,dy_2)\\ & +2(-y_0\,y_5+y_1\,y_4)(dy_0\,dy_5-dy_4\,dy_1)-2(y_7\,y_1+y_6\,y_0)(dy_6\,dy_0+dy_7\,dy_1)\\ & -2(y_1\,y_3+y_2\,y_0)(dy_3\,dy_1-dy_0\,dy_2)-2(y_7\,y_5+y_6\,y_4)(dy_5\,dy_7+dy_4\,dy_6)\\ & -2(y_7\,y_3+y_2\,y_6)(dy_2\,dy_6-dy_3\,dy_7)+2(-y_3\,y_4+y_5\,y_2)(dy_4\,dy_3-dy_2\,dy_5)\\ & +2(-y_6\,y_5+y_7\,y_4)(dy_6\,dy_5-dy_4\,dy_7)+2(-y_2\,y_7+y_3\,y_6)(dy_2\,dy_7-dy_3\,dy_6)\\ & -2(y_3\,y_5+y_4\,y_2)(dy_2\,dy_4+dy_3\,dy_5)-2(y_1\,y_5+y_0\,y_4)(dy_4\,dy_0+dy_3\,dy_5) \end{aligned}$$

We have $g(HM,\,HM)=\langle HM,\,HM\rangle$, and $\langle HM,\,VM\rangle=0$, where $\langle .\,,\,.\rangle$ is the usual inner product of $\mathbb{R}^8$.

- The Weitzenböck connection ∇ defined by (2.3.2) has coefficients

$$\nabla_{\underset{i}{X}}\underset{j}{X}=0,\ \text{ for } 1\leq i\leq 7,\ 1\leq j\leq 6.$$

As an illustration, some components of the torsion tensor T of ∇ are given by

$$T(\underset{1}{X},\,\underset{7}{X})=2(y_3\partial_{y_0}+y_2\partial_{y_1}-y_1\partial_{y_2}-y_0\partial_{y_3}+y_7\partial_{y_4}+y_6\partial_{y_5}-y_5\partial_{y_6}-y_4\partial_{y_7}),$$

$$T(\underset{6}{X},\,\underset{7}{X})=2(-y_6\partial_{y_0}+y_7\partial_{y_1}+y_4\partial_{y_2}-y_5\partial_{y_3}-y_2\partial_{y_4}+y_3\partial_{y_5}+y_0\partial_{y_6}-y_1\partial_{y_7})$$

and the curvature tensor of ∇ vanishes identically.

- The sR-connection defined by (2.3.3) has the properties:

$$\overset{\circ}{\nabla}_{\underset{i}{X}}\underset{j}{X}=-\overset{\circ}{\nabla}_{\underset{j}{X}}\underset{i}{X},\quad \overset{\circ}{\nabla}_{\underset{7}{X}}\underset{i}{X}=\frac{3}{2}\,T(\underset{i}{X},\,\underset{7}{X}),\ \ 1\leq i,j\leq 6.$$

For the coefficients of $\overset{\circ}{\nabla}$, we have, for example,

$$\begin{aligned}\overset{\circ}{\nabla}_{\underset{1}{X}} \underset{2}{X} = -\overset{\circ}{\nabla}_{\underset{2}{X}} \underset{1}{X} &= 2(y_7^2 + y_6^2 + y_5^2 + y_4^2)(y_1\,\partial_{y_0} - y_0\,\partial_{y_1} + y_3\,\partial_{y_2} - y_2\,\partial_{y_3}) \\ &\quad +2(y_0^2 + y_1^2 + y_2^2 + y_3^2)(y_5\,\partial_{y_4} - y_4\,\partial_{y_5} + y_7\,\partial_{y_6} - y_6\,\partial_{y_7}).\end{aligned}$$

The non vanishing components of the torsion tensor $\overset{\circ}{T}$ of $\overset{\circ}{\nabla}$ are

$$\overset{\circ}{T}(\underset{i}{X}, \underset{7}{X}) = -\frac{1}{2}\, T(\underset{i}{X}, \underset{7}{X}), \;\; 1 \leq i \leq 6.$$

It is to be noted that the other components of T, the components of $\overset{\circ}{T}$ and those of $\overset{\circ}{R}$ have not been written. They have been computed using Maple program and are so long and complicated. Similarly, we did not write the computations for the two other connections and their related torsion and curvature tensors as they are too long and boring.

The following result is also proved using Maple.

Proposition 2.4.2. *The parallelization sections* $\underset{i}{X}$, $i = 1, \ldots, 6$, *are Killing sections:* $\mathcal{L}_{\underset{i}{X}}\, g = 0$.

The next table gives the different parallelizable distributions defined on S^7, where we use the notation: $X_{k\ell} := [\underset{k}{X}, \underset{\ell}{X}]$. This table provides some sort of classification of sub-Riemannian parallelizable distributions on S^7.

Table 2.3: Parallelizable distributions of S^7

HM spanned by	Rank of HM	Independent commutators	HM is bracket generating	Step
$\underset{1}{X},\dots,\underset{6}{X}$	6	$X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{23}, X_{24}, X_{25}, X_{26}, X_{34}, X_{35}, X_{36}, X_{45}, X_{46}, X_{56}$	Yes	2
$\underset{1}{X},\dots,\underset{5}{X}$	5	$(X_{12}, X_{13}), (X_{12}, X_{23}), (X_{12}, X_{14}), (X_{12}, X_{24}), (X_{12}, X_{34}), (X_{12}, X_{15}), (X_{12}, X_{25}), (X_{12}, X_{35}), (X_{12}, X_{45})$	Yes	2
$\underset{1}{X},\dots,\underset{4}{X}$	4	$(X_{12}, X_{13}, X_{23}), (X_{12}, X_{13}, X_{14}), (X_{12}, X_{13}, X_{34}), (X_{12}, X_{13}, X_{24})$	Yes	2
$\underset{1}{X},\underset{2}{X},\underset{3}{X}$	3	(X_{12}, X_{13}, X_{23})	No	infinite
$\underset{1}{X},\underset{2}{X}$	2	X_{12}	No	infinite

We have 7 PD's of rank 6 (spanned by different choices of 6 sections from the 7 ones $\underset{1}{X},\dots,\underset{7}{X}$). Similarly, there are 21 PD's of rank 5, 35 PD's of rank 4, 35 PD's of rank 3 and 21 PD's of rank 2.

For example, the second row of this table concerns with the PD of rank 6 spanned by $\underset{1}{X},\dots,\underset{6}{X}$. It is non-integrable and bracket generating (BG) of step 2. The third column (intersecting the second row) gives the commutators independent with $\underset{1}{X},\dots,\underset{6}{X}$. That is, $\underset{1}{X},\dots,\underset{6}{X}$ together with X_{12} are independent and $\underset{1}{X},\dots,\underset{6}{X}$ together with X_{13} are independent, ..., etc. It should be noted that besides the above mentioned 7 PD's of rank 6, there are many other PD's of rank 6: we may take $\{\underset{1}{X},\dots,\underset{5}{X}, X_{12}\}$, $\{\underset{1}{X},\dots,\underset{5}{X}, X_{13}\}$, $\{\underset{1}{X},\dots,\underset{4}{X}, X_{12}, X_{13}\}, \dots$, etc.

The same discussion can be made for the other rows of the table where we consider PD's of rank 5, 4, 3, 2 on S^7. This gives many non-integrable PD's which are either BG or non BG.

Concluding Remarks

Some results obtained in this chapter may be further studied. For example, finding the first and second Bianichi identities of the sub-Riemannian connection. We may also impose certain conditions on the sub-Riemannian connection (for example, assuming the sR-connection to be semi-symmetric) and investigate the consequences.

More importantly, our sub-Riemannain approach may be applied to other famous parallellizale spaces like Lie groups and Heisenberg distribution. The rich mathematical structure of Lie groups will give rise to new and interesting results that have no analog in the present treatment. This is evident from our study of S^3 and S^7 in which new relations emerge that do not hold in the general case. We may also consider generalized versions of AP-geometry such as generalized or extended AP-space [104, 105].

3

Frobenius Integrability and Finsler Metrizability for 2-dimensional Sprays

In this chapter we introduce a constructive solution to the 2-dimensional case of the Finsler metrizability problem in case of non-flat spray. We associate to a given non-flat 2-dimensional spray two canonical geometric structures that contain all the data about the Finsler metrizability of the spray and, in the affirmative case, we build the Finsler function. One of these structures is a 3-dimensional regular distribution, called the *Berwald distribution*, whose integrability provides a candidate for the Finsler function that we are looking for. The second structure is a 2-form, whose rank gives the information about the regularity of the Finsler candidate. A key aspect in our approach is the use of a canonical frame, called the *Berwald frame*, associated to the given spray. The Berwald frame is used to express the integrability of the Berwald distribution as well as the rank of the 2-form. We thus reformulate and solve the Finsler metrizability problem in terms of some properties of the Berwald frame. In the integrable case, the Berwald distribution coincides with the holonomy distribution [77] and the number of solutions we obtain agrees with the metrizability freedom of a spray introduced in [39].

The Berwald frame has been first introduced, locally, for a 2-dimensional Finsler metric in [16]. An intrinsic formulation of the Berwald frame in the Finslerian setting has been provided in [100], see also [95, §9.9.1]. Such a frame has been rediscovered recently for a background Riemannian metric in [33] to give an alternative proof of the projective Finsler metrizability property of 2-dimensional sprays. In our case, we define the Berwald frame directly for an arbitrary spray and use its properties to obtain information about the Finsler metrizability of the given spray.

For the geodesic spray of a Finsler function, it is known that one can always construct an integrable distribution, transverse to the Liouville vector field, that is tangent to the indicatrix of the Finsler function [15]. In dimension 2, Theorems 3.5.1 and 3.5.3 provide a characterization of the Finsler metrizability in terms of the integrability of such a distribution. This distribution is the Berwald distribution and in the integrable case it is tangent to the indicatrix of the Finsler function that metrizes the given spray.

In the present chapter, we shall deal with the following items:

3.1. Geometric setting for semi-sprays and sprays

3.2. Short survey on Finsler functions

3.3. Historical note on the inverse problem

3.4. Berwald frame

3.5. Integrability of the Berwald distribution and Finsler metrizability

3.6. Examples of metrizable and non metrizable sprays

The main results of this chapter have been published [20] in: Differential Geometry and its Applications, **56** (2018), 308-324. DOI: 10.1016/j.difgeo.2017.10.002. arXiv: 1610.03949 [math.DG].

3.1 Geometric setting for semi-sprays and sprays

In this section we recall some definitions and concepts concerning semi-sprays and sprays that will be needed in this and the next chapter.

For an n-dimensional smooth manifold M, we denote by (TM, π, M), or simply TM, its tangent bundle and by $T_0M := TM \setminus \{0\}$ the total space of the tangent bundle with the null section removed. The Local coordinates (x^i) on M induce local coordinates (x^i, y^i) on TM. We will use the following notations: $C^\infty(TM)$ for the real algebra of smooth functions on TM, $\mathfrak{X}(TM)$ for the Lie algebra of smooth vector fields on TM and $\Lambda^k(TM)$ for the $C^\infty(TM)$-module of differentiable k-forms on TM.

The canonical submersion $\pi : TM \longrightarrow M$ induces a natural foliation on TM. The tangent spaces to the leaves of this foliation determine a regular n-dimensional distribution,

$$VTM : u \in TM \longrightarrow V_uTM = \operatorname{Ker} d_u\pi \subset T_uTM,$$

which is called the vertical distribution. The vertical distribution is an integrable, n-dimensional distribution and is locally generated by $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$. A vector field $X \in \mathfrak{X}(TM)$ is vertical if $X_u \in V_uTM,\ \forall u \in TM$.

Consider the Liouville (dilation) vector field $\mathcal{C} \in \mathfrak{X}(TM)$, which is a vertical vector field whose one-parameter group of diffeomorphisms is generated by the positive homotheties of the fibers, and the tangent structure (vertical endomorphism) J locally given respectively by:

$$\mathcal{C} = y^i \frac{\partial}{\partial y^i}, \quad J = \frac{\partial}{\partial y^i} \otimes dx^i.$$

We now introduce some different types of derivations, within the Frölicher-Nijenhuis formalism [49], which will be used in this chapter and the next one.

Consider a vector valued p-form P on TM. We will denote by

$$i_P : \Lambda^k(TM) \longrightarrow \Lambda^{k+p-1}(TM)$$

the derivation of degree $(p-1)$ given by

$(i_P\alpha)(X_1, \ldots, X_{k+p-1}) =$

$\frac{1}{p!(k-1)!}\sum_{\sigma\in S_{k+p-1}} \mathrm{sign}(\sigma)\alpha\left(P(X_{\sigma(1)}, \ldots, X_{\sigma(p)}), X_{\sigma(p+1)}, \ldots, X_{\sigma(k+p-1)}\right),$

where S_{k+p-1} is the permutation group of $\{1, \ldots, k+p-1\}$. Particularly, for the vertical endomorphism J and $\theta \in \Lambda^1(TM)$, we obtain $(i_J\theta)(X) = \theta(JX),\ \forall X \in \mathfrak{X}(TM)$. The derivation i_P is trivial on functions and hence it is uniquely determined by its action on $\Lambda^1(TM)$. It follows that i_P is a derivation of i_*-type.

We denote by

$$d_P : \Lambda^k(TM) \longrightarrow \Lambda^{k+p}(TM)$$

the derivation of degree p given by

$$d_P = [i_P, d] = i_P \circ d - (-1)^{p-1} d \circ i_P.$$

In particular,

$$(d_J\theta)(X,Y) = (JX)(\theta Y) - (JY)(\theta X) - \theta[JX,Y] - \theta[X,JY] - \theta(J[X,Y]),$$

$\forall X, Y \in \mathfrak{X}(TM),\ \forall\theta \in \Lambda^1(TM)$. Evidently $d_X = \mathcal{L}_X$ is the Lie derivative if $X \in \mathfrak{X}(TM)$. The derivation d_P commutes with the exterior derivative d and hence it is uniquely determined by its action on $C^\infty(TM)$. It follows that d_P is a d_*-type derivation.

For two vector valued forms K and P on TM of degrees k and p respectively, we consider the Frölicher-Nijenhuis bracket $[K, P]$. This is the vector valued $(k+p)$-form uniquely determined by

$$d_{[K,P]} = [d_K, d_P] = d_k \circ d_P - (-1)^{kp} d_P \circ d_K. \tag{3.1.1}$$

For example, $[X, J] = \mathcal{L}_X J = \mathcal{L}_X \circ J - J \circ \mathcal{L}_X$, $\forall X \in \mathfrak{X}(TM)$.

The definition of the tangent structure J leads to $[J, J] = 0$, while $d_J^2 = 0$ follows from the formula (3.1.1). Therefore, any d_J-exact form is d_J-closed and according to a Poincaré-type lemma [99], any d_J-closed form is locally d_J-exact.

A *semi-spray*, or a second order vector field, is a globally defined vector field on TM, $S \in \mathfrak{X}(TM)$, that satisfies $JS = \mathcal{C}$. Locally, a semi-spray can be expressed as

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

with G^i smooth functions on TM.

Any system of second order ordinary differential equations (SODE) in normal form

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \ i = 1, \ldots, n \tag{3.1.2}$$

can be identified with a semi-spray on TM, with local coefficients $G^i(x, y)$.

A semi-spray is called a *spray* if it is homogeneous of degree 2 with respect to the fiber coordinates y. This means that $[\mathcal{C}, S] = S \iff$ the functions $G^i(x, y)$ are homogeneous of degree 2, that is, $\mathcal{C}(G^i) = 2G^i$.

For a given spray S, one associates, using the Frölicher-Nijenhuis formalism, a geometric setting that includes a nonlinear connection, dynamical covariant derivative, almost complex structure and curvature tensors [21, 48, 49, 94].

A spray S induces a *nonlinear connection* through the endomorphism $\Gamma = [J, S]$ on T_0M [48]. The connection Γ is an *almost product structure*, which means that $\Gamma^2 = \mathrm{Id}$. Moreover, Γ gives rise to the following two projectors

$$h = \frac{1}{2}\left(\mathrm{Id} + [J, S]\right), \quad v = \frac{1}{2}\left(\mathrm{Id} - [J, S]\right). \tag{3.1.3}$$

The projector v corresponds to the vertical distribution VTM, while h induces a horizontal distribution HTM supplementary to the vertical distribution. Locally, the two projectors are expressed in the form:

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,$$

where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \quad \delta y^i := dy^i + N_j^i dx^j, \quad N_j^i := \frac{\partial G^i}{\partial y^j}.$$

The connection associated to a spray induces an *almost complex structure* on T_0M given by, [49, (3.14)],

$$\mathbb{F} = h \circ [S, h] - J = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i,$$

The *dynamical covariant derivative* ∇ generated by a spray S is a tensor derivation on T_0M whose action on functions and vector fields is given by [21, §3.2], [22]

$$\nabla f = S(f), \;\; \nabla X = h[S, hX] + v[S, vX], \;\; f \in C^\infty(T_0M), \;\; X \in \mathfrak{X}(T_0M). \quad (3.1.4)$$

A form on TM is said to be *semi-basic* if it vanishes whenever one of its arguments is vertical. A vector valued form on TM is called *semi-basic* if it takes vertical values and it vanishes whenever one of its arguments is vertical. For instance, the tangent structure J is a vector valued semi-basic 1-form.

The *Jacobi endomorphism* of a spray S is the semi-basic vector valued 1-form on T_0M defined by

$$\Phi := v \circ [S, h],$$

with local expression given by

$$\Phi = R_j^i(x, y) \frac{\partial}{\partial y^i} \otimes dx^j,$$

where

$$R_j^i(x, y) = 2 \frac{\partial G^i}{\partial x^j} - S\left(\frac{\partial G^i}{\partial y^j}\right) - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}.$$

A spray S is said to be *isotropic* if there exist a function $\rho \in C^\infty(T_0M)$ and a semi-basic 1-form $\alpha = \alpha_i(x,y)dx^i \in \Lambda^1(T_0M)$ such that

$$\Phi = \rho J - \alpha \otimes \mathcal{C} \text{ or, locally, } R^i_j = \rho\, \delta^i_j - \alpha_j\, y^i. \tag{3.1.5}$$

Throughout this chapter, we assume that M is a 2-*dimensional* orientable, smooth manifold. Consequently, the vertical distribution is a regular, integrable, 2-dimensional distribution and is locally generated by $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2})$. Moreover, the function

$$\rho = \mathrm{Tr}(\Phi) = R^1_1 + R^2_2,$$

which is called the *Ricci scalar*, is related to the semi-basic 1-form α by

$$\rho = i_S\alpha = \alpha_1 y^1 + \alpha_2 y^2. \tag{3.1.6}$$

The homogeneity property of a spray S is inherited by all its associated geometric structures. For example, the Jacobi endomorphism is 2^+-homogeneous, which implies that the Ricci scalar ρ is 2^+-homogeneous while the semi-basic 1-form α is 1^+-homogeneous.

It is known that 2-dimensional sprays are always isotropic, see [88, Lemma 8.1.10] or [95, Corollary 8.3.11]. This means that their Jacobi endomorphism is given by formula (3.1.5) with

$$\alpha_1 = \frac{R^2_2}{y^1} = -\frac{R^2_1}{y^2}, \quad \alpha_2 = \frac{R^1_1}{y^2} = -\frac{R^1_2}{y^1}.$$

In this work, we will focus our attention on non-flat sprays. This assumption means that the Jacobi endomorphism Φ is nowhere vanishing, which is equivalent to $\alpha \neq 0$ and implies that $\rho \neq 0$.

We will use the geometric setting described above to address the following metrizability problem for a given spray S. Does there exist a Finsler function whose geodesic spray is S? We will provide the answer to this problem using two geometric structures. The first structure is a distribution whose integrability will provide the metric candidate and the second structure is a 2-form which encodes information about the regularity of the metric.

For a spray S, the non-flatness assumption implies that the distribution

$$\mathcal{D} = \mathrm{Im}(h) \oplus \mathrm{Im}(\Phi) \tag{3.1.7}$$

is a regular, 3-dimensional distribution. We call $\mathcal{D}$ the *Berwald distribution* of the spray S. We will prove that the spray is metrizable if and only if $\mathcal{D}$ is integrable.

