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ORIGINAL RESEARCH ARTICLE

Bayesian analysis of multi-component stress-strength reliability using improved record values

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ABSTRACT

This research investigates statistical methods for estimating the reliability in a complex system composed of non-identical components with varying strengths in the presence of upper record ranked set samples. To model the behavior of these components, a specialized distribution in the shape of a bathtub is assumed. This distribution offers flexibility with adjustable levels of asymmetry, enabling its adaptation to different reliability scenarios. The study focuses on estimating the reliability of the system's bathtub-shaped distribution by employing two different approaches: classical and Bayesian. In the classical approach, the system's reliability is estimated using a maximum likelihood technique, and a simulation study is conducted to evaluate the accuracy of the estimates. The Bayesian approach, on the other hand, considers the use of the standard linear exponential (LINEX) loss function as an asymmetric loss function, as well as the squared error loss function as a symmetric loss function. Bayesian estimates of the system's reliability are obtained by utilizing two independent gamma prior distributions. Due to the complexity of these estimates, the Markov chain Monte Carlo method is employed since closed-form solutions cannot be obtained. Extensive simulations reveal that as the number of records increases, the measurement accuracy decreases. In most cases, the Bayesian estimates obtained using the LINEX loss function yield the lowest values. The theoretical findings are illustrated through examples drawn from real-world datasets, specifically focusing on a dataset concerning the timing of consecutive failures in aircraft air conditioning systems to demonstrate the proposed methodologies.

Keywords: non-identical component strength system; bathtub-shaped model; upper record ranked set sampling; Markov chain Monte Carlo; real data applications

MSC Classification: 62N05; 62D99; 62F15; 62F40; 94A20

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1. Introduction

The stress-strength (SS) model plays a critical role in reliability testing. It measures the reliability of a component by comparing the stress (Y) it experiences to its strength (X). If the stress remains below the component's strength, $\mathfrak{R} = P(Y < X)$, the system will continue to function. The concept of the SS model was first introduced by Birnbaum[1] who originally proposed this idea, and it was further developed by Birnbaum and McCarty[2]. Various techniques and assumptions about distributions have been explored in numerous studies to make inferences based on this model. For recent works on this topic, for some recent works, see Sadeghpour et al.[3], Hassan et al.[4], Alsadat et al.[5], and, Hassan et al.[6].

The fundamental concept of $\Re$ can be adapted to form a system with two or more components. The research of Bhattacharyya and Johnson[7] investigated the multi-component SS (MSS) model under the concept that q out of p system components, where $(1 \leq q \leq p)$ components resist a common random stress Y. This model is relevant to a variety of industries, including manufacturing, and logistical operations. An electrical power plant, for instance, can only generate the right amount of energy if at least six out of its eight generating units are running. The deck of suspension bridges is held up by a number of vertical cables that are suspended from the towers. Consider a suspension bridge made up of p pairs of vertical cables. Only a minimum number of vertical cables running through the deck that are not destroyed when put under stress from wind loading, high traffic, corrosion, etc. will ensure the survival of the bridge.

In a system consisting of p similar components, the system will continue to operate if a certain number q $(1 \leq q \leq p)$ or more of the components are functioning. Stress applied to the system, denoted as Y, follows a random variable with a cumulative distribution function (CDF) $G_Y(x)$. The strengths of the components, which represent the minimum stress required for failure, are independent and identically distributed (iid) random variables $X_1, X_2, \ldots, X_p$ with a CDF $F(x)$. To capture the realistic behavior of these CDFs, it is assumed that the derived functions have flexible shapes with varying levels of asymmetry. Then, according to Bhattacharyya and Johnson[7], the reliability of the q-out-of-p system, denoted by $\Re_{q,p}$, is given by

$$\Re_{q,p} = P\left[\text{at least } q \text{ of } \left(X_1, X_2, \ldots, X_p\right) \text{ exceed } Y\right] = \sum_{a=q}^{p} \binom{p}{a} \int_0^\infty [1 - F(x)]^a \, F(x)^{p-a} \mathrm{d}G_Y(x).$$

The evaluation of MSS models' reliability using different SS distributions and sampling techniques was a topic of study for many academics, for instance, Hassan and Basheikh[8], Rao et al.[9], Rao et al.[10], Dey et al.[11], Kızılaslan[12], Pak et al.[13], Kayal et al.[14], Hassan et al.[15], Kotb and Raqab[16], Jana and Bera[17], Azhad et al.[18], Hassan et al.[19], Yousef and Almetwally[20], Almetwally et al.[21] , and, Haj Ahmad et al.[22].

Due to the varied structures of system components, many real-world situations may invalidate the assumption that the strength distributions are identical. This often happens with systems that contain backup components. Even when formed of the same material, different things might have different strengths. For example, a metal that has been heat treated to provide acceptable mechanical properties may break in a variety of ways when it is quenched. The strengths of the components differ as a result. It appears that a more realistic model is one that at least includes non-identical random strengths for system components (see Kotz et al.[23]).

The mathematical setting is now described. Suppose that a system possesses multiple components of various kinds, say p components, of which p_1 are of kind 1, p_2 are of kind 2, …, and the remainder $p_n = p - \sum_{i=1}^{n-1} p_i$ components are of kind n. Suppose that the random strengths for the components of the i-th kind have a CDF $F_i(x)$ with $i = 1, 2, \ldots, n$. In addition, suppose that Y is a common stress with CDF $Q_Y(x)$. It is assumed that all the involved components are subjected to this stress. As long as the q-out-of-the-p components can resist the stress, the system will be operational. The research of Johnson[24] described the system reliability $\Re_{q_1,\ldots,q_n,p_1,\ldots,p_n}$ with non-identical component strengths as follows:

$$\Re_{q_1,\ldots,q_n,p_1,\ldots,p_n} = \sum_{j_1=q_1}^{p_1} \cdots \sum_{j_n=q_n}^{p_n} \left(\prod_{i=1}^{n} \binom{p_i}{j_i} \right) \int_0^\infty (1 - F_i(x))^{j_i} (F_i(x))^{p_i - j_i} \, dQ_Y(x), \tag{1}$$

where the summation is applied to all combinations $(j_1, j_2, \ldots, j_n)$ with $0 \leq j_i \leq p_i$ for $i = 1, 2, \ldots, n$ such $q \leq \sum_{i=1}^{n} j_i \leq p$. For a system to operate, each q_i specifies the minimal number of components of the i^{th} type.

When examining a system with two distinct types of components, the model Equation (1) can be stated as follows:

$$\Re_{q_1,q_2,p_1,p_2} = \sum_{j_1=q_1}^{p_1} \sum_{j_2=q_2}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} \int_0^\infty (1-F_1(x))^{j_1}(F_1(x))^{p_1-j_1}(1-F_2(x))^{j_2}(F_2(x))^{p_2-j_2}\, dQ_Y(x). \quad (2)$$

In many practical fields, including hydrology, sports, medicine, life testing, etc., record values (RVs) have recently become an important area of research. Chandler[25] proposed the fundamental concept of RVs. In statistics, records are defined as the extremes that follow one another in a series of random variables. The greatest (respectively smallest) number derived from a series of random variables is referred to as an upper (respectively lower) RV. The RVs have the advantage of requiring fewer measures than a whole sample, which is particularly advantageous for damaging studies when expensive measurements must be conducted (see Wu[26]).