Another distribution that can be associated to an arbitrary spray S has been introduced by Muzsnay in [77]. It is called the *holonomy distribution* $\mathcal{D}_{\mathcal{H}}$ and is generated by horizontal vector fields and their succesive Lie brackets. The holonomy distribution has been recently used in [39] to discuss the metrizability freedom of a spray. The two distributions are related by $\mathcal{D} \subset \mathcal{D}_{\mathcal{H}}$.

For an isotropic spray S with Jacobi endomorphism Φ given by formula (3.1.5), we consider the following 2-form:

$$\Omega = d\left(\frac{\alpha}{\rho}\right) + 2\left(i_{\mathbb{F}}\frac{\alpha}{\rho}\right) \wedge \frac{\alpha}{\rho}. \tag{3.1.8}$$

In the metrizability case, the rank of the 2-form Ω will provide information about the regularity of the Finsler function required. The idea of considering the 2-form Ω has its origin in the "Scalar Flag Curvature"-test proposed in [24, Theorem 3.1].

For the metrizability problem of a given spray, we pay attention to the non-Riemannian case by searching for Finslerian solutions.

3.2 Short survey on Finsler functions

We give here a short survey on some notions of Finsler geometry [5, 9, 13, 88, 94, 95] that will be used in the sequel.

Definition 3.2.1. *A continuous function $F : TM \longrightarrow \mathbb{R}$ is called a* Finsler function *if it satisfies the following conditions:*

(i) F is smooth and strictly positive on T_0M;

(ii) F is positively homogeneous of degree 1 in the fiber coordinates (1^+-homogeneous), which means that

$$F(x, \lambda y) = \lambda F(x, y),\ \forall \lambda \geq 0,\ (x, y) \in TM;$$

(iii) F is regular, which means that the following metric tensor *has maximal rank n on T_0M*

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

If we replace the regularity condition (iii) by the weak regularity condition $\text{rank}(g_{ij}) < n$*, then F is said to be a degenerate Finsler function.*

The regularity conditions can be reformulated in terms of the Hilbert 2-form $\omega_{F^2} := -dd_J F^2$ as follows. Since the function F satisfies (iii), it is a Finsler function if and only if ω_{F^2} has a maximal rank $2n$, that is, ω_{F^2} is a symplectic structure. Moreover, the function F satisfies the weak regularity condition and hence is a degenerate Finsler function if and only if the 2-form ω_{F^2} has rank $< 2n$.

A spray S is *Finsler metrizable* if there exists a (degenerate) Finsler function that satisfies the Euler-Lagrange equation

$$i_S dd_J F^2 = -dF^2. \tag{3.2.1}$$

Equation (3.2.1) expresses the fact that the base integral curves of the spray S (homogeneous SODE) are solutions to the Euler-Lagrange equations for F^2. In other words, S is the geodesic spray when F is a Finsler function. In the degenerate case, S is one of the geodesic sprays of F.

Consider a geodesic spray S of a (degenerate) Finsler function F and let Γ be the connection induced by S. The Hilbert 2-form of F can be expressed in terms of the connection Γ by

$$\omega_{F^2} = 2\, g_{ij}\, dx^i \wedge \delta y^j. \tag{3.2.2}$$

In this geometric setting, we can associate to a Finsler function F a pseudo-Riemannian metric on T_0M [48] by

$$2G(X,Y) = -\omega_{F^2}(X, \mathbb{F}Y), \tag{3.2.3}$$

locally

$$G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j.$$

If S is the geodesic spray of a Finsler function F, then we can use the metric G above to construct a distribution that is orthogonal to the Liouville vector field, the Berwald distribution. The integrability of this distribution, which is tangent to the indicatrix of F, has been shown in [15].

The main results of this chapter give a converse of this result. That is, if for a spray S, the Berwald distribution is integrable, then it will be tangent to the indicatrix of a Finsler function that metrizes the given spray.

Consider a (degenerate) Finsler function F, a spray S satisfying (3.2.1), and its Jacobi endomorphism Φ. We say that a spray S has *scalar flag curvature* $\kappa \in C^\infty(T_0M)$ if the Jacobi endomorphism Φ is of the form

$$\Phi = \kappa F^2 J - \kappa F d_J F \otimes \mathcal{C}. \tag{3.2.4}$$

When κ is a constant, we say that S has *constant flag curvature*, see for instance [23, 89]. The non-flatness assumption that we work with is equivalent to $\kappa \neq 0$.

The Euler-Lagrange equation (3.2.1) satisfied by the geodesic spray S of a (degenerate) Finsler function F is equivalent to $d_hF^2 = 0$. This implies that

$$dF^2 = d_vF^2 = i_{\mathbb{F}}d_JF^2.$$

It is known that if the geodesic spray of a Finsler function is isotropic then the Finsler function has scalar flag curvature, see [88, Lemma 8.2.2]. The material discussed above enable us to give an alternative proof of this result for the more general case that includes the degenerate Finsler functions [20].

Proposition 3.2.2. *Consider an isotropic spray S that is also a geodesic spray of a (degenerate) Finsler function F. Then the spray S has scalar flag curvature.*

Proof. The Euler-Lagrange equation (3.2.1) is equivalent to $d_hF^2 = 0$. From this we find that $S(F^2) = 0$. We also have

$$d_{[S,h]}F^2 = \mathcal{L}_S d_h F^2 - d_h \mathcal{L}_S F^2 = 0.$$

Using this equation, the fact that $dF^2 = d_vF^2$ and the definition of the Jacobi endomorphism, we conclude that

$$0 = d_{[S,h]}F^2 = i_{[S,h]}dF^2 = i_{[S,h]}d_vF^2 = i_{[S,h]}i_v dF^2 = i_{v\circ[S,h]}dF^2 = i_\Phi dF^2 = d_\Phi F^2.$$

We now use the assumption that S is isotropic so that the Jacobi endomorphism is given by formula (3.1.5). Consequently,

$$0 = d_\Phi F^2 = d_{\rho J - \alpha\otimes\mathcal{C}}F^2 = \rho d_J F^2 - \alpha\mathcal{L}_{\mathcal{C}}F^2 = \rho d_J F^2 - 2F^2\alpha,$$

whence

$$\frac{\alpha}{\rho} = \frac{d_JF^2}{2F^2} = \frac{d_JF}{F}. \tag{3.2.5}$$

Setting $\kappa = \frac{\rho}{F^2}$ and using formulate (3.2.5), the Jacobi endomorphism can be written as follows:

$$\Phi = \rho\left(J - \frac{\alpha}{\rho}\otimes\mathcal{C}\right) = \frac{\rho}{F^2}\left(F^2J - F^2\frac{d_JF}{F}\otimes\mathcal{C}\right) = \kappa\left(F^2J - Fd_JF\otimes\mathcal{C}\right).$$

This shows that the formula in (3.2.4) holds and hence the spray S has scalar flag curvature κ. □

Consider an isotropic spray S that is also a geodesic spray of a (degenerate) Finsler function F. In view of formula (3.2.5), we find that the 2-form Ω given by (3.1.8) is related to the Hilbert 2-form ω_{F^2} by

$$\Omega = d\left(\frac{1}{2F^2} d_J F^2\right) + \frac{1}{2F^4} dF^2 \wedge d_J F^2 = -\frac{1}{2F^2}\, dd_J F^2 = \frac{1}{2F^2}\,\omega_{F^2}. \quad (3.2.6)$$

Since 2-dimensional sprays are always isotropic, it follows, by Proposition 3.2.2, that Finsler metrizable 2-sprays have scalar flag curvature.

3.3 Historical note on the inverse problem

Consider a SODE in normal form

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \dot{x}\right) = 0,\ \ i = 1, \ldots, n.$$

A geometric approach to the study of the solutions of such a SODE consists in identifying the system (3.1.2) with a second order vector field or a semi-spray. In 1960 Ambrose et al. [2] introduced the notion of a semi-spray (spray) to give an intrinsic presentation of SODEs. The solutions of (3.1.2) are called the geodesics of the semi-spray. In some cases, the semi-spray (spray) can be obtained from a variational principle. To decide whether or not a SODE or semi-spray can be derived from a variational principle is known as the inverse problem of Lagrangian mechanics. Some variational semi-sprays (sprays) are metrizable. These semi-sprays (sprays) or SODEs can be viewed as the geodesic equations of a metric, see for instence [6].

In fact, the inverse problem of Lagrangian mechanics is far from being solved. It has been completely solved in dimension 1 by Darboux [35]. In dimension 2, Douglas [38] gave a characterization which states that a given semi-spray is Lagrange metrizable if and only if there exists a non-singular symmetric matrix g with entries $g_{ij}(x, y)$ satisfying Helmholtz conditions. For a non-degenerate metric whose components are $g_{ij}(x,y)$, the Helmholtz conditions are expressed locally in the form:

$$\frac{\partial g_{jk}}{\partial y^i} = \frac{\partial g_{ik}}{\partial y^j}, \quad \nabla g_{ij} = 0^1, \quad g_{ik}\, R^k_l = g_{lk}\, R^k_i.$$

Crampin in 1981 [32] proved that a semi-spray S is Lagrange metrizable if and only if there exists a 2-form ω with maximal rank such that $\forall H \in H(TM)$

$$\mathcal{L}_S\omega = 0,\ \ \omega_{|V(TM)\times V(TM)} = 0, \quad (i_H d\omega)\,|_{V(TM)\times V(TM)} = 0.$$

In 2009 Bucataru and Dahl [21] showed that a semi-spray S is Lagrange metrizable if and only if there exists a semi-basic, non-trivial 1-form $\theta \in \Lambda^1(T_0M)$ such that $\mathcal{L}_S\theta$ is closed or equivalently θ satisfies the following Helmholtz conditions:

$$d_h\theta = 0,\ d_\Phi\theta = 0,\ d_J\theta = 0,\ \nabla d\theta = 0.$$

In 2015 Bucataru and Constantinescu [19] generalized this work by studying the inverse problem of non-conservative Lagrangian mechanics[2]. They proved the same result, in a special case, under only three conditions. More precisely, a semi-spray S is Lagrangian if and only if there exists a semi-basic, non-trivial 1-form $\theta \in \Lambda^1(T_0M)$ satisfying the following Helmholtz conditions:

$$d_\Phi\theta = 0,\ d_J\theta = 0,\ \nabla d_v\theta = 0.$$

The geodesics of a Finsler function, parameterized so that the tangent vector has constant Finsler length, define a spray. However, not every spray can be obtained in this way. In the case when a given SODE is homogeneous, the problem is known as the Finsler metrizability problem [24, 49, 60, 70, 96]. A solution to this problem has been proposed in the analytic case by Muzsnay [77] by studying the formal

[1] $\nabla g_{ij} := S(g_{ij}) - g_{rj}\, N^r_i - g_{ir}\, N^r_j$.

[2] The notion of non-conservative Lagrangian system will be defined in the next chapter.

integrability of the associated Euler-Lagrange PDE. Muzsnay showed that a given spray can not be Finsler metrizable if the Liouville vector field $\mathcal{C}$ lies in $\mathcal{D}_{\mathcal{H}}$.

Recently, Elgendi and Muzsnay in [39] introduced the metrizability freedom concept. They proposed a method to determine whether a given spray is metrizable or not. Moreover, if the spray is metrizable, their method gives the number of Finsler functions that induce the given spray and the manner by which they can be constructed.

In 2016, Mestdag [70] used a technique of symmetry reduction which is called "Routh reduction" to show that the solutions of the system of Euler-Lagrange equations representing a strongly convex autonomous Lagrangian which lie on a specific energy level can be thought of as geodesics of an associated Finsler function.

3.4 Berwald frame

In [16] Berwald constructed a frame on T_0M canonically associated to a 2-dimensional Finsler manifold and used it to characterize projectively flat 2-dimensional Finsler manifolds and to classify them in some particular cases, namely, Landsberg spaces and Finsler spaces with the main scalar I being a function of position only. A detailed analysis of the role played by the Berwald frame for the geometry of a 2-dimensional Finsler space is presented in [9, Section 3.5]. In [33], Crampin rediscovered such a frame, which he associated to a given Riemannian metric, to give a new constructive proof of the fact that any 2-dimensional spray is projectively Finsler metrizable. Recently, many authors used the dual of Berwald frame to solve different problem in Finsler geometry, see for instance [18, 44].

In this section we will show that the *Berwald frame* can be associated to any 2-dimensional non-flat spray. We study the regularity conditions and the commutation formulae satisfied by the Berwald frame in the following three cases: a spray in general, a Finsler metrizable spray and, finally, a spray metrizable by a degenerate Finsler function.

3.4.1 Berwald frame for a spray

For a 2-dimensional spray S, we consider the geometric setting described in Section 3.1. As we already mentioned, if S is isotropic, then its Jacobi endomorphism is given by formula (3.1.5). We make the assumption that S is non-flat and thus the semi-basic 1-form α and the Ricci scalar ρ are nowhere vanishing on T_0M. If we allow to work with conic Finsler functions, then we will restrict the domain to some open cone $A \subset T_0M$, where α and ρ are not vanishing.

Consider a vector field $H \in \mathfrak{X}(T_0M)$ that satisfies the following three conditions:

$$[\mathcal{C}, H] = H, \quad h(H) = H, \quad \alpha(H) = 0. \tag{3.4.1}$$

The first two conditions of (3.4.1) mean that H is a 2^+-homogeneous horizontal vector field. The last condition above is equivalent to $\Phi(H) = \rho JH$, which means that H is (fibrewise) an eigenvector for the Jacobi endomorphism Φ corresponding to the non-vanishing eigenvalue ρ.

Conditions (3.4.1) do not determine the vector field H uniquely. Such vector field is determined only up to a 0^+-homogeneous function factor. We can fix such a vector field H by requiring that $\{S, H\}$ is compatible with a fixed orientation of M. Since $\alpha(S) = \rho \neq 0$ and $\alpha(H) = 0$, we find that H and S are two linearly independent vector fields that generate the horizontal distribution. It follows that $V := JH$ and $\mathcal{C} = JS$ are two linearly independent vector fields that generate the vertical distribution. Consequently, $\{H, S, V, \mathcal{C}\}$ is a frame on T_0M which is called the *Berwald frame*.

Lemma 3.4.1. *Consider a spray S and let $\{H, S, V, \mathcal{C}\}$ be a fixed Berwald frame.*

(i) The following formulae are satisfied:

$$[\mathcal{C}, V] = 0, \quad [S, H] = \nabla H + \rho V, \quad [S, V] = -H + \nabla V. \tag{3.4.2}$$

(ii) The rank of the 2-form Ω defined by (3.1.8) is given by

$$\operatorname{rank}(\Omega) = \begin{cases} 4, & \text{if } \alpha([H, V]) \neq 0; \\ 2, & \text{if } \alpha([H, V]) = 0. \end{cases} \tag{3.4.3}$$

Proof. (i) Since the tangent structure J is 0^+-homogeneous, it follows that $[\mathcal{C}, J] = -J$. If we evaluate both sides of this formula on the horizontal 2^+-homogeneous vector field H we obtain

$$-V = -J(H) = [\mathcal{C}, J](H) = [\mathcal{C}, JH] - J[\mathcal{C}, H] = [\mathcal{C}, V] - V.$$

Thus the first formula of (3.4.2) is true so that V is a 1^+-homogeneous vector field.

Using formulae (3.1.4) and (3.1.5), we have

$$[S, H] = h[S, hH] + v[S, hH] = \nabla H + \Phi(H) = \nabla H + \rho V.$$

From the properties of the dynamical covariant derivative, [21, Theorem 3.4], we conclude that $\nabla J = 0$, and hence

$$J(\nabla H) = \nabla J(H) = \nabla V.$$

Using the second formula in (3.4.2), we obtain $J[S, H] = \nabla V$. In order to prove the last formula in (3.4.2), we use the fact that H is horizontal and definition (3.1.3) of the horizontal projectoion h to obtain

$$2H = \operatorname{Id}(H) + [J, S](H) = H + [J(H), S] - J[H, S] = H + [V, S] + \nabla V.$$

This completes the proof of (i).

(ii) We give a matrix representation of the 2-form Ω with respect to the Berwald frame $\{H, S, V, \mathcal{C}\}$. To do so, we evaluate (3.1.8) on pairs of vector fields $X, Y \in \{H, S, V, \mathcal{C}\}$. Since α is a semi-basic 1-form, it follows that $\alpha(V) = \alpha(\mathcal{C}) = 0$. Using (3.1.6) and $\alpha(H) = 0$, we find that

$$\frac{\alpha}{\rho}(S) = 1, \ \ \frac{\alpha}{\rho}(X) = 0, \ \ \forall X \in \{H, V, \mathcal{C}\}.$$

Hence the exterior derivative of the the 1-form $\frac{\alpha}{\rho}$ is given by

$$d\left(\frac{\alpha}{\rho}\right)(X,Y) = -\frac{\alpha}{\rho}([X,Y]), \ \ \forall X,Y \in \{H,S,V,\mathcal{C}\}.$$

Using the commutation formulae (3.4.2), we obtain the following matrix representation of Ω with respect to $\{H,S,V,\mathcal{C}\}$

$$\Omega = \begin{pmatrix} 0 & \alpha(\nabla H)/\rho & -\alpha([H,V])/\rho & 0 \\ -\alpha(\nabla H)/\rho & 0 & 0 & 1 \\ \alpha([H,V])/\rho & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.4.4}$$

From (3.4.4), it is clear that the rank of Ω is given by formula (3.4.3). □

Definition 3.4.2. *A spray S is called* regular *if the 2-form Ω is an almost symplectic form, which means that it satisfies the condition that* $\text{rank}(\Omega) = 4$. *The spray is called* degenerate *if* $\text{rank}(\Omega) = 2$.

Remark 3.4.3. The regularity condition $\alpha([H,V]) \neq 0$ does not depend on the choice of the Berwald frame. Indeed, if we replace H by aH where $a \in C^\infty(T_0M)$ is a nowhere vanishing 0^+-homogeneous function, then V will be replaced by aV so

$$\alpha([aH,aV]) = a^2\alpha([H,V]).$$

3.4.2 Berwald frame for a Finsler function

Consider the geodesic spray S of a Finsler function F and the Berwald frame $\{H,S,V,\mathcal{C}\}$ determined the conditions (3.4.1). Assume further that H has the same length as S so that

$$G(H,H) = G(S,S) = F^2. \tag{3.4.5}$$

Remark 3.4.4. The three normalised vector fields $\{\frac{H}{F}, \frac{S}{F}, \frac{V}{F}\}$ were used in [13, §4.3] to provide an orthonormal frame for the projective sphere bundle SM of a 2-dimensional Finsler manifold.

We now extend the results of Lemma 3.4.1 by finding the commutation formulae for the Berwald frame that is uniquely associated to a Finsler function.

Lemma 3.4.5. *Consider the geodesic spray S of a Finsler function F and let $\{H, S, V, \mathcal{C}\}$ be its Berwald frame.*

(i) The Berwald frame satisfies the formulae

$$H(F^2) = 0, \quad V(F^2) = 0, \quad \nabla H = 0, \quad \nabla V = 0,$$

(ii) The Berwald distribution spanned by $\{H, S, V\}$ is integrable i.e.

$$[S, H] = \rho V, \quad [S, V] = -H, \quad [H, V] = S + IH + S(I)V. \tag{3.4.6}$$

(iii) The geodesic spray is regular.

Proof. First we prove the third part (iii) of the lemma. Using (3.2.6), (3.2.3) and the scaling condition (3.4.5), we obtain

$$\Omega(H, V) = -\frac{1}{2F^2}\omega_{F^2}(H, V) = -\frac{1}{2F^2}\omega_{F^2}(H, -\mathbb{F}H) = -\frac{1}{2F^2}2G(H, H) = -1. \tag{3.4.7}$$

From the matrix representation (3.4.4) it is clear that $\text{rank}(\Omega) = 4$, and so the geodesic spray S is regular.

(i) Equation (3.2.1) holds if and only if $d_h F^2 = 0$, if and only if $hX(F^2) = 0$, $\forall X \in \mathfrak{X}(T_0M)$. Consequently, the horizontal vector field H also satisfies $H(F^2) = 0$.

The geodesic spray S has scalar flag curvature with Jacobi endomorphism given by formula (3.2.4). It follows that condition (3.4.1) defining the horizontal vector field H can be written in the form

$$0 = 2\alpha(H) = 2\kappa F d_J F(H) = \kappa d_J F^2(H) = \kappa JH(F^2) = \kappa V(F^2),$$

which means that $V(F^2) = 0$, because $\kappa \neq 0$.

From the matrix representation (3.4.4) we obtain $\Omega(H, \mathcal{C}) = 0$ and $\Omega(S, \mathcal{C}) = 1$. Since the dynamical covariant derivative preserves the horizontal distribution, it follows that $\nabla H = a_1 S + a_2 H$, for a_1, $a_2 \in C^\infty(T_0M)$, is a horizontal vector field. Thus $\nabla V = \nabla JH = J\nabla H = a_1\mathcal{C} + a_2 V$. Using the properties of the dynamical covariant derivative that were proved in [21], we find that $\nabla\Omega = 0$ and $\nabla\mathcal{C} = 0$. Therefore

$$0 = (\nabla\Omega)(H, \mathcal{C}) = \nabla(\Omega(H, \mathcal{C})) - \Omega(\nabla H, \mathcal{C}) - \Omega(H, \nabla\mathcal{C}) = -\Omega(a_1 S + a_2 H, \mathcal{C}) = -a_1,$$
$$0 = (\nabla\Omega)(H, V) = \nabla(\Omega(H, V)) - \Omega(\nabla H, V) - \Omega(H, \nabla V) = -2a_2\Omega(H, V) = -2a_2,$$

and so $\nabla H = \nabla V = 0$.

(ii) In view of the above calculations and the last two commutation formulae in (3.4.2), we conclude that the first two commutation formulae in (3.4.6) are true.

We note that the Lie bracket of the vector fields H, V can be written in terms of the elements of the Berwald frame in the form

$$[H, V] = b_1 S + b_2 H + b_3 \mathcal{C} + b_4 V, \text{ for } b_1, b_2, b_3, b_4 \in C^\infty(T_0 M).$$

Using the regularity condition (3.4.7), we find that

$$b_1 = \frac{\alpha([H, V])}{\rho} = -\Omega(H, V) = 1.$$

From the first two commutation formulae of (3.4.6) and the Jacobi identity

$$[S, [H, V]] + [V, [S, H]] + [H, [V, S]] = 0,$$

we obtain

$$\begin{aligned} 0 &= [S, S + b_2 H + b_3 \mathcal{C} + b_4 V] + [V, \rho V] + [H, H] \\ &= -b_3 S + (S(b_2) - b_4) H + S(b_3)\mathcal{C} + (S(b_4) + b_2\rho + V(\rho)) V. \end{aligned}$$

Preceding calculations imply

$$b_3 = 0,\ b_4 = S(b_2) \text{ and } S(b_4) + b_2\rho + V(\rho) = 0.$$

Using the notation $I = b_2$, we conclude that the last commutation formula (3.4.6) is true and so the coefficient function I satisfies $S^2(I) + I\rho + V(\rho) = 0$. □

Remark 3.4.6. The three commutation formulae (3.4.6), viewed as derivations on 0^+-homogeneous functions, were obtained first by Berwald [16, (7.6)]. By comparing the last formula in (3.4.6) and the first formula in [16, (7.6)] we find that the function I is the main scalar of the Finsler function. In the Riemannian case, the main scalar I vanishes and the three commutation formulae (3.4.6) reduce to the commutation formulae [33, Lemma 1]. For a different derivation of the Berwald frame with the commutation formulae (3.4.6) we refer to [100], see also [95, §9.9.1], where the pull-back formalism is adopted.

3.4.3 Berwald frame for a degenerate Finsler function

Consider a geodesic spray S of a degenerate Finsler function F so that S is a solution of the equation (3.2.1). We consider a Berwald frame for S determined by the conditions (3.4.1). First we extend the results of Lemma 3.4.5.

Lemma 3.4.7. *Consider a geodesic spray S of a degenerate Finsler function F and let $\{H, S, V, \mathcal{C}\}$ be a Berwald frame.*

(i) The following formulae are valid:

$$\begin{aligned}
&H(F^2) = 0, \quad V(F^2) = 0, \quad \alpha\left(\nabla H\right) = 0, \quad d_vF\left(\nabla V\right) = 0,\\
&\alpha\left([S, H]\right) = d_vF\left([S, H]\right) = 0, \quad \alpha\left([S, V]\right) = d_vF\left([S, V]\right) = 0, \\
&\alpha\left([H, V]\right) = d_vF\left([H, V]\right) = 0.
\end{aligned} \tag{3.4.8}$$

(ii) The geodesic spray is degenerate.

Proof. (i) We obtain as above that $d_hF^2 = 0$ and hence $H(F^2) = 0$. The condition $\alpha(H) = 0$ implies $V(F^2) = 0$.

Since F is degenerate, $\text{rank}(\omega_{F^2}) = \text{rank}(\Omega) = 2$. Using the matrix representation (3.4.4) of the 2-form Ω with respect to the Berwald frame, we find that $\Omega(H, S) = 0$ and $\Omega(H, V) = 0$. Consequently,

$$\alpha\left([H, V]\right) = -\rho\,\Omega(H, V) = 0, \quad \alpha\left(\nabla H\right) = \rho\,\Omega(H, S) = 0.$$

Therefore $\alpha(\nabla H) = 0$ and hence $d_JF(\nabla H) = 0$. The dynamical covariant derivative preserves the horizontal distribution spanned by H and S. The condition $\alpha(\nabla H) = 0$ means that the vector field ∇H does not have a component along S and hence $\nabla H = a_1H$ for some function $a_1 \in C^\infty(T_0M)$. Using the fact that $\nabla J = 0$, it follows that $\nabla V = a_1V$, which means that the vertical vector field ∇V does not have a component along $\mathcal{C}$. The last commutation formulae (3.4.2) can be written now as follows:

$$[S, H] = a_1H + \rho V, \quad [S, V] = -H + a_1V. \tag{3.4.9}$$

Commutation formulae (3.4.9) show that the vector fields $[S, H]$ and $[S, V]$ have no components along S and $\mathcal{C}$ and therefore the corresponding formulae of (3.4.8) are true.

We have already seen that the degeneracy condition on a Finsler function implies that $\alpha([H, V]) = 0$. This means that the vector field $[H, V]$ has no component along S and hence can be expressed in the form

$$[H, V] = a_2H + a_3\mathcal{C} + a_4V,$$

for some functions $a_2, a_3, a_4 \in C^\infty(T_0M)$. Using the Jacobi identity for the three vector fields S, H, V and the above expressions for the Lie brackets $[S, H]$, $[S, V]$ and $[H, V]$, we conclude that $a_3 = 0$ and, therefore, the last formulae of (3.4.8) are true.

(ii) Since F is a degenerate Finsler function, $\text{rank}(g_{ij}) = 1$, which, in view of (3.2.2) and (3.2.6), is equivalent to $\text{rank}(\omega_{F^2}) = \text{rank}(\Omega) = 2$. Thus, the spray S is degenerate. □

3.5 Integrability of the Berwald distribution and Finsler metrizability

In this section we will prove that a 2-dimensional spray S is metrizable if and only if the Berwald distribution (3.1.7) is integrable. The regularity of the corresponding Finsler function depends on the rank of the 2-form (3.1.8). We treat separately the regular and degenerate cases.

3.5.1 Finsler metrizability

For a 2-dimensional, non-flat spray S, we consider the Berwald distribution $\mathcal{D}$ (3.1.7) and the 2-form Ω (3.1.8). The next theorem provides characterizations for the Finsler metrizability of a regular spray together with an algorithm that can be used to construct effectively a Finsler function that metrizes the spray.

Theorem 3.5.1. *We consider a 2-dimensional, non-flat spray S. The following conditions are equivalent:*

(i) S is Finsler metrizable;

(ii) S is regular and the Berwald distribution $\mathcal{D}$ is integrable;

(iii) there exists a closed 1-form $\omega \in \mathcal{D}^*$ such that

$$\operatorname{rank}\left(d_J\omega + 2\omega \wedge i_J\omega\right) = 4, \tag{3.5.1}$$

where $\mathcal{D}^*$ is the annihilator of $\mathcal{D} : \omega(X) = 0 \ \ \forall \omega \in \mathcal{D}^*,\ X \in \mathcal{D}$.

Proof. $(i) \Longrightarrow (ii)$. We assume that S is the geodesic spray of a Finsler function F. We consider the Berwald frame associated to the Finsler function F as it has been described in Section 3.4.2. From Lemma 3.4.5, the spray S is regular. By the commutation formulae (3.4.6) and Frobenius Theorem, the distribution $\mathcal{D} = \operatorname{span}\{H, S, V\}$ is integrable.

$(ii) \Longrightarrow (iii)$. We assume that S is a regular spray and the Berwald distribution $\mathcal{D} = \operatorname{span}\{H, S, V\}$ is integrable. Since $\operatorname{rank}\mathcal{D} = 3$, it follows that $\operatorname{rank}\mathcal{D}^* = 1$. This freedom allows us to choose a 0^+-homogeneous 1-form $\omega \in \mathcal{D}^*$ so that $\mathcal{L}_{\mathcal{C}}\omega = 0$.

We fix this 1-form with the normalisation condition $i_{\mathcal{C}}\omega = 1$. We will prove that the 1-form ω satisfies the two conditions (iii).

Since the Berwald distribution $\mathcal{D}$ is integrable, it follows that for any two vector fields $X, Y \in \mathcal{D}$ we have $[X, Y] \in \mathcal{D}$ and hence $\omega([X, Y]) = 0$. Therefore

$$d\omega(X, Y) = 0, \forall\, X, Y \in \mathcal{D}.$$

Using Cartan's formula, as well as the defining properties of ω, we have

$$i_{\mathcal{C}} d\omega = \mathcal{L}_{\mathcal{C}}\omega - d i_{\mathcal{C}}\omega = 0.$$

Thus $d\omega = 0$.

Since $\omega \in \mathcal{D}^*$ it follows that $\omega \circ \Phi = 0$. As the spray S is isotropic and its Jacobi endomorphism Φ is given by (3.1.5), we have

$$0 = \omega \circ (\rho J - \alpha \otimes \mathcal{C}) = \rho i_J \omega - (i_{\mathcal{C}}\omega)\,\alpha = \rho i_J\omega - \alpha.$$

The above calculations imply

$$i_J\omega = \frac{\alpha}{\rho} \quad \text{and so} \quad \omega = i_{\mathbb{F}}\frac{\alpha}{\rho}. \tag{3.5.2}$$

Using these two formulae, we conclude that the 2-form in formula (3.5.1) coincides with the 2-form (3.1.8). The regularity condition of the spray S implies that condition (3.5.1) holds.

$(iii) \Longrightarrow (i)$. Consider a closed 1-form $\omega \in \mathcal{D}^*$ that satisfies the condition (3.5.1). Again, the fact that $\omega \in \mathcal{D}^*$ implies $\omega \circ \Phi = 0$, and as proved above, formulae (3.5.2) are true. Locally, these two formulae can be written in the form

$$i_J\omega = \frac{\alpha_i dx^i}{\rho} \quad \text{and so} \quad \omega = \frac{\alpha_i \delta y^i}{\rho}.$$

From these two formulae, we obtain $i_{\mathcal{C}}\omega = 1$ and $\mathcal{L}_{\mathcal{C}}\omega = 0$.

Since the 1-form ω is closed, using Poincaré's Lemma, we conclude that there exists a locally defined function f on T_0M such that $\omega = df$. We set $F = \exp(f)$ and show that F is a Finsler function whose geodesic spray is S. Observe first that

$$\mathcal{C}(f) = i_{\mathcal{C}} df = i_{\mathcal{C}}\omega = i_{\mathcal{C}} i_{\mathbb{F}}\frac{\alpha}{\rho} = i_{\mathbb{F}\mathcal{C}}\frac{\alpha}{\rho} = i_S\frac{\alpha}{\rho} = 1,$$

which implies that

$$\mathcal{C}(F) = \mathcal{C}(\exp(f)) = \exp(f) = F,$$

and so F is 1^+-homogeneous.

In view of the two formulae (3.5.2), we find that the 2-form Ω, (3.1.8), is given by $\Omega = d_J\omega + 2\omega \wedge i_J\omega$. Using the assumption (3.5.1), we obtain $\operatorname{rank}(\Omega) = 4$. Then, from (3.5.2), we get

$$\frac{\alpha}{\rho} = i_J\omega = d_J f = \frac{d_J F}{F} = \frac{d_J F^2}{2F^2}.$$

Using this formula, we see that the 2-form Ω and the function F are related by (3.2.6). It follows that $\operatorname{rank}(\omega_{F^2}) = 4$, so that F^2 is regular and hence F is indeed a Finsler function.

It remains to check that S is the geodesic spray of the Finsler function. We show this by proving that $d_h F = 0$. We use again the condition $\omega \in \mathcal{D}^*$, which implies that $\omega \circ h = 0$. Since $\omega = df = \frac{dF}{F}$ it follows that $d_h f = 0$ and hence $d_h F = 0$. This completes the proof. □

Remark 3.5.2. Criterion (*iii*) in Theorem 3.5.1 shows "where to look" for a Finsler function in the case when a spray S is metrizable. It is enough to pick a closed 1-form ω from the 1-dimensional annihilator of the Berwald distribution $\mathcal{D}$. The two conditions $\mathcal{L}_{\mathcal{C}}\omega = 0$ and $i_{\mathcal{C}}\omega = 1$ show that the 1-form ω is unique. Condition $d\omega = 0$ is equivalent to the integrability of the distribution $\mathcal{D}$ and provides the Finsler function F that metrizes the spray S, through the condition $\omega = \frac{dF}{F}$. In this case the metrizability freedom we have for choosing the Finsler function is 1 and it agrees with the result of [39] for non-flat isotropic spray.

Criterion (*ii*) in Theorem 3.5.1 is more geometric and it shows, in the integrable case, that the Berwald distribution $\mathcal{D}$ is tangent to a hypersurface in TM that represents the indicatrix of the Finsler function that metrizes the spray.

3.5.2 Degenerate Finsler metrizability

In this section, we pay attention to the case when a 2-dimensional non-flat spray S is metrizable by a degenerate Finsler function. Similar to the regular case, we will show in the next theorem that all the information regarding the metrizability of a spray are encoded again into the Berwald distribution (3.1.7) and the 2-form (3.1.8).

Theorem 3.5.3. *Let S be a 2-dimensional, non-flat spray. The following conditions are equivalent:*

(i) *S is metrizable by a degenerate Finsler function.*

(ii) *The spray S is degenerate and the Berwald distribution $\mathcal{D}$ is integrable.*

(iii) *There exists a closed 1-form $\omega \in \mathcal{D}^*$ such that*

$$\operatorname{rank}(d_J\omega + 2\omega \wedge i_J\omega) = 2. \tag{3.5.3}$$

Proof. We use techniques and ideas similar to those used in the proof of Theorem 3.5.1. We will concentrate only on the aspects related to the degeneracy of the spray and the corresponding degenerate Finsler function.

$(i) \implies (ii)$. According to the second item of Lemma 3.4.7, the spray S is degenerate. From the commutation formulae (3.4.8), we obtain

$$d_vF([S,H]) = d_vF([S,V]) = d_vF([H,V]) = 0,$$

which shows that the vector fields $[S,H]$, $[S,V]$ and $[H,V]$ have no components along the Liouville vector field $\mathcal{C}$. Using Frobenius Theorem, we conclude that the Berwald distribution $\mathcal{D} = \operatorname{span}\{H,S,V\}$ is integrable.

$(ii) \implies (iii)$. Assume that S is a degenerate spray and the Berwald distribution $\mathcal{D} = \operatorname{span}\{H,S,V\}$ is integrable. We can either follow the arguments of Theorem 3.5.1 for the same implication, or we can just take

$$\omega = i_{\mathbb{F}}\frac{\alpha}{\rho} = \frac{\alpha_i \delta y^i}{\rho} \in \mathcal{D}^*. \tag{3.5.4}$$

With this choice, we have $i_{\mathcal{C}}\omega = 1$ and $\mathcal{L}_{\mathcal{C}}\omega = 0$ which implies that $i_{\mathcal{C}}d\omega = 0$. Using the assumption that the Berwald distribution $\mathcal{D}$ is integrable, it follows that there exists $\theta \in \Lambda^1(T_0M)$ such that $d\omega = \omega \wedge \theta$. We evaluate both sides of this formula on the Liouville vector field $\mathcal{C}$ to obtain $0 = i_{\mathcal{C}}d\omega = \theta - (i_{\mathcal{C}}\theta)\,\omega$, whence $\theta = (i_{\mathcal{C}}\theta)\,\omega$ and therefore $d\omega = 0$.

Using the expression of the 1-form ω we see that the 2-form in formula (3.5.3) coincides with the 2-form (3.1.8). The assumption that the spray S is degenerate implies that $\operatorname{rank}(\Omega) = 2$ and hence formula (3.5.3) is valid.

$(iii) \implies (i)$. Consider a closed 1-form $\omega \in \mathcal{D}^*$ that satisfies formula (3.5.3). Condition $\omega \in \mathcal{D}^*$ implies that ω is actually given by formula (3.5.4). Similarly, as in the proof of Theorem 3.5.1, there exists a locally defined function f on T_0M such that $\omega = df$. The function $F = \exp(f)$ is 1^+-homogeneous and satisfies $d_hF = 0$. Formula (3.5.3) implies that $\operatorname{rank}(\Omega) = 2$, and in view of formula (3.2.6), it follows that $\operatorname{rank}(\omega_{F^2}) = 2$. Therefore, F is a degenerate Finsler function that metrizes the given spray S. □

Remark 3.5.4. It should be noted that one can provide an equivalent reformulation of the Finsler metrizability criteria of Theorems 3.5.1 and 3.5.3 using the holonomy distribution $\mathcal{D}_{\mathcal{H}}$ introduced in [77] as follows. The spray S is metrizable by a degenerate Finsler function if and only if the Berwald distribution $\mathcal{D}$ is integrable, which is equivalent to $\mathcal{D} = \mathcal{D}_{\mathcal{H}}$. In this case, the metrizability freedom of the spray S, [39, Proposition 4.9], is

$$m_s = \operatorname{codim} \mathcal{D}_{\mathcal{H}} = \operatorname{codim} \mathcal{D} = 1.$$

Therefore, if a non-flat spray S is metrizable, then the corresponding degenerate Finsler function is unique, up to a multiplicative constant.

3.6 Examples of metrizable and non metrizable sprays

In this section we exemplify the main results of our work. We will use both criteria (ii) and (iii) in Theorems 3.5.1 and 3.5.3 to test the metrizability of the given spray and to show how one can find the corresponding (degenerate) Finsler function in the integrable case. Finally, we will give two examples of sprays which are not Finsler metrizable.

3.6.1 Metrizable sprays

In the next example we show how the algorithms described in the proof of Theorem 3.5.1 can be applied to test the Finsler metrizability of a spray and, in the affirmative case, to construct the Finsler function.