The mathematical foundation of ranked set sampling (RSS) was developed in McIntyre[27],and Dell and Clutter[28], who demonstrated that this sampling technique yields an effective estimate of the population mean. Numerous studies have proven the effectiveness of RSS and its variants in calculating a variety of population indicators (see Jiang and Gui[29] and, Alotaibi et al.[30]).

A new sampling strategy for generating record-breaking data was developed, called record-ranked set sampling (RRSS). In Salehi and Ahmadi[31], the RRSS system was established to help scientists in situations when the only observations that would be used are the most recent record-breaking data, such as athletic data, meteorological data, and Olympic data. Assuming there are n independent sequential sequences of continuous random variables, the i^{th} sequence sampling is stopped when the i^{th} RV is achieved. Only the last RV in each sequence is used as an observation for analysis. The last RV of the i^{th} sequence in this plane is denoted by $U_{i,i}$, then the available observations are $\underline{U}_{i,i} = (U_{1,1}, U_{2,2}, ,2U_{n,n})^T$. The following diagram can be used to illustrate this observational process:

$$\begin{array}{lllllll}
1: & \boldsymbol{U}_{(1)1} & & & & \rightarrow & \boldsymbol{U}_{1,1} = \boldsymbol{U}_{(1)1} \\
2: & \boldsymbol{U}_{(1)2} & \boldsymbol{U}_{(2)2} & & & \rightarrow & \boldsymbol{U}_{2,2} = \boldsymbol{U}_{(2)2} \\
\vdots & \vdots & \vdots & \ddots & \cdots \quad \vdots & \rightarrow & \vdots \\
n: & \boldsymbol{U}_{(1)n} & \boldsymbol{U}_{(2)n} & \boldsymbol{U}_{(3)n} & \cdots \quad \boldsymbol{U}_{(n)n} & \rightarrow & \boldsymbol{U}_{n,n} = \boldsymbol{U}_{(n)n}
\end{array}$$

where $U_{(i)j}$ is the i^{th} ordinary upper (lower) record in the j^{th} sequence. It is worth noting that these $U_{i,i}$'s are independent random variables but aren't always arranged in order.

Building more realistic models requires the Bayesian estimation of $\Re_{q_1,q_2,p_1,p_2}$ implying diverse distributions. See, for instance, Pandey et al.[32] and Paul and Uddin[33] for the Weibull and exponential distributions on the strength and stress variates, respectively. Also, Hassan and Basheikh[34] estimated $\Re_{q_1,q_2,p_1,p_2}$ for the non-identical MSS using the exponentiated Pareto distribution. Karam et al.[35] examined the estimation of $\Re_{q_1,q_2,p_1,p_2}$ when the component strengths and stress follow the inverse version of the Lomax distribution based on complete samples. A non-identical MSS system's estimation under adaptive hybrid progressive censoring samples was examined by Kohansal et al.[36], who considered the estimation of $\Re_{q_1,q_2,p_1,p_2}$ when the component strengths and stress follow the bathtub-shaped distribution (BShD). The choice of this distribution is motivated by its ability to capture diverse skewness properties in the reliability-type random variables. Çetinkaya[37] proposed an estimation of $\Re_{q_1,q_2,p_1,p_2}$ when component strengths and stress follow Weibull distributions in a generalized progressive hybrid censoring system. Under adaptive Type-II hybrid progressive censoring samples, Arshad et al.[38] created an estimation of the p-type non-identical MSS model from the proportional reversed hazard rate (PRHR) family using records. Hassen et al.[39] discussed an estimation of $\Re_{q_1,q_2,p_1,p_2}$ when component strengths and stress follow expoentiated Pareto using lower record data.

It is crucial to remember that much of the MSS reliability estimation research done so far has focused on complete or censored samples, and RVs have not been used frequently. Currently, there is no proof that an RRSS scheme exists, especially when it comes to the estimation of MSS systems with non-identical component strengths. In this study, we investigate the non-identical component strengths situation, where component strengths and stress follow a BShD, and we are interested in developing MSS models within the RRSS scheme. A maximum likelihood estimate (MLE) of $\Re_{q_1,q_2,p_1,p_2}$ is created for the upper RRSS (URRSS), and a simulation study is investigated. Linear exponential (LINEX) and squared error (SE) loss functions are used to build the Bayesian estimate (BE) of $\Re_{q_1,q_2,p_1,p_2}$. For $\Re_{q_1,q_2,p_1,p_2}$, the BEs cannot be reduced to simple closed forms; thus, we employ the Markov chain Monte Carlo (MCMC) approach. We also looked at actual data sets to demonstrate the applicability of our research.

The structure of the paper is organized as follows: A formulation of $\Re_{q_1,q_2,p_1,p_2}$ and its MLE under the URRSS, as well as a numerical analysis, are presented in Section 2. Section 3 discusses the BEs of $\Re_{q_1,q_2,p_1,p_2}$, under SE and LINEX loss functions. Section 4 of this article presents the MCMC method. For demonstration reasons, real data sets are provided in section 5. Section 6 has concluding observations.

2. Determination and classical estimation of $\Re_{q_1,q_2,p_1,p_2}$

In this part, a model description of $\Re_{q_1,q_2,p_1,p_2}$ is provided. The MLE of $\Re_{q_1,q_2,p_1,p_2}$ is obtained in the presence of the URRSS. Additionally, a numerical analysis is done.

2.1. Expression of $\Re_{q_1,q_2,p_1,p_2}$

When the strength and stress random variables follow the BShD, the formula for the MSS reliability is available. Before expressing it, a retrospective on the BShD is necessary.

The BShD, developed in Chen[40], has been regarded as one of the most significant distributions among the others in lifetime data analysis. It is a significant model that may be applied to research a wide range of issues in experiments on reliability and life testing. This significance is because of the shape properties of the related functions, demonstrating attractive asymmetric, skewness, and kurtosis levels. Several researchers discussed the BShD's studies and applications; for instance, see Wu[41], Rastogi et al.[42], Sarhan et al.[43], and, Ahmed[44]. The CDF of the BShD, for $x > 0$, with scale and shape parameters $\theta > 0$, and $\lambda > 0$ is provided by:

$$G(x) = 1 - \exp\left[\theta\left(1 - e^{x^{\lambda}}\right)\right].$$

The probability density function (PDF) of the BShD is:

$$g(x) = \lambda\theta x^{\lambda-1} \exp\left[\theta\left(1 - e^{x^{\lambda}}\right) + x^{\lambda}\right].$$

The associated survival function and hazard function (HF) are as follows, respectively:

$$S(x) = \exp\left[\theta(1 - e^{x^{\lambda}})\right],$$

and

$$h(x) = \lambda\theta x^{\lambda-1} e^{x^{\lambda}}.$$

Figure 1 illustrates some of the possible shapes of the PDF for selected parameters. It can be decreasing or unimodal. In addition, asymmetric shapes from the left to the right are observed, which is not so common in two-parameter lifetime distributions.