Example 3.6.1. *The regular case:*
In $M = \{(x^1, x^2) \in \mathbb{R}^2, x^2 > 0\}$, we consider the SODE

$$\frac{d^2x^1}{dt^2} - \frac{2}{x^2}\frac{dx^1}{dt}\frac{dx^2}{dt} = 0, \quad \frac{d^2x^2}{dt^2} + \frac{1}{x^2}\left(\left(\frac{dx^1}{dt}\right)^2 - \left(\frac{dx^2}{dt}\right)^2\right) = 0. \tag{3.6.1}$$

It it clear that $G^1 = -\frac{2y^1 y^2}{x^2}$, and $G^2 = \frac{1}{2\,x^2}\left(\left(y^1\right)^2 - \left(y^2\right)^2\right)$. By direct calculations, we find that the components of the induced nonlinear connection are given by

$$N_1^1 = -\frac{y^2}{x^2}, \quad N_2^1 = -\frac{y^1}{x^2}, \quad N_1^2 = \frac{y^1}{x^2}, \quad N_2^2 = -\frac{y^2}{x^2}. \tag{3.6.2}$$

The local components of the Jacobi endomorphism are

$$R_1^1 = -\frac{(y^2)^2}{(x^2)^2}, \quad R_2^1 = R_1^2 = \frac{y^1 y^2}{(x^2)^2}, \quad R_2^2 = -\frac{(y^1)^2}{(x^2)^2}.$$

Moreover, the two components of the semi-basic 1-form α and the Ricci scalar are

$$\alpha_1 = \frac{R_2^2}{y^1} = -\frac{y^1}{(x^2)^2}, \quad \alpha_2 = \frac{R_1^1}{y^2} = -\frac{y^2}{(x^2)^2}, \quad \rho = R_1^1 + R_2^2 = -\frac{1}{(x^2)^2}\{(y^1)^2 + (y^2)^2\}.$$

We first test the metrizability of the spray using criterion (iii) of Theorem 3.5.1. All the information about the Finsler metrizability of the spray (3.6.1) is encoded into the 1-form

$$\omega = i_{\mathbb{F}}\frac{\alpha}{\rho} = \frac{\alpha_1}{\rho}\delta y^1 + \frac{\alpha_2}{\rho}\delta y^2 = \frac{y^1 dy^1 + y^2 dy^2}{(y^1)^2 + (y^2)^2} - \frac{1}{x^2}dx^2.$$

We have

$$\Omega = \frac{-1}{(y^1)^2 + (y^2)^2}\left(dx^1 \wedge dy^1 + dx^2 \wedge dy^2\right),$$

so $\mathrm{rank}(\Omega) = 4$ and hence S is a regular spray. Moreover, $d\omega = 0$ and so $\omega = df$, where

$$f(x,y) = \frac{1}{2}\ln\left((y^1)^2 + (y^2)^2\right) - \ln(x^2).$$

It follows that S is metrizable by the Finsler function

$$F(x,y) = \exp(f(x,y)) = \frac{\sqrt{(y^1)^2 + (y^2)^2}}{x^2}, \tag{3.6.3}$$

which is the Poincaré metric on the half-plane M.

Now, we will check again the metrizability of the SODE (3.6.1) using the second criterion of Theorem 3.5.1. Consider a vector field $H \in \mathfrak{X}(T_0M)$ satisfying conditions (3.4.1). One can choose such a vector field to be

$$H = -y^2\,\frac{\delta}{\delta x^1} + y^1\,\frac{\delta}{\delta x^2}.$$

Therefore, the vector fields $\{H, S, V = JH\}$ generate the Berwald distribution $\mathcal{D}$. From the following Lie brackets, we can see directly that this distribution is integrable:

$$[H, V] = S, \quad [S, V] = -H, \quad [S, H] = \rho V.$$

Hence the distribution $\mathcal{D} = \operatorname{Im}(h) \oplus \operatorname{Im}(\Phi)$ is integrable and, consequently, S is Finsler metrizable by a regular Finsler function F. To find the explicit expression for F, we search for the integral manifold IM of the Berwald distribution $\mathcal{D}$. We will construct the manifold IM using the fact that it contains all horizontal curves and the curves tangent to the vertical vector field V.
A vertical curve $c_v(t) = (x^i, y^i(t))$ is tangent to the vector field V if and only if

$$\frac{dy^1}{dt} = -y^2, \quad \frac{dy^2}{dt} = y^1.$$

With the initial condition $c_v(0) = (x^i, y^i)$, we find that the curve

$$c_v(t) = (x^1, x^2, \, y^1 \cos t - y^2 \sin t, \, y^1 \sin t - y^2 \cos t)$$

belongs to the family of hypersurfaces

$$(y^1)^2 + (y^2)^2 = f(x^1, x^2), \tag{3.6.4}$$

for some arbitrary function f on the base manifold M.

We will restrict the family of hypersurfaces (3.6.4) by requiring them to contain also horizontal curves. A curve $c_h(t) = (x^i(t), y^i(t))$ is horizontal if and only if $v(\dot{c_h}(t)) = 0$ and hence it satisfies the system of second order ordinary differential equations

$$\frac{d^2x^i}{dt^2} + N^i_j \frac{dx^j}{dt} = 0. \tag{3.6.5}$$

For the nonlinear connection (3.6.2), the system (3.6.5) becomes

$$\frac{dy^1}{dt} - \left(\frac{y^2}{x^2}\frac{dx^1}{dt} + \frac{y^1}{x^2}\frac{dx^2}{dt} \right) = 0, \quad \frac{dy^2}{dt} + \frac{y^1}{x^2}\frac{dx^1}{dt} - \frac{y^2}{x^2}\frac{dx^2}{dt} = 0.$$

We multiply the first equation by y^1, the second equation by y^2 and add them to obtain

$$y^1\frac{dy^1}{dt} + y^2\frac{dy^2}{dt} - \frac{1}{x^2}\left((y^1)^2 + (y^2)^2\right)\frac{dx^2}{dt} = 0.$$

The last equation can be written as

$$\frac{d}{dt}\left((y^1)^2+(y^2)^2\right)-\frac{2}{x^2}\left((y^1)^2+(y^2)^2\right)\frac{dx^2}{dt}=0 \tag{3.6.6}$$

We want the horizontal curves to belong to the family of hypersurfaces (3.6.4). Therefore, if we substitute (3.6.4) into (3.6.6), we obtain

$$\frac{d}{dt}\left(f(x^1,x^2)\right)-\frac{2}{x^2}f(x^1,x^2)\frac{dx^2}{dt}=0,$$

which implies $f(x^1,x^2)=c(x^2)^2,\ c\in\mathbb{R}^*$. Thus the integral manifold of the Berwald distribution $\mathcal{D}$ is given by

$$IM=\{(x^i,y^i)\in TM\big|\ \frac{1}{(x^2)^2}\left((y^1)^2+(y^2)^2\right)=c\},$$

which represents the indicatrix of the Poincaré metric (3.6.3).

The following example represents a degenerate spray. We will test its metrizability and obtain the corresponding degenerate Finsler function using the methods provided by the two criteria of Theorem 3.5.3.

Example 3.6.2. *The degenerate case:*
In $M=\mathbb{R}^2$, consider the SODE

$$\frac{d^2x^1}{dt^2}+\frac{x^2}{1+(x^2)^2}\frac{dx^1}{dt}\frac{dx^2}{dt}=0,\quad \frac{d^2x^2}{dt}=0. \tag{3.6.7}$$

The coefficients of the nonlinear connection are

$$N_1^1=\frac{x^2y^2}{2(1+(x^2)^2)},\ N_2^1=\frac{x^2y^1}{2(1+(x^2)^2)},\ N_1^2=N_2^2=0.$$

The local components of the Jacobi endomorphism and the Ricci scalar are

$$R_1^1=\frac{(y^2)^2[(x^2)^2-2]}{4[(x^2)^2+1]^2},\quad R_2^2=0,\quad \rho=\frac{(y^2)^2[(x^2)^2-2]}{4[(x^2)^2+1]^2}.$$

The semi-basic 1-form $\frac{\alpha}{\rho}=\frac{\alpha_1}{\rho}dx^1+\frac{\alpha_2}{\rho}dx^2$ has the components

$$\frac{\alpha_1}{\rho}=\frac{R_2^2}{y^1\rho}=0,\quad \frac{\alpha_2}{\rho}=\frac{R_1^1}{y^2\rho}=\frac{1}{y^2}.$$

The information about the metrizability and the regularity of the spray are encoded into the 1-form

$$\omega = i_{\mathbb{F}}\frac{\alpha}{\rho} = \frac{\alpha_1}{\rho}\delta y^1 + \frac{\alpha_2}{\rho}\delta y^2 = \frac{1}{y^2}dy^2.$$

It follows that the corresponding 2-form (3.1.8) is given by

$$\Omega = -\frac{1}{(y^2)^2}dx^2 \wedge dy^2,$$

so rank $(\Omega) = 2$ and hence the spray S is degenerate. Since $\omega = df$, where $f(x, y) = \ln|y^2|$, it follows that $d\omega = 0$. Hence, the degenerate Finsler function that metrizes the given spray is given by

$$F(x, y) = \exp(f(x, y)) = |y^2|.$$

We now test the metrizability of the system (3.6.7) using the second criterion of Theorem *3.5.3*. For the given system, we construct a Berwald frame using the conditions (3.4.1). By choosing

$$H = -y^2\frac{\delta}{\delta x^1},$$

the Lie brackets of the vector fields S, H and $V = JH$ are given by

$$[H, V] = 0, \quad [S, V] = -H + \frac{y^2x^2}{2(1 + (x^2)^2)}V, \quad [S, H] = \frac{y^2x^2}{2(1 + (x^2)^2)}H + \rho V.$$

It follows that the Berwald distribution is integrable and hence the system (3.6.7) is metrizable by a degenerate Finsler function. If we compute the integral manifold IM of the Berwald distribution we find that

$$IM = \{(x^1, x^2, y^1, y^2) \in TM, y^2 = c\},$$

which represents the indicatrix of the degenerate Finsler function $F(x, y) = |y^2|$.

3.6.2 Non-metrizable sprays

The following is an example of a spray proposed by Elgendi and Muzsnay whose metrizability freedom is zero [39]. We will also show that the spray is not Finsler metrizable using different techniques that are provided by Theorems 3.5.1 and 3.5.3.

Example 3.6.3. In $M = \{(x^1, x^2) \in \mathbb{R}^2, x^2 > 0\}$, we consider the spray

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - 2\left(\varphi y^1 + \frac{y^1 y^2}{2x^2}\right)\frac{\partial}{\partial y^1} - 2\left(\varphi y^2 - \frac{(y^1)^2}{4}\right)\frac{\partial}{\partial y^2},$$

where we use the notation $\varphi = (x^2 (y^1)^2 + (y^2)^2)^{1/2}$.

The coefficients of the nonlinear connection are

$$N_1^1 = \frac{y^2}{2x^2} + \varphi + \frac{x^2 (y^1)^2}{\varphi},\ N_2^1 = \frac{y^1}{2x^2} + \frac{y^1 y^2}{\varphi},\ N_1^2 = -\frac{y^1}{2} + \frac{x^2 y^1 y^2}{\varphi},\ N_2^2 = \varphi + \frac{(y^2)^2}{\varphi},$$

while the local components of the corresponding Jacobi endomorphism and Ricci scalar are

$$R_1^1 = \frac{(y^2)^2[4(x^2)^2 + 1]}{4(x^2)^2},\ R_2^2 = \frac{(y^1)^2[4(x^2)^2 + 1]}{4(x^2)^2},\ \rho = ((y^1)^2 + (y^2)^2)\frac{[4(x^2)^2 + 1]}{4(x^2)^2}.$$

The spray S is isotropic and the semi-basic 1-form $\frac{\alpha}{\rho} = \frac{\alpha_1}{\rho}dx^1 + \frac{\alpha_2}{\rho}dx^2$ has the components

$$\frac{\alpha_1}{\rho} = \frac{y^1}{(y^1)^2 + (y^2)^2}, \quad \frac{\alpha_2}{\rho} = \frac{y^2}{(y^1)^2 + (y^2)^2}.$$

Having an expression for $\frac{\alpha}{\rho}$, we compute the 1-form

$$\omega = i_{\mathbb{F}}\frac{\alpha}{\rho} = \frac{\alpha_1}{\rho}\delta y^1 + \frac{\alpha_2}{\rho}\delta y^2.$$

In the above formula, if we replace $\delta y^i = dy^i + N_j^i dx^j$, then by a direct calculation we find that $d\omega \neq 0$. Therefore, the two Theorems *3.5.1* and *3.5.3* tell us that our spray S is not Finsler metrizable.

We can reach the same conclusion by using the Berwald frame. For the given spray S, we search for a horizontal vector field that satisfies equations (3.4.1). In other words, we search for

$$H = H^1 \frac{\delta}{\delta x^1} + H^2 \frac{\delta}{\delta x^2}$$

that satisfies $\alpha(H) = 0$ and $[\mathcal{C}, H] = H$. In order to check the second criterion of Theorems 3.5.1 and 3.5.3, we can take any solution of the above system. Such a solution is given by $H^1 = -y^2$ and $H^2 = y^1$. We then consider the vertical vector field $V = JH$. According to last formula in (3.4.2), we have $[S, V] = -H + \nabla V$. One can easily see that the dynamical covariant derivative ∇V has a non-vanishing component along the Liouville vector field $\mathcal{C}$. Therefore, the Berwald distribution $\mathcal{D} = \mathrm{span}\{H, S, V\}$ is not integrable and hence the spray S is not Finsler metrizable. It should be noted that the same conclusion follows using the metrizability criterion introduced by Muzsnay in [77]. This is because $\mathcal{C} \in \mathcal{D}_{\mathcal{H}}$.

The next and last example represents another spray that is not Finsler metrizable.

Example 3.6.4. In $M=\mathbb{R}^2$ consider the following system of SODEs:

$$\frac{d^2x^1}{dt^2}+\left(\frac{dx^1}{dt}\right)^2+\left(\frac{dx^2}{dt}\right)^2=0, \qquad \frac{d^2x^2}{dt^2}+4\frac{dx^1}{dt}\frac{dx^2}{dt}=0.$$

It is easy to see that the components of the induced nonlinear connection are given by:

$$N_1^1=y^1,\ N_2^1=y^2,\ N_1^2=2y^2,\ N_2^2=2y^1.$$

The above system can be identified with a spray S given by:

$$S=y^1\frac{\partial}{\partial x^1}+y^2\frac{\partial}{\partial x^2}-(y^1)^2+(y^2)^2)\frac{\partial}{\partial y^1}-4y^1y^2\frac{\partial}{\partial y^2}.$$

Proceeding as in the previous example, we compute the local components of the corresponding Jacobi endomorphism and the Ricci scalar:

$$R_1^1=-(y^2)^2,\ R_2^2=-2(y^1)^2,\ \rho=-2(y^1)^2-(y^2)^2.$$

The components of the semi-basic 1-form $\frac{\alpha}{\rho}$ are given by:

$$\frac{\alpha_1}{\rho}=\frac{2y^1}{2(y^1)^2+(y^2)^2},\quad \frac{\alpha_2}{\rho}=\frac{y^2}{2(y^1)^2+(y^2)^2}.$$

Since

$$\delta y^1=dy^1+y^1dx^1+y^2dx^2$$
$$\delta y^2=dy^2+2y^2dx^1+2y^1dx^2,$$

it follows that

$$\omega=\frac{1}{2(y^1)^2+(y^2)^2}(2y^1dy^1+y^2dy^2+(2(y^1)^2+2(y^2)^2)dx^1+4y^1y^2dx^2).$$

By direct calculations, we get:

$$\begin{aligned} d\omega &= -\frac{4(y^1)^2}{2(y^1)^2+(y^2)^2}dy^1\wedge dx^1+\left(\frac{2y^2}{2(y^1)^2+(y^2)^2}-\frac{2(y^2)^3}{(2(y^1)^2+(y^2)^2)^2}\right)dy^2\wedge dx^1\\ &+\left(\frac{4y^2}{2(y^1)^2+(y^2)^2}-\frac{16(y^1)^2y^2}{(2(y^1)^2+(y^2)^2)^2}\right)dy^1\wedge dx^2\\ &+\left(\frac{4y^1}{2(y^1)^2+(y^2)^2}-\frac{8y^1(y^2)^2}{(2(y^1)^2+(y^2)^2)^2}\right)dy^2\wedge dx^2.\end{aligned}$$

Notice that $d\omega \neq 0$ so that the third condition of our main theorem is not satisfied. This means that the spray is not Finsler metrizable.

To obtain the same result using Berwald distribution, we search for H that satisfies (3.4.1), or, equivalently,

$$y^1\frac{\partial(H^1+H^2)}{\partial y^1}+y^2\frac{\partial(H^1+H^2)}{\partial y^2}=H^1+H^2.$$

We choose the components of H to be linear functions namely

$$H^1=y^2 \text{ and } H^2=-2y^1.$$

We thus have the following Lie brackets:

$$[S,H]=\frac{4y^2[-3(y^1)^2+(y^2)^2]}{|y|^2+(y^1)^2}S-\frac{y^1[9(y^2)^2-2(y^1)^2}{|y|^2+(y^1)^2}H-[2(y^1)^2+(y^2)^2]V.$$

$$[S,V]=-H-\frac{y^1[9(y^2)^2-2(y^1)^2}{|y|^2+(y^1)^2}V-\frac{4y^2[-3(y^1)^2+(y^2)^2]}{|y|^2+(y^1)^2}\mathcal{C}.$$

$$[H,V]=-\frac{4y^2[-3(y^1)^2+(y^2)^2]}{|y|^2+(y^1)^2}\mathcal{C}+2S-\frac{y^2[-18(y^1)^2+(y^2)^2]}{|y|^2+(y^1)^2}V.$$

According to Frobenius theorem, we conclude that the distribution D is not integrable. It follows by the second condition of our main theorem that the spray is not Finsler metrizable.

Concluding Remarks

In this chapter we have defined the Berwald distribution associated to a spray in 2-dimensional manifold. We studied the Finsler metrizabilty problem in terms of that distribution by a simple method.

This work can be further extended. One way of doing this is to construct a distribution for $n=3$ similar to the Berwald distribution. In this case, however, we have two problems. The first problem is how to construct the six globally independent vector fields associated to the given spray that generate the required distribution.

The second problem is how to obtain the Finsler metric corresponding to the integrable distribution. We conjecture that the procedure is more complex but in general not far from the two dimensional case.

Another direction of research is to replace (in dimension two) the arbitrary parallelizable frame defined in [115] by the Berwald frame and investigate the consequences. Strangely enough, the construction of a Berwald frame associated to a flat spray is complicated and has not yet been achieved.

We end this chapter by posing the following question: Can our method (in dimension two) be applied to the case when the spray and the Finsler structure are replaced by a semi-spray and a Lagrangian structure, respectively? And if so, can this be generalized to an arbitrary dimension?

4

Alternative Lagrangians Obtained by Scalar Deformations

Euler-Lagrange equations in the conservative form represent physical systems in which mass, momentum and total energy are conserved. On the other hand, these quantities are not conserved in systems described by the Lagrange equations in the non-conservative form. According to classical mechanics, a system is completely described by the equations of motion. A given system may admit different Lagrangians which give rise to different energies. Conservative systems are easier to deal with than non-conservative systems.

In this chapter we are interested in the following situation: For a non-conservative system with a given 1-form, how to determine necessary and sufficient conditions for transforming this system into a conservative one by using the notion of a deformation and imposing appropriate conditions on this 1-form. Our result provides a class of SODEs of the form

$$\mathcal{L}_S d_J L - dL = \frac{S(E_L)}{\mathcal{C}(L)} d_J L,$$

which admit a pure Lagrangian description in terms of a deformed Lagrangian $\Phi(L)$;

i. e., $\mathcal{L}_S d_J \Phi(L) - d\Phi(L) = 0$. We give various examples of explicit deformations, some of which may have applications in both theoretical physics and applied mathematics. These examples start with a regular Lagrangians L and include both regular and singular deformed Lagrangians $\Phi(L)$.

In the present chapter, we shall deal with the following items:

4.1. A short survey on alternative Lagrangians

4.2. Deformations of Lagrangians

4.3. Some consequences of the main results

4.4. Examples of non-conservative Lagrangians

The main results of this chapter have been submitted for publication [31]. arXiv: 1712.01392 [math.DG].

4.1 A short survey on alternative Lagrangians

As we mentioned in the last chapter, semi-sprays are associated with a system of SODEs

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0,\ \ i = 1, \ldots, n, \tag{4.1.1}$$

and an approach to solve the inverse problem of Lagrangian mechanics is finding the metric associated to the semi-spray. Necessary and sufficient conditions for the existence of the multiplier matrix (g_{ij}) are known as Helmholtz conditions [19].

In some particular cases, there exist several different Lagrangians, called alternative Lagrangians, such that the system (4.1.1) is equivalent to the system of Euler-Lagrange equations of each of those Lagrangians. Alternative Lagrangians are used to construct constants of motion [27].

One can work with alternative Lagrangians in order to simplify computations. Given a geometric structure on a differentiable manifold, it is possible to consider more general structures on the same manifold related to the first one and then study the relations between the geometric objects induced by these different structures. For example, in [8] the authors considered a Finsler manifold (M, F) and using a differential deformation $\alpha : \mathbb{R}_+ \to \mathbb{R}$, they constructed a Lagrange manifold $(M, L = \alpha(F^2))$. In general the computations in the associated Lagrange manifold are simpler and one can often obtain useful information about the initial Finsler manifold. The examples presented here involve Antonelli's ecological metric and Synge metric [7, 9] which have various applications in biology and physics.

During the last few years non-standard Lagrangians are considered good candidates to explain non-conservative dynamical systems. Non-standard Lagrangians are Lagrangians that cannot be expressed as differences between kinetic energy terms and potential energy terms. They can be of exponential type, of power-law type, or even radical type [30]. They were used, for instance, to study second order Ricati and Abel equations [29] or non-inertial dynamics [41].

The above facts encouraged us to investigate alternative Lagrangians obtained as deformations of other Lagrangians.

Suppose that the system (4.1.1) is variational, i.e. is equivalent with the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0.$$

Consider a deformation $\Psi \circ L$ of the initial Lagrangian L, where $\Psi : \mathbb{R} \to \mathbb{R}$ is a differentiable function of class C^2. The equations of motion of $\Psi(L)$ have the form

$$\frac{d}{dt}\left(\frac{\partial \Psi(L)}{\partial \dot{x}^i}\right) - \frac{\partial \Psi(L)}{\partial x^i} = \sigma_i(x, \dot{x}),$$

where $\sigma_i(x, \dot{x})$ is a covariant tensor field that represents some external forces. Hence the initial system appears as a non-conservative mechanical system with respect to the alternative Lagrangian $\Psi(L)$. Most studied cases deal with dissipative and gyroscopic systems [26, 34, 37, 71].

In a recent paper [28], starting with a given Lagrangian L, the authors determined all deformations Ψ such that $\Psi(L)$ is dynamically equivalent to L, i.e. $\Psi(L)$ and L give the same dynamical vector field. This paper, was the starting point for the study developed here. The present work was also inspired by [8, 9, 26, 30, 41, 76, 84] in which the authors searched for alternative or non-standard Lagrangians.

Through this chapter we will use the notations and the results of the first two sections of the last chapter in addition to the following definitions.

A *Lagrangian* on TM is a smooth function $L : TM \to \mathbb{R}$ whose Hessian with respect to the fiber coordinates

$$g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} \tag{4.1.2}$$

is nontrivial. We say that L is a *regular Lagrangian* if the Poincaré-Cartan 2-form $dd_J L$ is a symplectic form on TM, where $d_J = i_J \circ d - d \circ i_J$. Locally, the regularity condition of a Lagrangian L is equivalent to the fact that the Hessian (4.1.2) of L has maximal rank n on TM.

For a Lagrangian L, we consider $E_L = \mathcal{C}(L) - L$ its *Lagrangian energy.*

For an arbitrary semi-spray S and a Lagrangian L, the following 1-form (*the Lagrange differential* [98]) is a semi-basic 1-form:

$$\delta_S L \;=\; \mathcal{L}_S d_J L - dL = \left\{ S\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} \right\} dx^i.$$

Definition 4.1.1. a) *Given a semi-spray S, we say that S is Lagrangian if there exists a (locally defined) Lagrangian L such that $\delta_S L = 0$. This means that the solutions of the system* (4.1.1) *are among the solutions of the Euler-Lagrange equations of some Lagrangian L : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$. If L is regular, these two systems are equivalent* [19].

b) *Consider a semi-spray S and assume that $\sigma \in \Lambda^1(TM)$ is a semi-basic* 1-*form. We say that S is of Lagrangian type with covariant force field σ if there exists a* (*locally defined*) *Lagrangian L such that $\delta_S L = \sigma$. In this case, the solutions of the system* (4.1.1) *are among the solutions of the system of Lagrange equations:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = \sigma(x^i, \dot{x}^i). \tag{4.1.3}$$

If the desired Lagrangian L is regular, then the two systems (4.1.1) *and* (4.1.3) *are equivalent* [19].

4.2 Deformations of Lagrangians

The system (4.1.3) is equivalent to $\delta_S L = \sigma$, where S is the semi-spray on TM with local coefficients $G^i(x, y)$ and σ is the semi-basic 1-form $\sigma = \sigma_i(x, y)dx^i$. Thus S is of a Lagrangian type with covariant force field σ.

In this section we determine necessary and sufficient conditions for the existence of non-constant, differentiable of class C^2 deformations $\Phi : \mathbb{R} \to \mathbb{R}$, such that the system (4.1.1) is equivalent to the system of Euler-Lagrange equations of $\Phi(L)$:

$$\frac{d}{dt}\left(\frac{\partial \Phi(L)}{\partial \dot{x}^i}\right) - \frac{\partial \Phi(L)}{\partial x^i} = 0 \Longleftrightarrow \delta_S \Phi(L) = 0.$$

From now on we assume that S is a Lagrangian second order vector field corresponding to the Lagrangian $\Phi(L)$.

Using Frölicher-Nijenhuis formalism and the theory of derivations, we obtain our first main result of this chapter.

Theorem 4.2.1. *Let $S \in \mathfrak{X}(TM)$ be a semi-spray of Lagrangian type with covariant force field σ and L the corresponding* (*local*) *Lagrangian on TM with $S(L) \neq 0$ and $\mathcal{C}(L) \neq 0$. Then there exists a non-constant, differentiable function $\Phi : \mathbb{R} \to \mathbb{R}$ of class C^2 such that S is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$ if and only if the following three conditions are satisfied:*

$$\sigma = \frac{S(E_L)}{\mathcal{C}(L)} d_J L, \tag{4.2.1}$$

$$\frac{\Phi''(L)}{\Phi'(L)} = -\frac{S(E_L)}{S(L)\mathcal{C}(L)}, \tag{4.2.2}$$

$$\left(\Phi''\frac{\partial L}{\partial y^j}\frac{\partial L}{\partial y^i} + \Phi'\frac{\partial^2 L}{\partial y^i \partial y^j}\right)_{i,j} \neq O_n. \tag{4.2.3}$$

Proof. We investigate the relation between the equations of motion for L and for its deformation $\Phi(L)$.

We first compute the Lagrange differential of the deformed Lagrangian $\Phi(L)$.

$$\begin{aligned}
\delta_S \Phi(L) &= \left[S\left(\frac{\partial \Phi(L)}{\partial y^i}\right) - \frac{\partial \Phi(L)}{\partial x^i}\right] dx^i = \left[S\left(\Phi'(L)\frac{\partial L}{\partial y^i}\right) - \Phi'(L)\frac{\partial L}{\partial x^i}\right] dx^i \\
&= \left[\Phi''(L)S(L)\frac{\partial L}{\partial y^i} + \Phi'(L)S\left(\frac{\partial L}{\partial y^i}\right) - \Phi'(L)\frac{\partial L}{\partial x^i}\right] dx^i.
\end{aligned}$$

Using $d_J L = \frac{\partial L}{\partial y^i} dx^i$, the above relation is equivalent to

$$\delta_S \Phi(L) = \Phi''(L)S(L)d_J L + \Phi'(L)\delta_S L.$$

Now suppose that S is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$.

The conditions $\delta_S L = \sigma$ and $\delta_S \Phi(L) = 0$ are simultaneously satisfied if and only if

$$\Phi''(L)S(L)d_J L + \Phi'(L)\sigma = 0. \tag{4.2.4}$$

Set $X = S$ and $K = J$ in the formula $i_X d_K = -d_K i_X + \mathcal{L}_{KX} + i_{[K,X]}$, ([49], Appendix A page 215) to obtain

$$i_S d_J + d_J i_S = \mathcal{L}_{JS} + i_{[J,S]}. \tag{4.2.5}$$

Applying i_S to equation (4.2.4), we get

$$\Phi''(L) = -\frac{S(E_L)}{S(L)\mathcal{C}(L)}\Phi'(L). \tag{4.2.6}$$

This formula is obtained using (4.2.5), noting that $i_S d_J L = \mathcal{L}_{\mathcal{C}}(L) = \mathcal{C}(L)$, and

$$\begin{aligned}
i_S \sigma &= i_S\left(\mathcal{L}_S d_J L - dL\right) = i_S\left(d i_S d_J L + i_S d d_J L - dL\right) \\
&= i_S\left(d\mathcal{C}(L) - dL\right) = i_S dE_L = \left(\mathcal{L}_{\mathcal{S}} - d i_S\right)(E_L) = S(E_L).
\end{aligned}$$

Equation (4.2.6) is the required condition (4.2.3). Substituting (4.2.6) in (4.2.4) we obtain the required expression for σ in (4.2.1).

For $\Phi(L)$ to be also a Lagrangian, we need to investigate its Hessian with respect to the fiber coordinates (y^i).

If we denote $\tilde{g}_{ij} = \frac{\partial^2 \Phi(L)}{\partial y^i \partial y^j}$, then $\tilde{g}_{ij} = \Phi'' \frac{\partial L}{\partial y^j} \frac{\partial L}{\partial y^i} + \Phi' g_{ij}$. Hence we are searching for a deformation Φ such that $\left(\Phi'' \frac{\partial L}{\partial y^j} \frac{\partial L}{\partial y^i} + \Phi' g_{ij} \right)$ is not a trivial matrix. Therefore, the non-constant function Φ has to satisfy the condition

$$\exists\, i, j \in \{1, \ldots, n\}, \quad g_{ij} \neq -\frac{\Phi''}{\Phi'} \frac{\partial L}{\partial y^j} \frac{\partial L}{\partial y^i}.$$

Evidently $\Phi(L)$ is a regular Lagrangian if and only if $rank \left(\Phi'' \frac{\partial L}{\partial y^j} \frac{\partial L}{\partial y^i} + \Phi' g_{ij} \right) = n$ on TM.

The converse is trivial. □

Remark 4.2.2. For $\sigma = 0$, i. e., studying the deformations of conservative mechanical system, we reobtain the result of [28]: $\Phi(L)$ is dynamically equivalent to L if and only if $\Phi(t) = at + b$, $a, b \in \mathbb{R}$, $a \neq 0$ or L is a constant of motion for S, i. e., $S(L) = 0$. In the second case, Φ can be any differentiable function.

We remark that having a semi-spray such that its equations of motion can be written as $\delta_S L = \sigma$, we are able to apply a simple test to check if S is Lagrangian with respect to an alternative Lagrangian $\Phi(L)$. For this we need to verify condition (4.2.1). In the affirmative case, we have to integrate equation (4.2.2) and obtain the deformation Φ. But this is possible only if the expression $-\frac{S(E_L)}{S(L)\mathcal{C}(L)}$ can be written as an integrable function of the initial Lagrangian L. Also, the matrix (4.2.3) has to be non trivial and of maximal rank n, depending on whether the Lagrangian $\Phi(L)$ is singular or regular.

We consider some classes of Lagrangian's deformations that satisfy the conditions

$$\delta_S L = \frac{S(E_L)}{\mathcal{C}(L)} d_J L, \qquad \exists\, f : \mathbb{R} \to \mathbb{R} : \, -\frac{S(E_L)}{S(L)\mathcal{C}(L)} = f(L)$$

and condition (4.2.3), where $f : \mathbb{R} \to \mathbb{R}$ is an integrable function.

If we suppose that Φ is a strictly increasing function, then

$$\Phi(L) = \int \exp \left(\int f(L) dL \right) dL, \;\; \delta_S \Phi(L) = 0.$$

Homogeneous Lagrangians

Suppose that S is a spray of Lagrangian type with covariant force field σ and the corresponding Lagrangian L. In this subsection, we assume that σ and L are both homogeneous of order p, that is $\mathcal{C}(L) = pL$ and $\mathcal{L}_{\mathcal{C}}\sigma = p\sigma$. We will work on T_0M, the tangent bundle with the zero section removed.

If a Lagrangian L is homogeneous of degree 1, i.e., $p = 1 \Leftrightarrow \mathcal{C}(L) = L$, then $E_L = 0$ and formula (4.2.1) implies $\sigma = 0$. This case was treated in [28]. Therefore we assume $p > 1$ in our second main result.

Theorem 4.2.3. *Let $S \in \mathfrak{X}(T_0M)$ be a spray of Lagrangian type with covariant force field σ, with σ a semi-basic 1-form on T_0M, homogeneous of order $p > 1$. Let L be the corresponding (local) Lagrangian on TM, also homogeneous of order p. Suppose that L has positive values. Then there exists a differentiable strictly increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ of class C^2 such that S is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$, if and only if*

$$\left(\frac{1-p}{p}\frac{\partial L}{\partial y^j}\frac{\partial L}{\partial y^i} + L\frac{\partial^2 L}{\partial y^i \partial y^j}\right) \neq O_n \tag{4.2.7}$$

and

$$d_J L \wedge \sigma = 0.$$

Moreover, the deformation Φ is given by

$$\Phi(L) = aL^{\frac{1}{p}} + b,\ \ a, b \in \mathbb{R}.$$

Proof. Suppose that S is a Lagrangian vector field with the corresponding Lagrangian $\Phi(L)$: $\delta_S\Phi(L) = 0$. From Theorem 4.2.1, using $\mathcal{C}(L) = pL$ and $E_L = (p-1)L$, we deduce

$$\frac{\Phi''}{\Phi'} = \left(\frac{1}{p} - 1\right)\frac{1}{L} \tag{4.2.8}$$

and

$$\sigma = \left(1 - \frac{1}{p}\right) S\left(\ln L\right) d_J L.$$

It follows that $d_J L \wedge \sigma = 0$.

Conversely, suppose that σ is homogeneous of order $p > 1$ and $d_J L \wedge \sigma = 0$. Applying i_S to this relation, we get

$$\mathcal{C}(L)\sigma - i_S\sigma d_J L = 0 \Longleftrightarrow pL\sigma - (p-1)S(L)d_J L = 0 \Longleftrightarrow \sigma = \left(1 - \frac{1}{p}\right)\frac{S\left(L\right)}{L} d_J L.$$

Hence $\sigma = \frac{S(E_L)}{\mathcal{C}(L)} d_J L$. Using again Theorem 4.2.1, we conclude that $\delta_S \Phi(L) = 0$.

For L with positive values and Φ a strictly increasing function, we can easily integrate the equation (4.2.8) and obtain

$$\Phi(L) = aL^{\frac{1}{p}} + b,\ \ a, b \in \mathbb{R}.$$

The fact that the Hessian matrix of $\Phi(L)$ is non-trivial is equivalent to the condition (4.2.7). □

Note that if we have started with a homogeneous Lagrangian L of order $p > 1$ with $b = 0$, then the deformed Lagrangian $\Phi(L)$ is always homogeneous of order 1, i. e., $\mathcal{C}\left(\Phi(L)\right) = \Phi(L)$.

4.3 Some consequences of the main results

All the cases considered in this section satisfy the conditions:

$$\delta_S L = \frac{S(E_L)}{\mathcal{C}(L)}\, d_J L,\ \ \exists\, f : \mathbb{R} \longrightarrow \mathbb{R} :\ -\frac{S(E_L)}{S(L)\mathcal{C}(L)} = f(L), \tag{4.3.1}$$

with f integrable.

1. When f is a constant function, $f(L) = \gamma \in \mathbb{R}^*$, then
$$\Phi(L) = \frac{1}{\gamma}\exp(\gamma L) + a,\ a \in \mathbb{R}.$$
A similar deformation of Lagrangians was presented in [30, 41].

2. When $f(L) = \frac{\gamma}{L+a},\ \ \gamma \in \mathbb{R}^* \backslash \{-1\},\ \ a \in \mathbb{R}$, then
$$\Phi(L) = \frac{1}{1+\gamma}(L+a)^{1+\gamma} + b,\ \ b \in \mathbb{R}.$$
It is the deformation treated in [8, 30, 41, 58].

(a) As a particular sub-case, $f(L) = \left(\frac{1}{p} - 1\right)\frac{1}{L},\ p \in \mathbb{N},\ p > 1$ so that

$$\Phi(L) = aL^{\frac{1}{p}} + b,\ a \in \mathbb{R}^*, b \in \mathbb{R}.$$

This deformation is obtained when one searches for Lagrangians that are homogeneous of order $p > 1$ as the Lagrangians studied in Finsler Geometry [7].

(b) The deformation $\Phi(L) = \frac{b}{2}L^2 + aL + c$ was used by T. Kawaguchi and R. Miron [58]. They studied the geometry of a generalized Lagrange space obtained from a Riemannian space, using a deformation of the Riemannian metric.

3. If $f(L) = -\frac{1}{L+a},\ a \in \mathbb{R}$, we get

$$\Phi(L) = b\ln(L + a) + c,\ \ b \in \mathbb{R}^*, c \in \mathbb{R}.$$

Such a deformation appeared in [8].

4. If $f(L) = -\frac{2c}{cL+d},\ c, d \in \mathbb{R}$, we have

$$\Phi(L) = \frac{aL + b}{cL + d},\ a, b \in \mathbb{R},\ ad - bc \neq 0.$$

This deformation was appreaed in [8].

4.4 Examples of non-conservative Lagrangians

Most of the following examples produce regular Lagrangians $\Phi(L)$. Some calculations in this section were done by Maple.

We start with an example corresponding to the first class of Lagrangian's deformations enumerated above.

Example 4.4.1. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$\frac{d^2x^1}{dt^2} = 0, \qquad \frac{d^2x^2}{dt^2} = 0, \qquad \frac{d^2x^3}{dt^2} + x^1y^1 = 0. \tag{4.4.1}$$

The associated semi-spray is

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} - x^1y^1\frac{\partial}{\partial y^3}.$$

We consider the regular Lagrangian

$$L(x,y) := C_2 + \frac{1}{\gamma}\ln\left[\gamma\left(x^1y^1 + x^2y^2 + x^3y^3 + (y^1)^2 + (y^2)^2\right) - C_1\right],$$

with C_1, C_2, $\gamma \in \mathbb{R}^*$, defined on an open set of T_0M such that

$$x^1y^1 + x^2y^2 + x^3y^3 + (y^1)^2 + (y^2)^2 > \frac{C_1}{\gamma}$$

and the semi-basic 1-form

$$\sigma = \frac{\gamma\left[x^1x^3y^1 - (y^1)^2 - (y^2)^2 - (y^3)^2\right]\left[(x^1 + 2y^1)dx^1 + (x^2 + 2y^2)dx^2 + x^3dx^3\right]}{\left[\gamma(x^1y^1 + x^2y^2 + x^3y^3 + (y^1)^2 + (y^2)^2) - C_1\right]^2}.$$

By direct computations, we find that the system (4.4.1) is equivalent to $\delta_S L = \sigma$ with L and σ are as defined above.

One can prove that

$$\begin{aligned}
S(L) &= \exp\left(\gamma(C_2 - L)\right)\left[(y^1)^2 + (y^2)^2 + (y^3)^2 - x^1x^3y^1\right],\\
d_JL &= \exp\left(\gamma(C_2 - L)\right)\left[(x^1 + 2y^1)dx^1 + (x^2 + 2y^2)dx^2 + x^3dx^3\right],\\
\mathcal{C}(L) &= \exp\left(\gamma(C_2 - L)\right)\left[x^1y^1 + x^2y^2 + x^3y^3 + 2(y^1)^2 + 2(y^2)^2\right],\\
S(E_L) &= -\gamma\,\mathcal{C}(L)\,S(L), \qquad \delta_S L = -\gamma\,S(L)\,d_JL.
\end{aligned}$$

Making use of the above formulae, one can easily verify that both conditions (4.3.1) are satisfied with $f(L) = \gamma$. Hence, the deformed Lagrangian is given by

$$\Phi(L)(x,y) = \frac{1}{\gamma}\exp\left(\gamma C_2\right)\left[\gamma\left(x^1y^1 + x^2y^2 + x^3y^3 + (y^1)^2 + (y^2)^2\right) - C_1\right].$$

One can easily see that $\Phi(L)$ is a regular Lagrangian.

The next example corresponds to the second class of Lagrangian's deformations.