Figure 2 illustrates some of possible shapes of the HF for selected values of the parameters. Asymmetric bathtub shapes are observed.

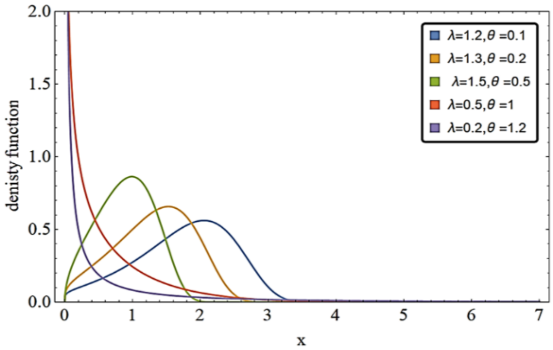

Figure 1. Some PDF representations of the BShD.

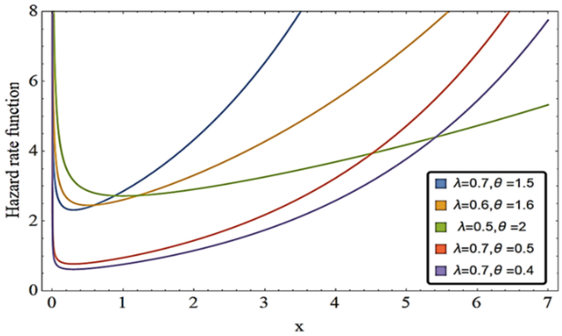

Figure 2. Some HF representations of the BShD.

Overall, based on its interesting features, and the existing work on the BShD, it is ideal to consider in our reliability setting.

From all of the p system components in the model Equation (2), we suppose that the first p_1 of first kind component strengths follow the BShD(λ, θ_1), while the remaining $p_2 = p - p_1$ of kind 2 component strengths follow the BShD(λ, θ_2). In addition, we suppose that Y follows the BShD(λ, θ_3), independently. The respective CDFs are described as

$$F_i(x) = 1 - \exp\left[\theta_i\left(1 - e^{x^\lambda}\right)\right],\ x, \lambda, \theta_i > 0, i = 1, 2, \tag{3}$$

$$Q_Y(x) = 1 - \exp\left[\theta_3\left(1 - e^{y^\lambda}\right)\right],\ y, \lambda > 0, \theta_3 > 0. \tag{4}$$

Based on the expressions in Equation (2) by Equations (3) and (4), the formula of $\Re_{q_1,q_2,p_1,p_2}$ for such a system is as follows:

$$\Re_{q_1,q_2,p_1,p_2} = \sum_{j_1=q_1}^{p_1} \sum_{j_2=q_2}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} \int_0^\infty \left[\exp\left(\theta_1(1-e^{x^\lambda})\right)\right]^{j_1} \left[1-\exp\left(\theta_1(1-e^{x^\lambda})\right)\right]^{(p_1-j_1)} \left[\exp\left(\theta_2(1-e^{x^\lambda})\right)\right]^{j_2}$$
$$\times \left[1-\exp\left(\theta_2(1-e^{x^\lambda})\right)\right]^{(p_2-j_2)} \theta_3 \lambda x^{\lambda-1} \exp\left(\theta_3(1-e^{x^\lambda}) + x^\lambda\right) dx.$$

Let us now work on the integral term. To this end, let $z = \exp\left(1 - e^{x^\lambda}\right)$, $dz = -\lambda x^{\lambda-1} e^{x^\lambda} \exp\left(1 - e^{x^\lambda}\right) dx$, and then inserting in previous equation, we get

$$\Re_{q_1,q_2,p_1,p_2} = \sum_{j_1=q_1}^{p_1} \sum_{j_2=q_2}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} \theta_3 \int_0^1 z^{\theta_1 j_1}(1 - z^{\theta_1})^{(p_1-j_1)} z^{\theta_2 j_2}(1 - z^{\theta_2})^{(p_2-j_2)} z^{\theta_3 - 1} dz.$$

Using the classical binomial expansion two times, the following expression is established:

$$\mathfrak{R}_{q_1,q_2,p_1,p_2} = M_{j_1,j_2,i_1,i_2}\theta_3 \int_0^1 z^{\theta_1(i_1+j_1)+\theta_2(i_2+j_2)+\theta_3-1}\,dz = \frac{M_{j_1,j_2,i_i,i_2}\theta_3}{\theta_1(i_i+j_1)+\theta_2(i_2+j_2)+\theta_3}, \tag{5}$$

where,

$$M_{j_1,j_2,i_1,i_2} = \sum_{j_1=q_1}^{p_1}\sum_{j_2=q_2}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\sum_{i_1=0}^{p_1-j_1}\sum_{i_2=0}^{p_2-j_2}\binom{p_1-j_1}{i_1}\binom{p_2-j_2}{i_2}(-1)^{i_1+i_2}.$$

Note that the expression in Equation (5) depends on θ_1, θ_2 and θ_3.

2.2. MLE of $\mathfrak{R}_{q_1,q_2,p_1,p_2}$ via URRSS

In this part, we investigate the MLE of $\mathfrak{R}_{q_1,q_2,p_1,p_2}$ using the URRSS data from the BShD. To compute the MLE of $\mathfrak{R}_{q_1,q_2,p_1,p_2}$, denoted by $\hat{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$, the MLEs of $\theta_1,\theta_2,\theta_3$ and λ, denoted by $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\lambda}$, must be computed first.

Suppose that $\underline{r_{i,i}} = \left(r_{1,1},\ldots,r_{n,n}\right)^T$ represents the n observed random vector $\underline{R}_{i,i} = (R_{1,1},\ldots,R_{n,n})^T$ of the URRSS from BShD(λ,θ_1). Let $\underline{t_{j,j}} = \left(t_{1,1},\ldots,t_{m,m}\right)^T$ be the observed m random vector of URRSS $\underline{T}_{j,j} = \left(T_{1,1},\ldots,T_{m,m}\right)^T$ from BShD(λ,θ_2). Also, suppose that $\underline{s_{u,u}} = \left(s_{1,1},\ldots,s_{w,w}\right)^T$ is the observation of the random vector $\underline{S}_{u,u} = \left(S_{1,1},\ldots,S_{w,w}\right)^T$ of size w from the URRSS obtained from BShD(λ,θ_3). Note that $\underline{R}_{i,i}$, $\underline{T}_{j,j}$ and $\underline{S}_{u,u}$ are independent.

The joint PDF of the URRSS of size n, according to Arnold et al.[45], is defined by:

$$L\left(\underline{u_{i,i}}|\eta\right) = \prod_{i=1}^{n}\frac{\left[-\ln(1-H(u_{i,i};\eta))\right]^{i-1}}{(i-1)!}h(u_{i,i};\eta); \qquad \eta\in\Theta, \tag{6}$$

where, $\underline{u_{i,i}} = (u_{1,1},u_{2,2},\ldots,u_{n,n})^T$ is the observed values of $\underline{U_{i,i}}$, η is real valued parameter and Θ is the parameter space, $h(u_{i,i};\eta)$ is the PDF of the URRSS and $H(u_{i,i};\eta)$ is CDF of the URRSS. Hence, the observed URRSS data $\underline{r_{i,i}}, \underline{t_{j,j}}and\underline{s_{u,u}}$, given η, based on Equation (6), are given as below:

$$L_1\left(\underline{r_{i,i}}|\lambda,\theta_1\right) = (\lambda\theta_1)^n\prod_{i=1}^{n}\frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!}\exp\left(\theta_1\tau_{i,i}+(r_{i,i})^{\lambda}\right)(r_{i,i})^{\lambda-1},$$

$$L_2\left(\underline{t_{j,j}}|\lambda,\theta_2\right) = (\lambda\theta_2)^m\prod_{j=1}^{m}\exp\left(\theta_2\varphi_{j,j}+(t_{j,j})^{\lambda}\right)(t_{j,j})^{\lambda-1}\frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(j-1)!},$$

$$L_3\left(\underline{s_{u,u}}|\lambda,\theta_3\right) = (\lambda\theta_3)^w\prod_{u=1}^{w}\frac{\left[-\theta_3\delta_{u,u}\right]^{u-1}}{(u-1)!}\exp\left(\theta_3\delta_{u,u}+(s_{u,u})^{\lambda}\right)(s_{u,u})^{\lambda-1},$$

where $\tau_{i,i} = 1-e^{(r_{i,i})^{\lambda}}$, $\varphi_{j,j} = 1-e^{(t_{j,j})^{\lambda}}$, $\delta_{u,u} = 1-e^{(s_{u,u})^{\lambda}}$, $i=1,\ldots,n,\ j=1,\ldots,m,\ u=1,\ldots,w.$

The joint likelihood function (LF) of $\eta=(\theta_1,\theta_2,\theta_3)$, based on the URRSS, is given by:

$$L\left(\underline{r_{i,i}},\underline{t_{j,j}},\underline{s_{u,u}}|\eta\right) = {\theta_1}^n{\theta_2}^m{\theta_3}^w\lambda^{n+m+w}\prod_{i=1}^{n}\frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!}\exp\left(\theta_1\tau_{i,i}+(r_{i,i})^{\lambda}\right)(r_{i,i})^{\lambda-1}$$
$$\times\prod_{j=1}^{m}\frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(j-1)!}\exp\left(\theta_2\varphi_{j,j}+(t_{j,j})^{\lambda}\right)(t_{j,j})^{\lambda-1}$$

$$\times \prod_{u=1}^{w} \frac{[-\theta_3 \delta_{u,u}]^{u-1}}{(u-1)!} \exp(\theta_3 \delta_{u,u} + (s_{u,u})^{\lambda}) (s_{u,u})^{\lambda-1},$$

Consequently, the joint log-LF, denoted by $\ln \ell$, is derived as:

$$\ell \propto n \ln \theta_1 + m \ln \theta_2 + w \ln \theta_3 + (n+m+w) \ln \lambda + \sum_{i=1}^{n} ((i-1) \ln(-\theta_1 \tau_{i,i}) + (\lambda - 1)\, ln\, r_{i,i})$$
$$+ \sum_{i=1}^{n} (\theta_1 \tau_{i,i} + (r_{i,i})^{\lambda}) + \sum_{j=1}^{m} ((j-1) \ln(-\theta_2 \phi_{j,j}) + (\lambda - 1) \ln t_{j,j} + \theta_2 \phi_{j,j} + (t_{j,j})^{\lambda}) \quad (7)$$
$$+ \sum_{u=1}^{w} ((u-1) \ln[-\theta_3 \delta_{u,u}] + (\lambda - 1) \ln s_{u,u} + \theta_2 \delta_{u,u} + (s_{u,u})^{\lambda}).$$

Given that λ is known, the partial derivatives of log-LF Equation (7), for θ_1, θ_2 and θ_3, are:

$$\frac{\partial \ell}{\partial \theta_1} = n\theta_1^{-1} + \sum_{i=1}^{n} \left[\frac{(i-1)}{\theta_1} + \tau_{i,i}\right], \quad (8)$$

$$\frac{\partial \ell}{\partial \theta_2} = m\theta_2^{-1} + \sum_{j=1}^{m} \left[\frac{(j-1)}{\theta_2} + \varphi_{j,j}\right], \quad (9)$$

$$\frac{\partial \ell}{\partial \theta_3} = w\theta_3^{-1} + \sum_{u=1}^{w} \left[\frac{(u-1)}{\theta_3} + \delta_{u,u}\right]. \quad (10)$$

Then, the MLEs of θ_1, θ_2 and θ_3, denoted by $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$, are obtained by setting Equations (8)–(10) to be zero and solving them numerically using optimization algorism as conjugate-gradient optimization. Therefore, based on invariance property, we obtain the MLE of $\Re_{q_1,q_2,p_1,p_2}$, say $\hat{\Re}_{q_1,q_2,p_1,p_2}$, by inserting $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_3$in Equation (5) as follows:

$$\hat{\Re}_{q_1,q_2,p_1,p_2} = \frac{M_{j_1,j_2,i_i,i_2} \hat{\theta}_3}{\hat{\theta}_1(i_1+j_1)+\hat{\theta}_2(i_2+j_2)+\hat{\theta}_3}. \quad (11)$$

Note that $\hat{\Re}_{q_1,q_2,p_1,p_2}$ obtained in Equation (11) dependes on MLEs of parmaters θ_1, θ_2 and θ_3.

2.3. Simulation study

In this part, the numerical efficiency of the MLEs for the MSS system are analyzed. Absolute biases (ABs) and mean squared errors (MSEs) are used as measurements to evaluate the precision of the estimates for several arbitrary parameter values and record numbers. The numerical research is carried out in the manner described below:

- Create the URRSS from the BShD, where (n, m, w) = (3, 3, 3), (3, 3, 5), (3, 5, 3), (5, 5, 5), (5, 5, 6), (6, 5, 5), (7, 7, 7) and (7, 8, 7).
- The parameters values of $(\theta_1, \theta_2, \theta_3)$ = (2, 1, 3), (1, 0.3, 2), (2, 0.2, 3) and (2, 0.2, 2) for $\lambda = 3$ in all situations.
- The real values of $\Re_{q_1,q_2,p_1,p_2}$ have the following values at
  a. (q_1, q_2, p_1, p_2) = (2, 2, 3, 3) are 0.563, 0.67, 0.615 and 0.495.
  b. (q_1, q_2, p_1, p_2) = (1, 2, 3, 3) are 0.725, 0.852, 0.838 and 0.737.
  c. (q_1, q_2, p_1, p_2) = (2, 1, 3, 3) are 0.611, 0.7, 0.618 and 0.499.
  d. (q_1, q_2, p_1, p_2) = (3, 3, 3, 3) are 0.25, 0.339, 0.559 and 0.233.
- We use 10,000 repetitions to compute the ABs and MSEs of $\hat{\Re}_{q_1,q_2,p_1,p_2}$.