Example 4.4.2. Consider the Liénard-type second-order nonlinear differential equation of the form [47, 50]

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + h(x) = 0, \tag{4.4.2}$$

where g and h are real smooth functions of x defined on an interval. Equation (4.4.2) plays an important role in many areas of applied sciences, cardiology, neurology, biology, mechanics, seismology, chemistry, physics, and cosmology.

If the functions g, h satisfy the Chiellini integrability condition

$$\frac{d}{dx}\left(\frac{h}{g}\right) = kg,$$

with $k \in \mathbb{R}^*$, then it is possible to construct exact solutions for a first-order Abel equation of the first kind associated to (4.4.2).

We consider the Liénard equation (4.4.2) with g, h satisfying Chiellini's condition where k is of the form $-\alpha(\alpha+1)$ and $\alpha \in \mathbb{R}^*$.

Hence equation (4.4.2) can be written as $\delta_S L = \sigma$. Here,

$$S = y\frac{\partial}{\partial x} - (g(x)y + h(x))\frac{\partial}{\partial y},$$

is a semi-spray defined on a 1-dimensional real smooth manifold,

$$\sigma = 2\left(\frac{1}{\alpha}h(x) - g(x)y\right)dx$$

is a semi-basic 1-form and, finally, the Lagrangian is

$$L(x, y) = \left(y - \frac{1}{\alpha}\frac{h(x)}{g(x)}\right)^2.$$

We can show that σ satisfies the conditions (4.3.1) by the following straightforward computations:

$$\begin{aligned} \mathcal{C}(L) &= 2y\left(y - \frac{1}{\alpha}\frac{h(x)}{g(x)}\right), \qquad E_L = y^2 - \frac{1}{\alpha^2}\left(\frac{h(x)}{g(x)}\right)^2, \\ S(L) &= 2\alpha g(x)L, \qquad S(E_L) = -g(x)\mathcal{C}(L), \qquad d_J L = -\frac{1}{g(x)}\sigma. \end{aligned}$$

Moreover, $f(L) = -\frac{1}{2\alpha L}$ so that the deformed Lagrangian is given by

$$\Phi(L) = \frac{2\alpha\, a}{2\alpha+1}L^{\frac{2\alpha+1}{2\alpha}} + b, \;\; a, b \in \mathbb{R}.$$

We know from Theorem 4.2.1 that for any $a, b \in \mathbb{R}$, $\Phi(L)$ verifies $\delta_S \Phi(L) = 0$. In particular, for $b = 0$ and $a = \frac{\alpha}{2(\alpha+1)}$, we get

$$\Phi(L)(x, y) = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)} \left(y - \frac{1}{\alpha} \frac{g(x)}{f(x)} \right)^{\frac{2\alpha+1}{\alpha}}.$$

This Lagrangian was obtained in [47] using the Jacobi Last Multiplier method.

The next two examples concern the third class of Lagrangian's deformations.

Example 4.4.3. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$\frac{d^2x^1}{dt^2} = 0, \qquad \frac{d^2x^2}{dt^2} + 2x^2 = 0, \qquad \frac{d^2x^3}{dt^2} = 0, \tag{4.4.3}$$

and the associated semi-spray

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - 2x^2 \frac{\partial}{\partial y^2}.$$

The SODE (4.4.3) is equivalent to $\delta_S L = \sigma$ for the regular Lagrangian

$$L(x^2, y^1, y^2, y^3) := F_1(y^1)\, F_2(y^3)\, \exp\left(C_2 \left[\frac{1}{2}(y^2)^2 - (x^2)^2 \right] + C_1 y^2 \right),$$

where $F_1(y^1)$, $F_2(y^3)$ are smooth real functions depending only on y^1, y^3 respectively, C_1, $C_2 \in \mathbb{R}^+$, and the semi-basic 1-form on TM defined by

$$\begin{aligned} \sigma &= -2x^2(2C_2 y^2 + C_1) \exp\left(C_2 \left[\frac{1}{2}(y^2)^2 - (x^2)^2 \right] + C_1 y^2 \right) \times \\ &\quad \times \left(F_1' F_2 dx^1 + (C_1 + C_2 y^2) F_1 F_2 dx^2 + F_2' F_1\, dx^3 \right). \end{aligned}$$

Using the following computations

$$\begin{aligned} & d_J L = L \left(\frac{F_1'}{F_1}(y^1) dx^1 + (C_1 + C_2 y^2) dx^2 + \frac{F_2'}{F_2}(y^3) dx^3 \right), \\ & \mathcal{C}(L) = \left[y^1 \frac{F_1'}{F_1}(y^1) + y^2 (C_2\, y^2 + C_1) + y^3 \frac{F_2'}{F_2}(y^3) \right] L, \\ & S(L) = -2x^2(2C_2\, y^2 + C_1) L, \qquad S(E_L) = \frac{\mathcal{C}(L) S(L)}{L}, \qquad \delta_S L = \frac{S(E_L)}{\mathcal{C}(L)} d_J L, \end{aligned}$$

one can check that σ satisfies the first part of condition (4.3.1). Moreover, $f(L) = -\frac{1}{L}$, and thus the deformed Lagrangian is given by

$$\Phi(L)(x,y) = a \ln\left(F_1(y^1)\, F_2(y^3)\right) + a\left(C_2\left[\frac{1}{2}(y^2)^2 - (x^2)^2\right] + C_1 y^2\right) + b,\ \ a, b \in \mathbb{R},\ a \neq 0.$$

It should be noted that if both functions $F_i'' F_i - F_i'^2$, $i \in \{1, 2\}$ are non-identically zero then $\Phi(L)$ is a regular Lagrangian.

Example 4.4.4. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$\frac{d^2x^1}{dt^2} = 0, \qquad \frac{d^2x^2}{dt^2} + 2 = 0, \qquad \frac{d^2x^3}{dt^2} = 0, \tag{4.4.4}$$

and its associated semi-spray

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - 2\frac{\partial}{\partial y^2}.$$

The SODE (4.4.4) is equivalent to $\delta_S L = \sigma$ with the Lagrangian

$$L(x^2, y^1, y^2) := \sqrt{2C_2 y^1 + C_4}\ \exp\left(C_1\left(x^2 - \frac{1}{4}(y^2)^2\right) - C_3 y^2\right),\ \ C_i \in \mathbb{R}$$

defined on an open set such that

$$2C_2 y^1 + C_4 > 0$$

and the semi-basic 1-form

$$\begin{aligned}\sigma \ &= \ 2(C_1 y^2 + C_3)\sqrt{2C_2 y^1 + C_4}\ \exp\left(C_1\left(x^2 - \frac{1}{4}(y^2)^2\right) - C_3 y^2\right) \times \\ &\quad \times \left[\frac{C_2}{2C_2 y^1 + C_4} dx^1 - \frac{1}{2}(C_1 y^2 + 2C_3) dx^2\right].\end{aligned}$$

We check that σ satisfies the first condition of (4.3.1) by computing:

$$\begin{aligned} d_J L &= L\left[\frac{C_2}{2C_2 y^1 + C_4} dx^1 - \frac{1}{2}(C_1 y^2 + 2C_3) dx^2\right], \\ S(L) &= 2(C_1 y^2 + C_3)\sqrt{2C_2 y^1 + C_4}\ \exp\left(C_1\left(x^2 - \frac{1}{4}(y^2)^2\right) - C_3 y^2\right) \\ &= 2(C_1 y^2 + C_3) L, \end{aligned}$$

$$\mathcal{C}(L) = \left[\frac{C_2 y^1}{2C_2 y^1 + C_4} - \frac{1}{2}(C_1 y^2 + 2C_3) y^2\right] L,$$
$$S(E_L) = S(L)\left[\frac{C_2 y^1}{2C_2 y^1 + C_4} - \frac{1}{2}(C_1 y^2 + 2C_3) y^2\right] = \frac{\mathcal{C}(L) S(L)}{L},$$
$$\delta_S L = S(L)\left[\frac{C_2}{2C_2 y^1 + C_4} dx^1 - \frac{1}{2}(C_1 y^2 + 2C_3) dx^2\right] = \frac{S(L)}{L} d_J L.$$

It follows that $f(L) = -\frac{1}{L}$, so that the deformed Lagrangian is

$$\Phi(L)(x,y) = a\,(C_1[x^2 - \frac{1}{4}(y^2)^2] - C_3 y^2) + \frac{a}{2}\ \ln(2C_2 y^1 + C_4) + b,\ \ a, b \in \mathbb{R},\ a \neq 0.$$

It should be noted that $\Phi(L)$ is a singular Lagrangian.

The following two examples correspond to the forth class of Lagrangian's deformations.

Example 4.4.5. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$\frac{d^2x^1}{dt^2} = 0, \qquad \frac{d^2x^2}{dt^2} = 0, \qquad \frac{d^2x^3}{dt^2} + 2G^3(x,y) = 0, \tag{4.4.5}$$

and its associated semi-spray

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} - 2G^3(x,y)\frac{\partial}{\partial y^3}.$$

The SODE (4.4.5) is equivalent to $\delta_S L = \sigma$ with Lagrangian

$$L(x,y) := -\frac{d}{c} - \frac{1}{c^2\, C_1\, (x^1y^1 + x^2y^2 + x^3y^3 + (y^1y^2)^2 + C_2)}$$

and semi-basic 1-form

$$\sigma = -\frac{2\ [(y^1)^2 + (y^2)^2 + (y^3)^2 - 2x^3G^3]\,[(x^1 + 2y^1(y^2)^2)\,dx^1 + (x^2 + 2y^2\,(y^1)^2)\,dx^2 + x^3dx^3]}{c^2\, C_1\, (x^1y^1 + x^2y^2 + x^3y^3 + (y^1y^2)^2 + C_2)^3},$$

where c, d, $C_1 \in \mathbb{R}^*$ and $C_2 \in \mathbb{R}$. We choose the domain of L such that

$$x^1y^1 + y^2(x^2 + (y^1)^2y^2) + x^3y^3 + C_2 \neq 0.$$

We verify that σ satisfies the first part of condition (4.3.1) as follows:

$$d_J L = C_1\,(cL + d)^2\left[\left(x^1 + 2y^1(y^2)^2\right)dx^1 + \left(x^2 + 2y^2\,(y^1)^2\right)dx^2 + x^3dx^3\right],$$

$$\mathcal{C}(L) = (cL+d)\left[-\frac{1}{c} + 3C_1(y^2y^1)^2(cL+d)\right],$$
$$S(L) = C_1(cL+d)^2\left[(y^1)^2+(y^2)^2+(y^3)^2-2x^3G^3\right],$$
$$S(E_L) = \frac{2c}{(cL+d)}C(L)S(L), \qquad \delta_S L = \frac{2c}{(cL+d)}S(L)d_JL.$$

Therefore, both conditions (4.3.1) are satisfied, with $f(L) = -\frac{2c}{cL+d}$. Hence, the deformed Lagrangian is given by

$$\begin{aligned}\Phi(L)(x,y) \;&=\; -b\,c\,C_1\left[x^1y^1+x^2y^2+x^3y^3+\left(y^1y^2\right)^2+C_2\right]+\\ &\quad \frac{a}{c}\left[1+c\,d\,C_1\left(x^1y^1+x^2y^2+x^3y^3+\left(y^1y^2\right)^2+C_2\right)\right].\end{aligned}$$

In this case L is a regular Lagrangian and $\Phi(L)$ is singular.

Example 4.4.6. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$\frac{d^2x^1}{dt^2}=0, \qquad \frac{d^2x^2}{dt^2}=0, \qquad \frac{d^2x^3}{dt^2}=0. \tag{4.4.6}$$

The associated flat spray is

$$S = y^1\frac{\partial}{\partial x^1}+y^2\frac{\partial}{\partial x^2}+y^3\frac{\partial}{\partial x^3}.$$

The SODE (4.4.6) is equivalent to $\delta_S L = \sigma$ with Lagrangian

$$L(x,y) := -\frac{d}{c} - \frac{1}{C_1\,c^2\left[x^1y^1+y^2x^2+x^3y^3+(y^1y^2y^3)^2+C_2\right]},$$

and semi-basic 1-form on TM

$$\begin{aligned}\sigma \;=\; &-\frac{2\,\left[(y^1)^2+(y^2)^2+(y^3)^2\right]}{C_1\,c^2\left[x^1y^1+y^2x^2+x^3y^3+(y^1y^2y^3)^2+C_2\right]^3}\times\\ &\times\left[\left(x^1+2y^1(y^2y^3)^2\right)dx^1+\left(x^2+2y^2(y^1y^3)^2\right)dx^2+\left(x^3+2y^3(y^1y^2)^2\right)dx^3\right],\end{aligned}$$

where c, $C_1\in\mathbb{R}^*$ and d, $C_2\in\mathbb{R}$. We choose the domain of L such that

$$x^1y^1+y^2x^2+x^3y^3+(y^1y^2y^3)^2+C_2\neq 0.$$

By direct computations, we get

$$d_JL = C_1\,(cL+d)^2\Big[\left(x^1+2y^1(y^2y^3)^2\right)dx^1+\left(x^2+2y^2(y^1y^3)^2\right)dx^2$$

$$
\begin{aligned}
&+ \left(x^3 + 2y^3(y^1y^2)^2\right) dx^3\Big],\\
\mathcal{C}(L) &= C_1\,(cL+d)^2\left[5\left(y^1y^2y^3\right)^2 - C_2 - \frac{1}{C_1\,(cL+d)}\right],\\
S(L) &= C_1\,(cL+d)^2\left[(y^1)^2+(y^2)^2+(y^3)^2\right],\\
S(E_L) &= 2c\,{C_1}^2\,(cL+d)^3\,\left[(y^1)^2+(y^2)^2+(y^3)^2\right]\left[5\left(y^1y^2y^3\right)^2 - C_2 - \frac{1}{C_1\,(cL+d)}\right]\\
&= \frac{2c}{(cL+d)}\mathcal{C}(L)S(L),\\
\delta_S L &= \frac{2c}{(cL+d)}S(L)d_JL.
\end{aligned}
$$

Hence, σ satisfies the conditions (4.3.1) and $f(L) = -\frac{2c}{cL+d}$. Therefore, the deformed Lagrangian is

$$
\begin{aligned}
\Phi(L)(x,y) = &-b\,c\,C_1\left[x^1y^1 + x^2y^2 + x^3y^3 + \left(y^1y^2y^3\right)^2 + C_2\right]\\
&+ \frac{a}{c}\left[1 + c\,d\,C_1\left(x^1y^1 + x^2y^2 + x^3y^3 + \left(y^1y^2y^3\right)^2 + C_2\right)\right].
\end{aligned}
$$

In this case L and $\Phi(L)$ are both regular Lagrangians.

The last example represents one case of homogeneous Lagrangians.

Example 4.4.7. Let M be a real 3-dimensional smooth manifold. Consider the SODE

$$
\frac{d^2x^1}{dt^2} - \left[(y^1)^2+(y^2)^2+(y^3)^2\right] = 0, \qquad \frac{d^2x^2}{dt^2} = 0, \qquad \frac{d^2x^3}{dt^2} = 0, \tag{4.4.7}
$$

and its associated spray

$$
S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} + \left[(y^1)^2+(y^2)^2+(y^3)^2\right]\frac{\partial}{\partial y^1}.
$$

We consider the Lagrangian

$$
L(x,y) = \frac{1}{2}\exp(2x^1)\left[(y^1)^2+(y^2)^2+(y^3)^2\right].
$$

It is clear that (4.4.7) is equivalent to $\delta_S L = \sigma$, with

$$
\sigma = 2y^1\exp(2x^1)\left(y^1dx^1 + y^2dx^2 + y^3dx^3\right).
$$

It is also clear that both L and σ are homogeneous of order 2 and $\sigma = 2y^1 d_J L$ which implies that $\sigma \wedge d_J L = 0$. Applying Theorem 4.2.3, we obtain the deformed Lagrangian

$$\Phi(L) = a\sqrt{L} + b = \frac{a}{\sqrt{2}} \exp(x^1) \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + b, \ a, b \in \mathbb{R}.$$

One can easily show that $\Phi(L)$ is a singular Lagrangian. Actually, this example represents a special case of the conformal transformation of the ecological metric [7,8], which is given by $\overline{L}(x, y) = \exp(p\,\phi(x))[(y^1)^p + \cdots + (y^n)^p]$, where $n = dim(M)$, by taking $n = 3$, $p = 2$, $\phi(x) = x^1$.

Concluding Remarks

In this chapter, we have introduced and investigated alternative Lagrangians obtained by scalar deformation of a non-conservative mechanical system. We have also studied the homogeneous case in which both the Lagrangian and the covariant force field are homogeneous. Moreover, many classes of deformed Lagrangians, studied by various geometers, have been obtained as special cases from our main results.

Some natural questions arise concerning this work. For example, what is the effect of our deformation on geometric objects that live in a Lagrange space? In particular, what are the consequences of this deformation on the geometric objects that live in Finsler geometry? Last, but not least, what are the physical consequences of transforming a non-conservative system associated to a given semi-spray to a conservative one associated to the same semi-spray?

5 Semi-concurrent vector fields in Finsler geometry

Concurrent vector fields in Riemannian geometry and related topics have been studied before by many authors, see for example [80, 86, 106].

In 1950, Tachibana [97] generalized the notion of a concurrent vector field from Riemannian geometry to Finsler geometry and characterized the spaces admitting this kind of vector fields.

In 1974, Mastumoto and Eguchi [67] discussed the geometric consequences of the existence of concurrent vector fields on Finsler manifolds. They showed that a concurrent vector field controls the geometry of the underlying manifold.

Two years later, in 1976, Hashiguchi [51] treated a special kind of conformal change in Finsler geometry introducing the concept of C-conformal change. Hashiguchi was able to show that some of the results obtained by Matsumoto and Eguchi in the case when a Finsler manifold admits a concurrent vector field also hold when this space admit a C-conformal change.

In 2012, Youssef et. al. [108], have introduced the concept of B-condition which generalizes all the above mentioned concepts. They have shown that some of the results previously obtained remain valid in this more general setting.

In 2012, Peyghan and Tayebi [81] have proved that if ${M_1}_{f_2} \times_{f_1} M_2$ is a doubly warped Finsler manifold, with f_1 constant on M_1 and f_2 constant on M_2, then ${M_1}_{f_2} \times_{f_1} M_2$ is a Berwaldian manifold if and only if M_1 is Riemannain, M_2 is Berwaldian and $\dfrac{\partial f_2}{\partial x^i} C^{ij}_k = 0$.

Recently, in 2015, Faghfouri and Hosseninoghi [42] have treated the question: Does there exist a non-constant smooth function f on a Finsler manifold M such that $\dfrac{\partial f}{\partial x^i} C^{ij}_k = 0$? They showed that any two dimensional Finsler manifold admitting such a kind of function is necessarily Riemannian. They conjectured that this should hold for a Finsler manifold of arbitrary dimension. This problem, in a more general setting, is one of the main objects of the present chapter.

In this chapter, we investigate the relation between all the above mentioned concepts. We focus our attention to the most general concept, which we call *semi-concurrent vector field*. According to Tashibana's theorem [97], we prove that a regular Finsler manifold which admits a concurrent vector field is Riemannian. We study some special Finsler manifolds admitting a semi-concurrent vector field. Various examples of non-Riemannain conic Finsler spaces admitting semi-concurrent vector fields are given. We investigate the cases in which an n-dimensional Finsler manifold (M, F), admitting a non-constant smooth function f on M such that $\frac{\partial f}{\partial x^i}\frac{\partial g^{ij}}{\partial y^k} = 0$, is a Riemannian manifold, giving an answer to the question of [42]. Finally, we conjuncture that there is no regular Finsler metric admitting a semi-concurrent vector field. In other words, a Finsler metric admitting a semi-concurrent vector field is necessarily either Riemannian or conic Finslerian.

In the present chapter, we shall deal with the following items:

5.1. Notations and preliminaries

The main results of this chapter have been submitted for publication [112]. arXiv: 1802.02405 [math.DG].