- The results are presented in **Table 1** and are displayed in **Figures 3–8**.
- The MSEs and ABs of MSS reliability estimates at $(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$ are the smallest compared to others for different values of (q_1, q_2, p_1, p_2) (**Table 1**). The MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ at $(\theta_1, \theta_2, \theta_3)$ = (2, 1, 3) are the smallest in almost all cases (**Table 1**).
- As the URRSS numbers increase, the MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for all values of $(\theta_1, \theta_2, \theta_3)$ decrease (**Figure 3**).
- **Figure 4** demonstrates that the set of values (2, 1, 3) has the smallest MSEs at different sample sizes for $(q_1, q_2, p_1, p_2) = (2, 2, 3, 3)$.

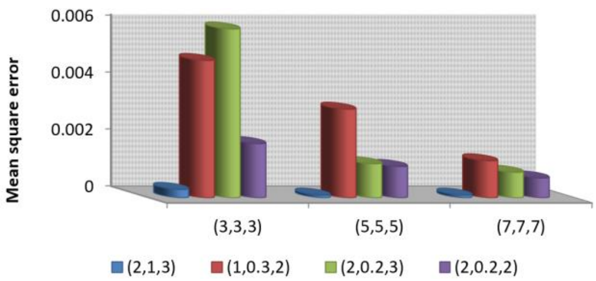

Figure 3. MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1, \theta_2, \theta_3)$ values at $(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$ and $n = m = w$.

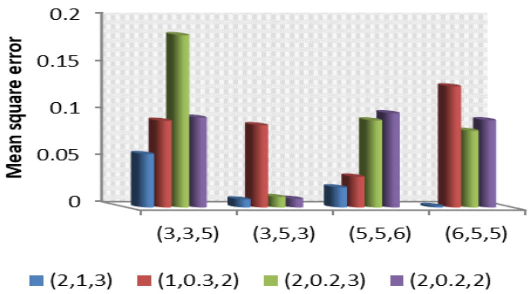

Figure 4. MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1, \theta_2, \theta_3)$ values at $(q_1, q_2, p_1, p_2) = (2, 2, 3, 3)$.

- **Figure 5** indicates that, as the numbers of *n*, *m* and *w* increase, the ABs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ decrease for all actual values of $(\theta_1, \theta_2, \theta_3)$.
- **Figure 6** demonstrates that the MSEs of MSS reliability estimates at (q_1, q_2, p_1, p_2) = (3, 3, 3, 3) are larger than the MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for other values of (q_1, q_2, p_1, p_2) at $(\theta_1, \theta_2, \theta_3) = (2, 1, 3)$.

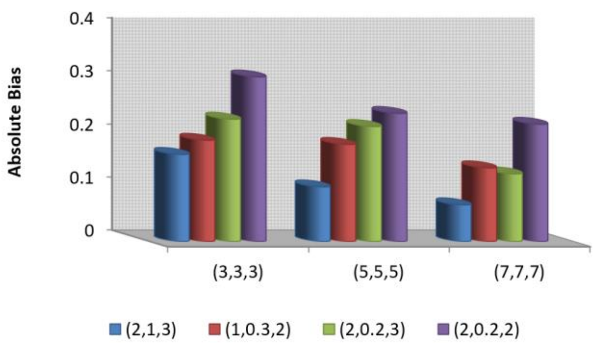

Figure 5. ABs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1, \theta_2, \theta_3)$ values at (q_1, q_2, p_1, p_2) = (3, 3, 3, 3) and $n = m = w$.

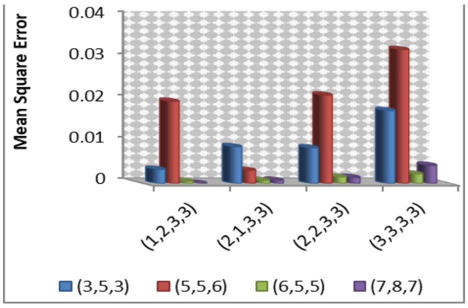

Figure 6. MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different values of *n*, *m*, *w* at $(\theta_1,\theta_2,\theta_3)=(2,1,3)$.

- **Figure 7** illustrates that the ABs of MSS reliability estimates at $(q_1,q_2,p_1,p_2)=(1,2,3,3)$ are smaller than the ABs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for others values of (q_1,q_2,p_1,p_2) for all true parameter values.
- **Figure 8** illustrates that the MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ decrease when the true value of $\Re_{q_1,q_2,p_1,p_2}$ increases for all values of (q_1,q_2,p_1,p_2).

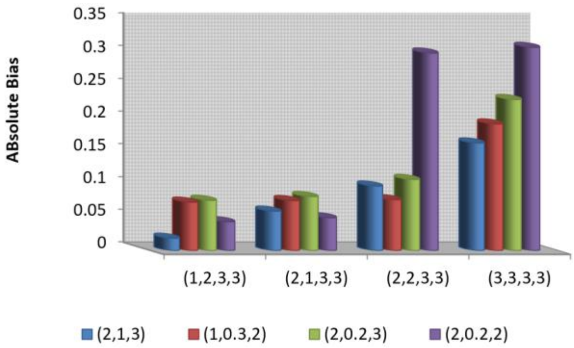

Figure 7. ABs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different $(\theta_1,\theta_2,\theta_3)$ and (q_1,q_2,p_1,p_2) values at *n* = *m* = *w* = 3.

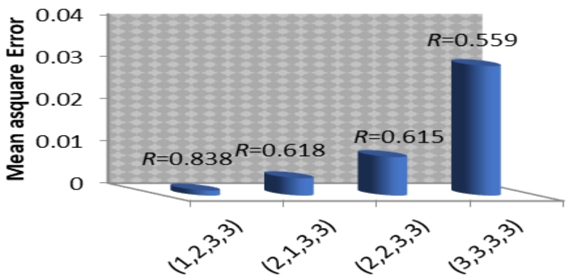

Figure 8. MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for $(\theta_1,\theta_2,\theta_3)=(2,0.2,3)$ at *n* = *m* = *w* = 5.

Table 1. Numerical results of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ for different values of $(\theta_1,\theta_2,\theta_3)$.

$(\theta_1,\theta_2,\theta_3)=(2,1,3)$					$(\theta_1,\theta_2,\theta_3)=(1,0.3,2)$				
(q_1,q_2,p_1,p_2)	Real $\Re_{q_1,q_2,p_1,p_2}$	(n,m,w)	AB	MSE	(q_1,q_2,p_1,p_2)	Real $\Re_{q_1,q_2,p_1,p_2}$	(n,m,w)	AB	MSE
(1, 2, 3, 3)	0.725	(3, 3, 3)	0.0193	0.0003	(1, 2, 3, 3)	0.852	(3, 3, 3)	0.0711	0.0048
		(3, 3, 5)	0.2079	0.0432			(3, 3, 5)	0.2139	0.0888
		(3, 5, 3)	0.0585	0.0034			(3, 5, 3)	0.0177	0.0008
		(5, 5, 5)	0.0120	0.0001			(5, 5, 5)	0.0611	0.0031
		(5, 5, 6)	0.1406	0.0197			(5, 5, 6)	0.1150	0.0137
		(6, 5, 5)	0.0249	0.0006			(6, 5, 5)	0.0981	0.0090
		(7, 7, 7)	0.0054	0.0001			(7, 7, 7)	0.0312	0.0013
		(7, 8, 7)	0.0099	0.0001			(7, 8, 7)	0.1518	0.0237

Table 1. (*Continued*).