5.1 Notations and preliminaries

Let (M, F) be an n-dimensional smooth connected Finsler manifold; F being the Finsler function (Finsler metric or Lagrangian). Let (x^i) be the coordinates of any point of the base manifold M and (y^i) a supporting element at the same point. We use the following terminology and notations:

∂_i: partial differentiation with respect to x^i,

$\dot{\partial}_i$: partial differentiation with respect to y^i (basis vector fields of the vertical bundle),

$g_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2 = \dot{\partial}_i\dot{\partial}_j E$: the Finsler metric tensor, where $E := \frac{1}{2}F^2$ is the energy function,

$l_i := \dot{\partial}_i F = g_{ij}l^j = g_{ij}\frac{y^j}{F}$: the normalized supporting element; $l^i := \frac{y^i}{F}$,

$l_{ij} := \dot{\partial}_i l_j$,

$h_{ij} := Fl_{ij} = g_{ij} - l_i l_j$: the angular metric tensor,

$C_{ijk} := \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$: the Cartan tensor,

$C^i_{jk} := g^{ri}C_{rjk} = \frac{1}{2}g^{ri}\dot{\partial}_k g_{rj}$: the h(hv)-torsion tensor,

G^i: the components of the geodesic spray associated with (M, F),

$N^i_j := \dot{\partial}_j G^i$: the Barthel (or Cartan nonlinear) connection associated with (M, F),

$\delta_i := \partial_i - N^r_i\dot{\partial}_r$: the basis vector fields of the horizontal bundle,

$G^i_{jh} := \dot{\partial}_h N^i_j = \dot{\partial}_h\dot{\partial}_j G^i$: the coefficients of Berwald connection,

$\Gamma^i_{jk} := \frac{1}{2} g^{ir}(\delta_j g_{kr} + \delta_k g_{jr} - \delta_r g_{jk})$: the Christoffel symbols with respect to δ_i,

$(\Gamma^i_{jk}, N^i_j, C^i_{jk})$: The Cartan connection.

For the Cartan connection $(\Gamma^i_{jk}, N^i_j, C^i_{jk})$, we define
$X^i_{j|k} := \delta_k X^i_j + X^m_j \Gamma^i_{mk} - X^i_m \Gamma^m_{jk}$: the horizontal covariant derivative of X^i_j,
$X^i_j|_k := \dot{\partial}_k X^i_j + X^m_j C^i_{mk} - X^i_m C^m_{jk}$: the vertical covariant derivative of X^i_j.

Now, we give the definition we shall adopt for a Finsler manifold.

Definition 5.1.1. *A Finsler structure on a manifold M is a function*

$$F : TM \to \mathbb{R}$$

with the following properties:

(a) *$F \geqslant 0$ and $F(0) = 0$.*

(b) *F is C^∞ on the slit tangent bundle $T_0M := TM \backslash \{0\}$.*

(c) *$F(x, y)$ is positively homogenous of degree one in y: $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in TM$ and $\lambda > 0$.*

(d) *The Hessian matrix $g_{ij}(x, y) := \dot{\partial}_i \dot{\partial}_j E$ is positive-definite at each point of T_0M, where $E := \frac{1}{2} F^2$ is the energy function of the Lagrangian F.*

The pair (M, F) is called a Finsler manifold and the symmetric bilinear form $g = g_{ij}(x, y) dx^i \otimes dx^j$ is called the Finsler metric tensor of the Finsler manifold (M, F).
Sometimes, a function F satisfying the above conditions is said to be a regular Finsler metric.

• *When the metric tensor g is non-degenerate at each point of T_0M, the pair (M, F) is called a pseudo-Finsler manifold.*

• *When F satisfies the conditions (a)-(d) but only on an open conic subset A of TM (for every $v \in A$ and $\mu > 0, \mu v \in A$), the pair (A, F) is called conic Finsler manifold. If, moreover, the metric tensor g is non-degenerate at each point of A, the pair (A, F) is called conic pseudo-Finsler manifold.*

For more details about conic Finsler and conic pseudo-Finsler metrics we refer, for example, to [57].

In the following, we give the definitions of the special Finsler manifolds we shall use in the sequel.

Definition 5.1.2. *A Finsler manifold (M, F) is said to be Berwaldian if the Berwald tensor $G^h_{ijk} := \dot{\partial}_i G^h_{jk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k G^h$ vanishes.*

It is to be noted that [65],

$$G^h_{ijk} = 0 \Longleftrightarrow C^h_{ij|k} = 0 \Longleftrightarrow G^h(x,\, y) \text{ is quadratic in}\, y \in T_x M.$$

Definition 5.1.3. *A Finsler manifold (M, F) is called Landsbergian if the Landsberg tensor $L_{ijk} := \frac{1}{2} F \ell_h G^h_{ijk}$ vanishes.*

It is to be noted that [65],

$$L_{ijk} = 0 \Longleftrightarrow y^i\, C^h_{jk|i} = 0.$$

Definition 5.1.4. [68] A Finsler manifold (M, L) of dimension $n \geq 2$ is said to be C_2-like if the Cartan tensor C_{ijk} satisfies

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k,$$

where $C_i := C_{ijk} g^{jk}$ and $C^2 := C_i C^i$.

Definition 5.1.5. [64] A Finsler manifold (M, F) of dimension $n \geq 3$ is called C-reducible if the Cartan tensor C_{ijk} has the form:

$$C_{ijk} = \frac{1}{n+1}(h_{ij}\, C_k + h_{ki}\, C_j + h_{jk}\, C_i). \tag{5.1.1}$$

Definition 5.1.6. [69] A Finsler manifold (M, L) of dimension $n \geq 3$ is called semi-C-reducible if the Cartan tensor C_{ijk} is written in the form:

$$C_{ijk} = \frac{r}{n+1}(h_{ij} C_k + h_{ki} C_j + h_{jk} C_i) + \frac{t}{C^2} C_i C_j C_k, \tag{5.1.2}$$

where r and t are scalar functions such that $r + t = 1$.

5.2 Semi-concurrent vector fields

Let (M, F) be an n-dimensional smooth Finsler manifold.

Definition 5.2.1. [97] A vector field $X^i(x)$ on M is said to be concurrent (with respect to Cartan connection) if it satisfies

$$X^h(x)\, C_{hij} = 0, \quad X^i{}_{|j} = -\delta^i_j. \tag{5.2.1}$$

The condition (5.2.1) will be called C-condition.

Definition 5.2.2. [51] The manifold M fulfils the C-conformal condition if there exists on M a conformal transformation $\overline{F} = e^{\sigma(x)}F$ such that

$$\sigma_h(x)\, C^h_{ij} = 0, \tag{5.2.2}$$

where $\sigma_h := \frac{\partial \sigma}{\partial x^h}$. The condition (5.2.2) will be called CC-condition.

Definition 5.2.3. [42] Assume that there exists a non-constant smooth function f on M such that

$$f_i(x)\frac{\partial g^{ij}}{\partial y^k} = 0, \tag{5.2.3}$$

where $f_i := \frac{\partial f}{\partial x^i}$. The condition (5.2.3) will be called F-condition.

Definition 5.2.4. *A vector field $B^i(x)$ on M is said to be semi-concurrent if it satisfies*

$$B^h(x)\, C_{hij} = 0. \tag{5.2.4}$$

The condition (5.2.4) *will be called the SC-condition.*

Lemma 5.2.5. *If a Finsler manifold satisfies the F-condition* (5.2.3)*, then it satisfies the SC-condition* (5.2.4)*.*

Proof. Assume that (M, F) satisfies (5.2.3) so that

$$f_i(x)g^{ij} = B^j(x),$$

where B^j are smooth functions on M. From which,

$$f_i = B^j g_{ij}.$$

Differentiating the above relation with respect to y^k, we get

$$B^i(x)\, C_{ijk} = 0.$$

This means that (M, F) satisfies the SC-condition (5.2.4). □

Remark 5.2.6. The converse of the above result is not true in general. In fact, if (M, F) satisfies the SC-condition, then, by (5.2.4), $\frac{\partial}{\partial y^k}(B^i\, g_{ij}) = 0$. Then, by integration, $B^i\, g_{ij} = \lambda_j(x)$ and $B^i = \lambda_j(x)\, g^{ij}$. Now, by differentiation both sides with respect to y^k, we find that $\lambda_j(x)\, C^{ij}_k = 0$. Therefore, the F-condition (5.2.3) is satisfied only in the case when $\lambda_j(x)$ represents the gradient of a non-constant function $f \in C^\infty(M)$. This shows in particular that the SC-condition is weaker than the F-condition.

In view of Lemma 5.2.5, one can observe that the above mentioned conditions (5.2.1)-(5.2.4) are interrelated as follows:

$$CC\text{-condition} \Longrightarrow F\text{-condition} \Longrightarrow SC\text{-condition},$$

$$C\text{-condition} \Longrightarrow SC\text{-condition}.$$

Consequently, the SC-condition is the weakest condition and hence the most general one. In the following we shall be concerned mainly with the SC-case: $B^i\, C_{ijk} = 0$. In fact, if a problem is solved in the SC-case, it would be also solved in the CC-, F- and C-cases. Moreover, the non-existence of a semi-concurrent vector field (the SC-condition is not satisfied) implies the non-existence of concurrent vector fields and the non-satisfaction of both the CC-condition and the F-condition.

Property 5.2.7. *In the F-case, the functions $f^i(x)$ defined by $f^i(x) := f_k g^{ik}$ are functions of x only. Indeed,*

$$\frac{\partial f^i}{\partial y^j} = \frac{\partial}{\partial y^j}(f_k(x) g^{ik}) = f_k(x)\frac{\partial g^{ik}}{\partial y^j} = 0.$$

So when we lower (or raise) the index of f^i (or f_i) the result is always functions of x only. one can easily show that the same property is valid for the other three conditions .

Lemma 5.2.8. *For the nonzero functions B^i satisfying* (5.2.4), *if the scalars α and α' satisfy*

$$\alpha\, B^i + \alpha' y^i = 0, \tag{5.2.5}$$

then $\alpha = \alpha' = 0$, which means that the two vector fields $B^i(x)$ and y^i are independent.

Proof. Contraction of (5.2.5) by y_i and B_i, respectively, gives rise to the system:

$$B_0\alpha + F^2\alpha' = 0,$$

$$B^2\alpha + B_0\alpha' = 0,$$

where $B_0 := B_i\, y^i = B^i\, y_i$ and $B^2 = B_i\, B^i$. We regard this system as a system of algebraic equations in the unknowns α and α'. Let us show first that B_0 and $B^2\, F^2 - B_0^2$ are nonzero.

Seeking for a contradiction, assume that $B_0 = 0$, so $B_i\, y^i = 0$. Differentiation with respect to y^j gives $B_j = 0$. Since M is connected, it follows that $B(x)$ is a constant function on M, which is a contradiction.

Now, assume that $B^2\, F^2 - B_0^2 = 0$, then we have $B^2 - \frac{B_0^2}{F^2} = 0$. Differentiation both sides with respect to y^i, we get

$$0 = -\frac{1}{F^2}\left(2B_0\,\frac{\partial B_0}{\partial y^i}\right) + \frac{2}{F^3}\,B_0^2\,\frac{\partial F}{\partial y^i} = -\frac{2}{F^2}B_0\,B_i + \frac{2}{F^3}\,B_0^2\,l_i, \quad \text{or} \quad B_i = \frac{B_0}{F}\,l_i.$$

Again, differentiating $B_i = \frac{B_0}{F}\,l_i$ with respect to y^j, we obtain

$$\frac{B_j}{F}\,l_i - \frac{B_0}{F^2}\,l_i\,l_j + \frac{B_0}{F^2}\,h_{ij} = 0,$$

and using $B_i = \frac{B_0}{F}l_i$, we get $\frac{B_0}{f^2}\,h_{ij} = 0$, which is a contradiction. Hence, we have

$$B_0 \neq 0, \;\; B^2\,F^2 - B_0^2 \neq 0. \tag{5.2.6}$$

Finally, since $B^2F^2 - B_0^2 \neq 0$, the above system has only the trivial solution; that is, $\alpha = \alpha' = 0$. □

Theorem 5.2.9. *Let (M, F) be a Finsler manifold. In each of the following cases*

(a) *(M, F) is two-dimensional,*

(b) *(M, F) is three-dimensional such that $F(x, -y) = F(x, y)$,*

(c) *(M, F) is C-reducible,*

(d) *(M, F) is Berwaldian with* $\det(B^i{}_{|j}) \neq 0$,

if the Finsler manifold (M, F) satisfies the SC-condition (5.2.4)*, then it is Riemannian.*

Proof.
(a) The Cartan tensor C_{ijk} of a two-dimensional Finsler manifold is given by

$$FC_{ijk} = J\eta_i\eta_j\eta_k,$$

where η_i is an orthogonal vector to y^i and J is the Berwald main scalar [16]. Contracting by B^i, we have

$$J\,B^i\eta_i\eta_j\eta_k = 0,$$

If $B^i\,\eta_i = 0$, then this leads to $B^i = \mu y^i$, which contradicts Lemma 5.2.8. Hence, $J = 0$ and so $C_{ijk} = 0$.

(b) Making use of Lemma 5.2.8, the proof can be carried out in a similar manner as in [67] for concurrent vector fields.

(c) It is to be noted first that the condition $B^i\,C_{ijk} = 0$ leads to $B^i\,C_i = 0$. As (M, F) is C-reducible, then, by (5.1.1), we have

$$C_{ijk} = h_{ij}C_k + h_{ki}C_j + h_{jk}C_i.$$

Contracting the above equation by B^iB^j, we get $B^i\,B^j\,h_{ij}C_k = 0$. We have the following implication

$$\begin{aligned} B^iB^jh_{ij}C_k = 0 \;&\Longrightarrow\; B^iB^j(g_{ij} - l_il_j)C_k = 0 \\ &\Longrightarrow\; (B^2 - B_0^2/F^2)C_k = 0 \\ &\Longrightarrow\; (B^2F^2 - B_0^2)C_k = 0 \\ &\Longrightarrow\; C_k = 0, \text{ in view of (5.2.6).} \end{aligned}$$

(d) As (M, F) is Berwaldian, $C_{ijk|h} = 0$ and as $\det(B^i{}_{|j}) \neq 0$, then $B^i{}_{|j}\,Z^i_k = \delta^i_k$, where (Z^i_k) denotes the inverse of $(B^i{}_{|j})$. Now, by the SC-condition, $B^i\,C_{ijk} = 0$. Then we have the following implications:

$$\begin{aligned} B^i\,C_{ijk} = 0 \;&\Longrightarrow\; (B^i\,C_{ijk})_{|h} = 0 \\ &\Longrightarrow\; B^i{}_{|h}\,C_{ijk} = 0 \\ &\Longrightarrow\; Z^h_r\,B^i{}_{|h}\,C_{ijk} = 0 \\ &\Longrightarrow\; C_{rjk} = 0. \end{aligned}$$

This completes the proof. □

Corollary 5.2.10. *If B^i is a concurrent vector field, then (d) is true without any condition, since in this case $B^i{}_{|j} = -\delta^i_j$. Moreover, the same result is also true for the Landsbergian case. This retrieves some results of Matsumoto* [67].

Remark 5.2.11. It is to be noted that part **(a)** of the above theorem generalizes the main result of [42]. The last is retrieved from **(a)** by letting B^i be a gradient of a non-constant function on M.

Theorem 5.2.12. *A semi-C-reducible Finsler manifold* (M, F) *satisfying the SC-condition* (5.2.4) *is either Riemannian or C2-like.*

Proof. Contracting (5.1.2) by B^iB^j and using the fact that $B^2F^2 - B_0^2 \neq 0$ (5.2.6), we get $rB^iB^jh_{ij}C_k = 0$, hence,

$$\begin{aligned} rB^iB^jh_{ij}C_k = 0 &\implies rB^iB^j(g_{ij} - l_il_j)C_k = 0 \\ &\implies r(B^2 - B_0^2/F^2)C_k = 0 \\ &\implies r(B^2F^2 - B_0^2)C_k = 0 \\ &\implies rC_k = 0, \end{aligned}$$

then either $r = 0$, which implies that the space is C_2-like, or $C_i = 0$, which implies that the space is Riemannian. □

Remark 5.2.13. It should be noted that if g is not positive definite, then the condition $C_i = 0$ does not necessarily imply that $(M,\ F)$ is Riemannain [36]. This can be shown by the following example (where the calculations have been performed using Maple program [111]).

Take $M = \mathbb{R}^3$, and $F = f(x)(y_1\, y_2\, y_3)^{\frac{1}{3}}$. The Finsler function F is not defined on the whole $T\mathbb{R}^3$, it is defined on the domain $D = T\mathbb{R}^3 - \{(x_i, y_i) \in T\mathbb{R}^3 \,|\, y_i \neq 0\}$.

The components of the metric are:

$$\begin{aligned} g_{11} &= -\frac{1}{9}\,\frac{f(x)\,(y_2\,y_3)^2}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}, & g_{12} &= \frac{2}{9}\,\frac{f(x)\,y_1\,y_2\,y_3^2}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}, \\ g_{13} &= \frac{2}{9}\,\frac{f(x)y_1\,y_2^2\,y_3}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}, & g_{22} &= -\frac{1}{9}\,\frac{f(x)\,(y_1\,y_3)^2}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}, \\ g_{23} &= \frac{2}{9}\,\frac{f(x)y_1^2\,y_2\,y_3}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}, & g_{33} &= -\frac{1}{9}\,\frac{f(x)\,(y_1\,y_2)^2}{(y_1\,y_2\,y_3)^{\frac{4}{3}}}. \end{aligned}$$

Hence, the components of Cartan tensor are:

$$C_{111} = \frac{2}{27}\,\frac{f(x)\,(y_2\,y_3)^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}, \qquad C_{112} = -\frac{1}{27}\,\frac{f(x)\,y_1\,y_2^2\,y_3^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}},$$

$$\begin{aligned}
C_{113} &= -\frac{1}{27}\frac{f(x)\,y_1\,y_2^3\,y_3^2}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}, & C_{122} &= -\frac{1}{27}\frac{f(x)\,y_1^2 y_2\,y_3^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}},\\
C_{123} &= \frac{2}{27}\frac{f(x)\,(y_2\,y_3)^2\,y_1^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}, & C_{133} &= -\frac{1}{27}\frac{f(x)\,y_1^2\,y_2^3\,y_3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}},\\
C_{222} &= \frac{2}{27}\frac{f(x)\,(y_1\,y_3)^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}, & C_{223} &= -\frac{1}{27}\frac{f(x)\,y_1^3\,y_2\,y_3^2}{(y_1\,y_2\,y_3)^{\frac{7}{3}}},\\
C_{233} &= -\frac{1}{27}\frac{f(x)\,y_1^3\,y_2^2\,y_3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}, & C_{333} &= \frac{2}{27}\frac{f(x)\,(y_1\,y_2)^3}{(y_1\,y_2\,y_3)^{\frac{7}{3}}}.
\end{aligned}$$

We note that, $C_i = 0$, for all i, although the space is not Riemannian.

Moreover, in the above example, although $B^i C_i = 0$, we do not have $B^i\,C_{ijk} = 0$. For example,

$$B^i\,C_{i11} = \frac{1}{27}\frac{f(x)\;(2B^1\,y_2\,y_3 - B^2\,y_1\,y_3 - B^3\,y_1\,y_2)}{y_1^2\;(y_1\,y_2\,y_3)^{\frac{1}{3}}} \neq 0.$$

Therefore, the above space does not admit a semi-concurrent vector field.

Remark 5.2.14. As a by-product, the above example shows the necessity of the condition $F(-y) = F(y)$ in part (2) of Theorem 5.2.9.

The T-tensor is defined by [69]

$$T_{hijk} = F\,C_{hij}|_k + C_{hij}\,l_k + C_{hik}\,l_j + C_{hjk}\,l_i + C_{ijk}\,l_h.$$

It is well-known that if (M, F) is Riemannian, then the T-tensor vanishes. But the converse is not true in general. The next result shows that the converse is true in the case where (M, F) satisfies the SC-condition (5.2.4).

Theorem 5.2.15. *A Finsler manifold satisfying the SC-condition is Riemannian if and only if the T-tensor T_{hijk} vanishes.*

Proof. We first show that the vertical covariant derivative of B^i vanishes identically. Indeed,

$$B^i|_k = \dot{\partial}_k B^i + B^m C^i_{mk} = \dot{\partial}_k B^i = 0,$$

since B^i are functions of x only.