$(\theta_1, \theta_2, \theta_3) = (2, 1, 3)$					$(\theta_1, \theta_2, \theta_3) = (1, 0.3, 2)$				
(q_1, q_2, p_1, p_2)	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**	(q_1, q_2, p_1, p_2)	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**
(2, 1, 3, 3)	0.611	(3, 3, 3)	0.0605	0.0036	(2, 1, 3, 3)	0.7	(3, 3, 3)	0.0742	0.0066
		(3, 3, 5)	0.0720	0.0066			(3, 3, 5)	0.3001	0.0911
		(3, 5, 3)	0.0941	0.0088			(3, 5, 3)	0.0833	0.0064
		(5, 5, 5)	0.0323	0.0010			(5, 5, 5)	0.0648	0.0044
		(5, 5, 6)	0.0477	0.0032			(5, 5, 6)	0.1166	0.0143
		(6, 5, 5)	0.0289	0.0008			(6, 5, 5)	0.1990	0.0391
		(7, 7, 7)	0.0185	0.0003			(7, 7, 7)	0.0319	0.0010
		(7, 8, 7)	0.0284	0.0008			(7, 8, 7)	0.2632	0.0680
(2, 2, 3, 3)	0.563	(3, 3, 3)	0.0988	0.0097	(2, 2, 3, 3)	0.66	(3, 3, 3)	0.0774	0.0061
		(3, 3, 5)	0.2378	0.0565			(3, 3, 5)	0.3221	0.0917
		(3, 5, 3)	0.0926	0.0085			(3, 5, 3)	0.2910	0.0870
		(5, 5, 5)	0.0729	0.0053			(5, 5, 5)	0.0658	0.0044
		(5, 5, 6)	0.1456	0.0212			(5, 5, 6)	0.1875	0.0326
		(6, 5, 5)	0.0401	0.0016			(6, 5, 5)	0.3571	0.1281
		(7, 7, 7)	0.0307	0.0009			(7, 7, 7)	0.0451	0.0013
		(7, 8, 7)	0.0387	0.0014			(7, 8, 7)	0.2760	0.0759
(3, 3, 3, 3)	0.25	(3, 3, 3)	0.1644	0.0270	(3, 3, 3, 3)	0.339	(3, 3, 3)	0.1910	0.0369
		(3, 3, 5)	0.1746	0.0304			(3, 3, 5)	0.3471	0.1212
		(3, 5, 3)	0.1326	0.0175			(3, 5, 3)	0.3110	0.0873
		(5, 5, 5)	0.1033	0.0106			(5, 5, 5)	0.1832	0.0332
		(5, 5, 6)	0.2007	0.0321			(5, 5, 6)	0.1888	0.0393
		(6, 5, 5)	0.0481	0.0023			(6, 5, 5)	0.3669	0.1418
		(7, 7, 7)	0.0695	0.0048			(7, 7, 7)	0.1391	0.0210
		(7, 8, 7)	0.0663	0.0043			(7, 8, 7)	0.2798	0.0801

$(\theta_1, \theta_2, \theta_3) = (2, 0.2, 3)$					$(\theta_1, \theta_2, \theta_3) = (2, 0.2, 2)$				
(q_1, q_2, p_1, p_2)	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**	(q_1, q_2, p_1, p_2)	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**
(1, 2, 3, 3)	0.838	(3, 3, 3)	0.0770	0.0059	(1, 2, 3, 3)	0.737	(3, 3, 3)	0.0444	0.0019
		(3, 3, 5)	0.2815	0.0792			(3, 3, 5)	0.3023	0.0131
		(3, 5, 3)	0.0968	0.0093			(3, 5, 3)	0.0188	0.0079
		(5, 5, 5)	0.0359	0.0012			(5, 5, 5)	0.0312	0.0011
		(5, 5, 6)	0.2560	0.0655			(5, 5, 6)	0.0510	0.0029
		(6, 5, 5)	0.2732	0.0746			(6, 5, 5)	0.2870	0.0813
		(7, 7, 7)	0.0176	0.0009			(7, 7, 7)	0.0199	0.0007
		(7, 8, 7)	0.2673	0.0714			(7, 8, 7)	0.2303	0.0651
(2, 1, 3, 3)	0.618	(3, 3, 3)	0.0828	0.0068	(2, 1, 3, 3)	0.499	(3, 3, 3)	0.0505	0.0025
		(3, 3, 5)	0.4055	0.1644			(3, 3, 5)	0.3030	0.0923
		(3, 5, 3)	0.0975	0.0098			(3, 5, 3)	0.0591	0.0071
		(5, 5, 5)	0.0712	0.0041			(5, 5, 5)	0.0432	0.0017
		(5, 5, 6)	0.2115	0.0701			(5, 5, 6)	0.0549	0.0041
		(6, 5, 5)	0.2750	0.0776			(6, 5, 5)	0.3001	0.0901

Table 1. (*Continued*).

$(\theta_1, \theta_2, \theta_3) = (2, 0.2, 3)$					$(\theta_1, \theta_2, \theta_3) = (2, 0.2, 2)$				
$(q_1, q_2, p_1 p_2)$	Real $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	AB	MSE	$(q_1, q_2, p_1 p_2)$	Real $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	AB	MSE
		(7, 7, 7)	0.0165	0.0012			(7, 7, 7)	0.0398	0.0010
		(7, 8, 7)	0.2801	0.0732			(7, 8, 7)	0.2413	0.0677
(2, 2, 3, 3)	0.615	(3, 3, 3)	0.1088	0.0118	(2, 2, 3, 3)	0.495	(3, 3, 3)	0.3014	0.0908
		(3, 3, 5)	0.4611	0.1823			(3, 3, 5)	0.3050	0.0951
		(3, 5, 3)	0.1113	0.0112			(3, 5, 3)	0.0611	0.0092
		(5, 5, 5)	0.1053	0.0091			(5, 5, 5)	0.2617	0.0712
		(5, 5, 6)	0.2981	0.0921			(5, 5, 6)	0.0560	0.0097
		(6, 5, 5)	0.3011	0.0812			(6, 5, 5)	0.3010	0.0923
		(7, 7, 7)	0.0859	0.0032			(7, 7, 7)	0.2501	0.0619
		(7, 8, 7)	0.2855	0.0749			(7, 8, 7)	0.2610	0.0710
(3, 3, 3, 3)	0.559	(3, 3, 3)	0.2307	0.0532	(3, 3, 3, 3)	0.233	(3, 3, 3)	0.3101	0.0968
		(3, 3, 5)	0.4801	0.1897			(3, 3, 5)	0.3270	0.1201
		(3, 5, 3)	0.1210	0.0134			(3, 5, 3)	0.200	0.0231
		(5, 5, 5)	0.2169	0.0306			(5, 5, 5)	0.2415	0.0065
		(5, 5, 6)	0.4101	0.1023			(5, 5, 6)	0.0760	0.0975
		(6, 5, 5)	0.4151	0.1249			(6, 5, 5)	0.3111	0.0618
		(7, 7, 7)	0.1279	0.0172			(7, 7, 7)	0.2211	0.0652
		(7, 8, 7)	0.3101	0.0802			(7, 8, 7)	0.4012	0.0135

3. Bayesian estimate of $\Re_{q_1,q_2,p_1,p_2}$

In this part, we look in this section at the BE of $\Re_{q_1,q_2,p_1,p_2}$ under the assumption that θ_1, θ_2, and θ_3 are random variables. It can be constructed that the suggested prior distributions for θ_1, θ_2, and θ_3 follow gamma distributions, defined with the following PDF:

$$\pi_i(\theta_i) \propto {\theta_i}^{a_i-1} e^{-b_i\theta_i}, i = 1, 2, 3,$$

where a_i, b_i are hyperparameters. For the independence of the parameters, the joint prior distribution of $\eta = (\theta_1, \theta_2, \theta_3)$ is given as

$$\pi(\eta) \propto {\theta_1}^{a_1-1}{\theta_2}^{a_2-1}{\theta_3}^{a_3-1} e^{-(b_1\theta_1+b_2\theta_2+b_3\theta_3)}.$$

Based on the URRSS samples, the joint PDF of $\eta = (\theta_1, \theta_2, \theta_3)$ is

$$\pi^*(\eta|\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}})$$
$$\propto {\theta_1}^{n+a_1-1}{\theta_2}^{m+a_2-1}{\theta_3}^{w+a_3-1}\lambda^{n+m+w} e^{-(b_1\theta_1+b_2\theta_2+b_3\theta_3)} \prod_{i=1}^{n} \frac{\left[-\theta_1\tau_{i,i}\right]^{i-1}}{(i-1)!} e^{\theta_1\tau_{i,i}+(r_{i,i})^\lambda}(r_{i,i})^{\lambda-1}$$
$$\times \prod_{j=1}^{m} \frac{\left[-\theta_2\varphi_{j,j}\right]^{j-1}}{(i-1)!} e^{\theta_2\varphi_{j,j}+(t_{j,j})^\lambda}(t_{j,j})^{\lambda-1} \prod_{u=1}^{w} \frac{\left[-\theta_3\delta_{u,u}\right]^{u-1}}{(u-1)!} e^{\theta_3\delta_{u,u}+(s_{u,u})^\lambda}(s_{u,u})^{\lambda-1}.$$

As a result, we may formulate the posterior PDF of $\eta = (\theta_1, \theta_2, \theta_3)$ as:

$$\pi^*\left(\eta|\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right) = \frac{L\left(\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}|\eta\right)\pi(\eta)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}|\eta\right)\pi(\eta)d\theta_1 d\theta_2 d\theta_3}.$$

The BE of $\Re_{q_1,q_2,p_1,p_2}$, denoted by $\hat{\Re}_{q_1,q_2,p_1,p_2}$, is its posterior mean, which results from the SE loss function assumption. It is defined as

$$\hat{\Re}_{q_1,q_2,p_1,p_2} = E(\Re_{q_1,q_2,p_1,p_2}) = \int_0^\infty \int_0^\infty \int_0^\infty \Re_{q_1,q_2,p_1,p_2} \pi^* \left(\eta | \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right) d\theta_1 d\theta_2 d\theta_3. \tag{12}$$

Additionally, the BE of $\Re_{q_1,q_2,p_1,p_2}$ for the LINEX loss function indicated by $\breve{\Re}_{q_1,q_2,p_1,p_2}$, is as follows:

$$\breve{\Re}_{q_1,q_2,p_1,p_2} = \frac{-1}{v} \log E(e^{-v\Re_{q_1,q_2,p_1,p_2}}) = \frac{-1}{v} \log\left[\int_0^\infty\int_0^\infty\int_0^\infty e^{-v\Re_{q_1,q_2,p_1,p_2}} \pi^* \left(\eta | \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right) d\theta_1 d\theta_2 d\theta_3\right]. \tag{13}$$

Due to the posterior PDF $\pi^* \left(\eta | \underline{r_{i,i}}, \underline{t_{j,j}}, \underline{s_{u,u}}\right)$ possesses a composite structure, it is challenging to derive an explicit formula for Equations (12) and (13). Hence, we compute these integrations using the Metropolis-Hastings (M-H) method and the MCMC methodology to derive the BEs.

4. Methodology

The MCMC simulation is implemented to investigate the efficiency of $\Re_{q_1,q_2,p_1,p_2}$'s MSS. The BEs under different loss functions are produced using gamma priors. Using the ABs and MSEs, the $\Re_{q_1,q_2,p_1,p_2}$'s BE results were evaluated. The various URRSS options are (*n*, *m*, *w*) = (3, 3, 3), (3, 3, 5), (3, 5, 3), (5, 5, 5), (5, 5, 6), (6, 5, 5), (7, 7, 7) and (7, 8, 7). The different choices of (q_1, q_2, p_1, p_2) = (2, 2, 3, 3), (1, 2, 3, 3), (2, 1, 3, 3) and (3, 3, 3, 3).

The values of hyper-parameter are prior I: (1, 1, 1, 2, 2, 2) and prior II: (3, 3, 3, 1, 1, 1). Considering (*v* = −2, 2), and the obtained results are based on 5000 replications.

The key challenge with the MCMC is getting the BEs of $\Re_{q_1,q_2,p_1,p_2}$ specified loss functions using the M-H technique after simulating samples based on the posterior PDF. It is known to converge to the intended distribution using acceptance/rejection criteria. According to Arshad et al.[38], the M-H algorithm functions as follows:

a) Put an initial parameter value of $\Re^0{}_{q_1,q_2,p_1,p_2}$ and the sample number N. Then choose an arbitrary probability density $g(.|\theta)$, where θ is the vector of fixed parameters.
b) For *i* = 2 to N, set $\Re_{q_1,q_2,p_1,p_2} = \Re^{i-1}{}_{q_1,q_2,p_1,p_2}$.
c) Obtain *u* via the uniform (0, 1) distribution.
d) Determine a candidate parameter $\Re^*{}_{q_1,q_2,p_1,p_2}$ from the proposal PDF.
e) If $u \le \frac{\pi(\theta^*) g(\theta | \theta^*)}{\pi(\theta) g(\theta^* | \theta)}$, then set $\Re^i{}_{q_1,q_2,p_1,p_2} = \Re^*{}_{q_1,q_2,p_1,p_2}$; otherwise, set $\Re^i{}_{q_1,q_2,p_1,p_2} = \Re_{q_1,q_2,p_1,p_2}$.
f) Go to step (b) and repeat the previous steps *N* times with *i* = *i* + 1.