Let the T-tensor vanish, then

$$F\, C_{hij}|_k + C_{hij}\, l_k + C_{hik}\, l_j + C_{hjk}\, l_i + C_{ijk}\, l_h = 0.$$

Contracting by B^i, and taking into account that $B^i|_k = 0$, we find that $\frac{B_0}{F}\, C_{hjk} = 0$. Since $B_0 \neq 0$ by (5.2.6), it follows that $C_{hjk} = 0$. □

Let us write

$$T_{ij} := T_{ijhk}\, g^{hk} = F\, C_i|_j + l_i\, C_j + l_j\, C_i.$$

We have the following immediate result.

Corollary 5.2.16. *A Finsler manifold satisfying the SC-condition is Riemannian if and only if the tensor T_{ij} vanishes.*

Remark 5.2.17. Theorem 5.2.15 and corollary 5.2.16 generalize the corresponding results of Masumoto-Eguchi [67] for concurrent vector fields.

5.3 Special case: Concurrent vector fields

As far as the authors know the first two papers which introduced the concept of a concurrent vector field on Finsler manifolds are Tachibana [97] and Masumoto-Eguchi [67]. Tachibana claimed that a necessary and sufficient condition for a Finsler manifold to admit a concurrent vector field is that its line element is expressible in the form

$$ds^2 = (dx^n)^2 + (x^n)^2 H(x^\alpha, dx^\alpha), \quad \alpha = 1, \ldots, n-1, \tag{5.3.1}$$

where $H(x^\alpha, dx^\alpha)$ is the square of the line element of an arbitrary $(n-1)$-dimensional Finsler manifold.

Theorem 5.3.1. [Tachibana]
A Finsler manifold (M, F) *admits a concurrent vector field if and only if* F *satisfies* (5.3.1).

Matsumoto and Eguchi [67] remarked however that the proof of Tachibana's theorem is not clear. In his book [63], Matsumoto argued that, for metrics of the form (5.3.1), the vector field $(0, \ldots, 0, -X^n)$ is certainly concurrent, but he could not see that the necessity of Tachibana's theorem should hold. In the following we prove that the form (5.3.1) implies that the metric is actually Riemannain.

Theorem 5.3.2. *A (regular) Finsler metric of the form* (5.3.1) *is Riemannian.*

Proof. Equation (5.3.1) can be written, in terms of the energy function E, in the form

$$E = (y^n)^2 + (x^n)^2 H(x^\alpha, y^\alpha), \ \ \alpha = 1, \ldots, n-1.$$

Since E is a Finslerian energy function, then it is smooth on the whole of T_0M and particularly on the direction $(0, \ldots, 0, y^n)$. Consequently, H is also smooth on T_0M and particularly on the direction $(0, \ldots, 0, y^n)$. But H does not depend on y^n and hence the section $(0, \ldots, 0, y^n)(\equiv \{(0_x, \ldots, 0_x, y^n);\, x \in M\})$ can be identified with the zero section of the $(n-1)$-dimensional space. Now, H is smooth, and particularly C^2 on the null section $(0, \ldots, 0, y^n) \equiv (0, \ldots, 0)$ and homogenous of degree 2, then H is a polynomial of degree 2. Hence, H is quadratic in y, which means that E is Riemannian. □

As a direct consequence of Theorem 5.3.2, we have

Theorem 5.3.3. *Assuming that Tachibana's theorem is true, a Finsler metric admitting a concurrent vector field is Riemannian. Consequently, there is no regular Finsler non-Riemannian metric admitting a concurrent vector field.*

A natural question arises: *Is a conic Finsler metric of the form* (5.3.1) *admitting a concurrent vector field Riemannian?* The following example gives a negative answer. It gives a non-Riemannian conic Finsler metric of the form (5.3.1) which admits a concurrent vector field.

Example 5.3.4. Let $M = \mathbb{R}^3$ and E be given by

$$E = y_3^2 + x_3^2 H = y_3^2 + x_3^2 \left(\sqrt{y_1^2 + x_1^2 y_2^2} + \epsilon y_2\right)^2.$$

where H is a 2-dimensional Finsler metric of Randers type given by

$$H = \left(\sqrt{y_1^2 + x_1^2 y_2^2} + \epsilon y_2\right)^2.$$

One can easily show that E is not smooth on the directions $(0, 0, \pm 1)$. Hence, E is defined on the conic set $D \subset TM$,

$$D = TM - \{(x_i, y_i) \in TM |\, y_1^2 + y_2^2 \neq 0\}.$$

The components g_{ij} of the metric tensor are given by

$$g_{11} = \frac{x_3^2\,(\epsilon\, x_1^4\, y_2^5 + \epsilon\, x_1^2\, y_1^2\, y_2^3 + \sqrt{y_1^2 + x_1^2\, y_2^2}\,(x_1^2\, y_2^2\,(y_1^2 + x_1^2\, y_2^2) - x_1^2\, y_1^2\, y_2^2 + 2\, y_1^2\,(y_1^2 + x_1^2\, y_2^2) - y_1^4))}{(y_1^2 + x_1^2\, y_2^2)^{5/2}},$$

$$g_{22} = \frac{x_3^2\,(2\,\epsilon\, x_1^4\, y_2^3 + 3\,\epsilon\, x_1^2\, y_1^2\, y_2 + x_1^2\,(y_1^2 + x_1^2\, y_2^2)^{3/2} + \epsilon^2\,(y_1^2 + x_1^2\, y_2^2)^{3/2})}{(y_1^2 + x_1^2\, y_2^2)^{3/2}},$$

$$g_{12} = \frac{\epsilon\, x_3^2\, y_1^3}{(y_1^2 + x_1^2\, y_2^2)^{3/2}}, \qquad g_{33} = 1.$$

The components C_{ijk} of the Cartan tensor are given by

$$C_{111} = -\frac{3}{2}\frac{\epsilon\, x_1^2\, x_3^2\, y_1\, y_2^3}{(y_1^2 + x_1^2\, y_2^2)^{5/2}}, \qquad C_{112} = \frac{3}{2}\frac{\epsilon\, x_1^2\, x_3^2\, y_1^2\, y_2^2}{(y_1^2 + x_1^2\, y_2^2)^{5/2}}$$

$$C_{122} = -\frac{3}{2}\frac{\epsilon\, x_1^2\, x_3^2\, y_1^3\, y_2}{(y_1^2 + x_1^2\, y_2^2)^{5/2}}, \qquad C_{222} = \frac{3}{2}\frac{\epsilon\, x_1^2\, x_3^2\, y_1^4}{(y_1^2 + x_1^2\, y_2^2)^{5/2}}.$$

This metric admits a concurrent vector field given by $B^1(x) = B^2(x) = 0, B^3(x) = x_3$. Moreover, if we let $B^3 = f(x)$, an arbitrary function of x, then $B = (0, 0, f(x))$ is a semi-concurrent vector field. Clearly, the given metric is not Riemannian.

5.4 Examples in dimension 4

As has been shown, the problem mentioned in the introduction is completely solved for the 2-dimensional case and also for several specific cases, where the Finsler manifold under consideration is subject to certain conditions. It turns out that in the general case ($\dim M \geq 3$ and no additional restrictions), a Finsler metric of the form (5.3.1) admitting a concurrent vector field is necessarily Riemannian, whereas a *conic* Finsler metric of the the same form (5.3.1) admitting a concurrent vector field is not necessarily Riemannian. In what follows we present some examples of Riemannian and conic Finslerian metrics admitting semi-concurrent vector fields. In the examples considered all calculations are preformed using Maple program [111].

Let us consider the manifold $M = \mathbb{R}^4$. A general form of a Finsler metric admitting a semi-concurrent vector field is given by:

$$\begin{aligned} E = {} & y_1(F_1(x)y_2 + y_4)F_2\left(\frac{(A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_7)y_1 + A_5y_2 + A_6y_3}{y_1}\right) \\ & + F_3\left(x, -\frac{A_5y_2 + A_6y_3}{y_1}\right)y_1^2 - F_4(x)(A_5y_2^2 + A_6y_2y_3) + F_5(x)y_1^2 \\ & + F_6(x)y_1y_2 + F_7(x)y_2^2 + F_8(x)y_4^2, \end{aligned} \tag{5.4.1}$$

where $A_1, \ldots, A_7$ are arbitrary constants and $F_1, \ldots, F_8$ are arbitrary smooth functions on TM or a subset of TM such that E is an energy function.

By appropriate choices of $A_1, \ldots, A_7; F_1, \ldots, F_8$, the energy function E may be Riemannian or conic (pseudo) Finslerian which admits semi-concurrent vector fields, as shown below.

To find the required semi-concurrent vector field $B = (B_1, B_2, B_3, B_4)$, we first find the metric components g_{ij} corresponding to the above energy function. From this we calculate the Cartan tensor components C_{ijk}. The required components of the semi-concurrent vector field B are then obtained by solving the system of equations $B^h C_{hij} = 0$. These turn out to be

$$\begin{aligned} B^1 &= 0 \\ B^2 &= f(x);\ f(x) \neq 0 \\ B^3 &= -\frac{A_5}{A_6} f(x) \\ B^4 &= -f(x) F_1(x). \end{aligned}$$

The next examples, corresponding to different choices of the A_i's and F_i's represent some special classes of (5.4.1).

Example 5.4.1. Set

$$A_1 = A_2 = A_3 = A_4 = A_7 = 0;\ F_1(x) = F_4(x) = 0,\ F_2(u) = u,\ F_3(x,u) = u^2$$

in (5.4.1) to obtain

$$E = y_4(A_5 y_2 + A_6 y_3) + (A_5 y_2 + A_6 y_3)^2 + F_5(x) y_1^2 + F_6(x) y_1 y_2 + F_7(x) y_2^2 + F_8(x) y_4^2.$$

This choice yields the metric components

$$g = \begin{pmatrix} F_5(x) & \frac{1}{2}F_6(x) & 0 & 0 \\ \frac{1}{2}F_6(x) & A_5^2 + F_7(x) & A_5 A_6 & \frac{1}{2}A_5 \\ 0 & A_5 A_6 & A_6^2 & \frac{1}{2}A_6 \\ 0 & \frac{1}{2}A_5 & \frac{1}{2}A_6 & F_8(x) \end{pmatrix}.$$

Clearly, the above matrix has rank 4, and so g is non-degenerate. Consequently, g is a pseudo-Riemannian metric. This metric is positive definite if the leading principal minors of the above matrix are all positive. By some computations, the leading principal minors of g are

$$F_5, \quad A_5^2 F_5 + F_5 F_7 - \frac{F_6^2}{4}, \quad \frac{A_6^2}{4}(4F_5F_7 - F_6^2), \quad \frac{A_6^2}{16}(4F_5F_7 - F_6^2)(4F_8 - 1).$$

Hence, the metric g is is Riemannian if $F_5 > 0,\ F_5F_7 > \frac{F_6^2}{4},\ F_8 > \frac{1}{4}$.

Example 5.4.2. Set

$$A_1 = A_2 = A_3 = A_4 = A_5 = A_7 = 0;\ F_1(x) = F_4(x) = 0,\ F_2(u) = u,\ F_3(x, u) = u^4$$

in (5.4.1) so that

$$E = A_6 y_3 y_4 + \frac{A_6^4\, y_3^4}{y_1^2} + F_5(x) y_1^2 + F_6(x) y_1 y_2 + F_7(x) y_2^2 + F_8(x) y_4^2.$$

This energy function represents a conic Finslerian metric whose conic domain has the form

$$D = \{(x, y) \in TM \mid y_1 \neq 0\}.$$

The metric g is given by

$$g = \begin{pmatrix} \frac{3A_6^4 y_3^4 + F_5(x) y_1^4}{y_1^4} & \frac{1}{2}F_6(x) & -\frac{4A_6^4 y_3^3}{y_1^3} & 0 \\ \frac{1}{2}F_6(x) & F_7(x) & 0 & 0 \\ -\frac{4A_6^4 y_3^3}{y_1^3} & 0 & \frac{6A_6^4 y_3^2}{y_1^2} & \frac{1}{2}A_6 \\ 0 & 0 & \frac{1}{2}A_6 & F_8(x) \end{pmatrix}.$$

As the matrix g has rank 4, the metric tensor g is thus pseudo-Finslerian. It can be shown that the leading principal minors of g are:

$$\frac{3A_6^4 y_3^4 + 5y_1^4}{y_1^4},\ \frac{12A_6^4 F_7 y_3^4 - F_6^2 y_1^4 + 20F_7 y_1^4}{4y_1^4},\ \frac{A_6^4 y_3^2 (4A_6^4 F_7 y_3^4 + 3F_6^2 y_1^4 + 24F_6 y_1^3 y_3 + 60F_7 y_1^4)}{2y_1^6},$$

$$\frac{A_6^2 (32A_6^6 F_7 F_8 y_3^6 - 12A_6^4 F_7 y_1^2 y_3^4 - 24A_6^2 F_6^2 F_8 y_1^4 y_3^2 + 480A_6^2 F_7 F_8 y_1^4 y_3^2 + F_6^2 y_1^6 - 20F_7 y_1^6)}{16y_1^6}.$$

Therefore, g is conic Finslerian if the above leading principal minors are all positive. The non-vanishing components of Cartan tensor are

$$C_{111} = -\frac{6\, A_6^4\, y_3^4}{y_1^5},\ \ C_{113} = \frac{6\, A_6^4\, y_3^3}{y_1^4},\ \ C_{133} = -\frac{6\, A_6^4\, y_3^2}{y_1^3},\ \ C_{333} = \frac{6\, A_6^4\, y_3}{y_1^2}$$

In this case, the semi-concurrent vector fields B is given by

$$B^1 = B^3 = B^4 = 0,\ B^2 = f(x);\ f(x) \neq 0.$$

Example 5.4.3. The following choice of some arbitrary constants and functions in (5.4.1), namely

$$A_1 = A_2 = A_3 = A_4 = A_7 = 0;\ F_2(u) = u^2,\ F_3(x, u) = u,\ F_4 = 0,$$

represents a more nontrivial example of a conic pseudo-Finsler metric defined by

$$\begin{aligned} E &= \frac{(F_1(x)\, y_2 + y_4)(A_5\, y_2 + A_6\, y_3)^2}{y_1} - (A_5\, y_2 + A_6\, y_3)y_1 + F_5(x)\, y_1^2 \\ &\quad + F_6(x)\, y_1\, y_2 + F_7(x)\, y_2^2 + F_8(x)\, y_4^2. \end{aligned}$$

The non-vanishing components of the metric tensor are given by

$$\begin{aligned} g_{11} &= \frac{\left({A_5}^2{y_2}^2 + {A_6}^2\, {y_3}^2\right)(F_1(x)\, y_2 + y_4) + 2A_5\, A_6\, F_1(x)\, y_2 y_3(y_2 + y_4) + 5{y_1}^3}{{y_1}^3} \\ g_{12} &= \frac{F_6(x){y_1}^2 - {A_5}^2 y_2(3F_1(x)y_2 + 2y_4) - 2A_5 A_6 y_3(2F_1(x)y_2 + y_4) - {A_6}^2 F_1(x){y_3}^2}{2{y_1}^2} \\ g_{22} &= \frac{3{A_5}^2\, F_1(x)\, y_2 + 2\, A_5\, A_6\, F_1(x)\, y_3 + {A_5}^2\, y_4 + F_7(x)\, y_1}{y_1} \\ g_{13} &= -\frac{1}{2}\frac{A_6\,(2A_5 F_1(x)\, {y_2}^2 + 2A_6\, F_1(x)\, y_2 y_3 + 2A_5\, y_2\, y_4 + 2A_6\, y_3\, y_4 + {y_1}^2)}{{y_1}^2}, \\ g_{23} &= \frac{A_6\ (2A_5\, F_1(x)\, y_2 + A_6\, F_1(x)\, y_3 + A_5\, y_4)}{y_1}, \qquad g_{14} = -\frac{1}{2}\,\frac{(A_5\, y_2 + A_6\, y_3)^2}{{y_1}^2} \\ g_{24} &= \frac{A_5\,(A_5\, y_2 + A_6\, y_3)}{y_1}, \qquad g_{33} = \frac{{A_6}^2\ (F_1(x)\, y_2 + y_4)}{y_1} \\ g_{34} &= \frac{A_6\,(A_5\, y_2 + A_6\, y_3)}{y_1}, \qquad g_{44} = F_8(x) \end{aligned}$$

Therefore, the non-vanishing components of Cartan tensor are given by

$$\begin{aligned} C_{111} &= -\frac{3}{2}\,\frac{A_5 {y_2}^2(F_1(x)[A_5 y_2 + 2A_6 y_3] + A_5 y_4) + {A_6}^2 {y_3}^2(F_1(x)y_2 + y_4) + 2A_5 A_6 y_2 y_3 y_4}{{y_1}^4} \\ C_{112} &= \frac{1}{2}\,\frac{F_1(x)\ \left(3A_5^2\, {y_2}^2 + 4\, A_5\, A_6\, y_2\, y_3 + {A_6}^2\, {y_3}^2\right) + 2{A_5}^2\, y_2\, y_4 + 2\, A_5\, A_6\, y_3\, y_4}{{y_1}^3} \\ C_{113} &= \frac{A_6\ (A_5\, F_1(x){y_2}^2 + A_6\, F_1(x)\, y_2\, y_3 + A_5\, y_2\, y_4 + A_6\, y_3\, y_4)}{{y_1}^3} \\ C_{122} &= -\frac{1}{2}\,\frac{A_5\ (3\, A_5\, F_1(x)\, y_2 + 2\, A_6 F_1(x)\, y_3 + A_5\, y_4)}{y1^2} \\ C_{123} &= -\frac{1}{2}\,\frac{A_6\ (2\, A_5\, F_1(x)\, y_2 + A_6\, F_1(x)\, y_3 + A_5\, y_4)}{y_1^2} \end{aligned}$$

$$C_{124} = -\frac{1}{2}\frac{A_5\ (A_5\,y_2 + A_6\,y_3)}{y_1^2}, \qquad C_{133} = -\frac{1}{2}\frac{A_6^2(F_1(x)\,y_2 + y_4)}{y_1^2}$$
$$C_{134} = -\frac{1}{2}\frac{A_6\,(A_5\,y_2 + A_6\,y_3)}{y_1^2}, \qquad C_{222} = \frac{3}{2}\frac{A_5^2\,F_1(x)}{y_1}$$
$$C_{223} = \frac{A_6\,A_5\,F_1(x)}{y_1}, \qquad C_{224} = \frac{1}{2}\frac{A_5^2}{y_1}$$
$$C_{233} = \frac{1}{2}\frac{A_6^2\,F_1(x)}{y1}, \qquad C_{234} = \frac{1}{2}\frac{A_6\,A_5}{y_1}$$
$$C_{114} = \frac{1}{2}\frac{A_5^2{y_2}^2 + 2\,A_5\,A_6\,y_2\,y_3 + {A_6}^2\,{y_3}^2}{{y_1}^3}, \qquad C_{334} = \frac{1}{2}\frac{A_6^2}{y1}.$$

The semi-concurrent vector field B is given by

$$B(x) = f(x)\frac{\partial}{\partial x^2} - \frac{A_5}{A_6}f(x)\frac{\partial}{\partial x^3} - f(x)F_1(x)\frac{\partial}{\partial x^4},\ \ f(x) \neq 0.$$

Note that in this example most of the components of Cartan tensor and three of the components of the vector field B are alive.

All the above examples are shown to be either Riemannian or conic (pseudo) Finslerian. The only two choices in (5.4.1) that produce a regular metric are the following: $A_5 = A_6 = 0$ or $F_2(u) = u$, $F_3(x,u) = u^2$. But these choices yield a quadratic energy, which means that the metric is Riemannian. We conclude that, in dimension 4, no choice of A_i and F_i in (5.4.1) can yield a regular Finsler metric.

All the examples presented in this chapter, among other evidences, motivate us to announce the following conjecture.

Conjuncture

There is no regular Finsler non-Riemannian metric that admits a semi-concurrent vector field. In other words, a Finsler metric admitting a semi-concurrent vector field is necessarily either Riemannian or conic Finslerian.

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