The following findings are drawn from the study's outputs, which are shown by **Figures 9–14** and are provided in **Tables 2** and **3**:

- The MSEs and ABs of BEs via the SE and LINEX decrease when the record numbers *n*, *m*, *w* increase for all true values of (q_1, q_2, p_1, p_2) (**Tables 2** and **3**).
- The ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ and $\hat{\Re}_{q_1,q_2,p_1,p_2}$ via the SE and LINEX loss functions have the smallest values at (q_1, q_2, p_1, p_2) = (1, 2, 3, 3) (**Tables 2** and **3**).
- At true value $\Re_{q_1,q_2,p_1,p_2} = 0.611$, the MSEs of BEs decrease as the number of records increases (see **Figure 9**).
- At true value $\Re_{q_1,q_2,p_1,p_2} = 0.611$, the ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ and $\hat{\Re}_{q_1,q_2,p_1,p_2}$ decrease when the number of records increases via prior I (see **Figure 10**).

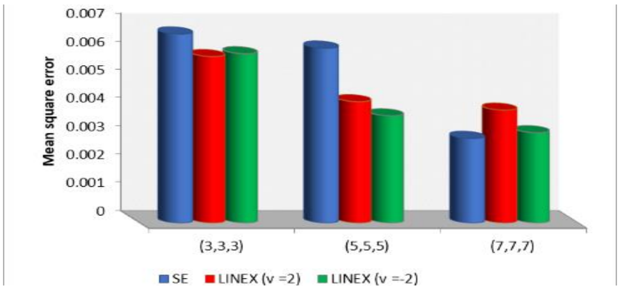

Figure 9. MSEs of BEs at (q_1, q_2, p_1, p_2) = (1, 2, 3, 3) for prior I.

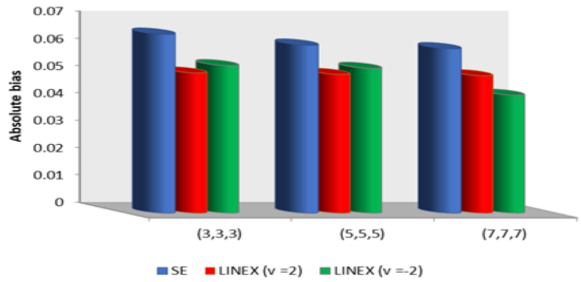

Figure 10. ABs of BEs at (q_1, q_2, p_1, p_2) = (2, 1, 3, 3) for prior I.

- At true value $\mathfrak{R}_{q_1,q_2,p_1,p_2} = 0.5639$, the MSEs of $\breve{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$ under the LINEX (–2), take the smallest value via prior I (see **Figure 11**).
- At true value $\mathfrak{R}_{q_1,q_2,p_1,p_2} = 0.5639$, the ABs of $\breve{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$ under the LINEX (–2), are the least for varied record numbers, except at (n, m, w) = (3, 3, 5) via prior I (see **Figure 12**).

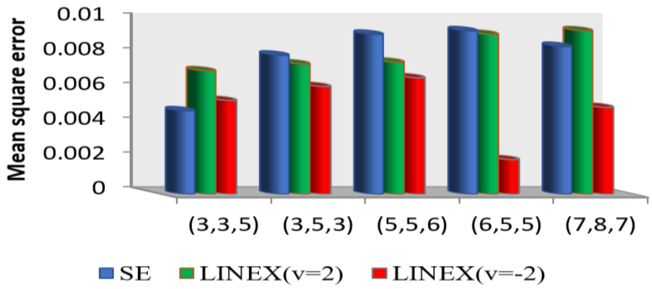

Figure 11. MSEs of BEs at (q_1, q_2, p_1, p_2) = (2, 2, 3, 3) for prior II.

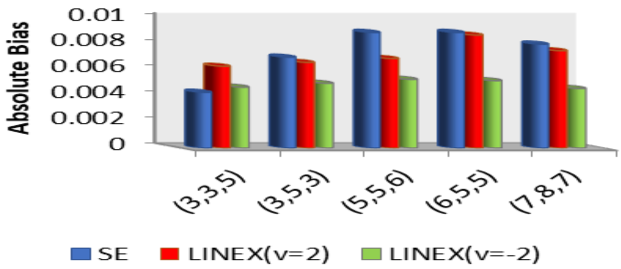

Figure 12. ABs of BEs at (q_1, q_2, p_1, p_2) = (2, 1, 3, 3) for prior II.

- The MSEs of $\breve{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$ inside the LINEX loss function, as opposed to the similar MSEs of $\hat{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$ via the SE loss function, possess the smallest values in nearly all situations (see **Figure 13**).
- At true value $\mathfrak{R}_{q_1,q_2,p_1,p_2} = 0.25$, the ABs of $\breve{\mathfrak{R}}_{q_1,q_2,p_1,p_2}$ within $v = -2$ are the least for the records $n = m = w$ via prior II (see **Figure 14**).

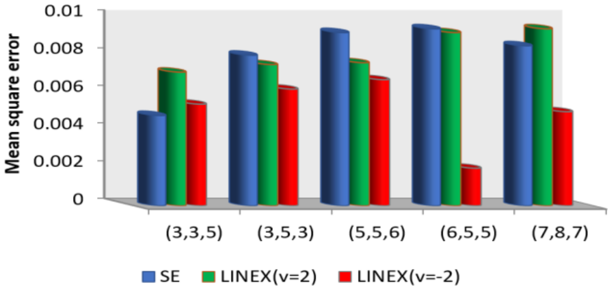

Figure 13. MSEs of BEs at (q_1, q_2, p_1, p_2) = (2, 2, 3, 3) for prior I.

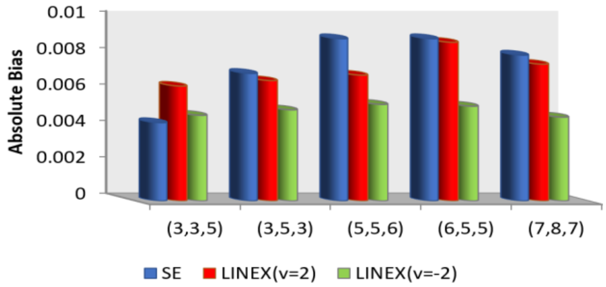

Figure 14. ABs of BEs at (q_1, q_2, p_1, p_2) = (3, 3, 3, 3) for prior II.

Table 2. Numerical results of BEs for prior I.

$(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$					$(q_1, q_2, p_1, p_2) = (2, 1, 3, 3)$			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**
SE	0.725	(3, 3, 3)	0.0324	0.0024	0.611	(3, 3, 3)	0.0157	0.0022
LINEX (2)			0.0214	0.0014			0.0147	0.0011
LINEX (−2)			0.0189	0.001			0.0134	0.0008
SE		(3, 3, 5)	0.0359	0.003		(3, 3, 5)	0.0143	0.0025
LINEX (2)			0.0245	0.0027			0.0121	0.0031
LINEX (−2)			0.0224	0.0026			0.0111	0.0023
SE		(3, 5, 3)	0.0131	0.0034		(3, 5, 3)	0.0121	0.003
LINEX (2)			0.0121	0.0028			0.0164	0.0026
LINEX (−2)			0.0341	0.0021			0.0147	0.0019
SE		(5, 5, 5)	0.0301	0.0021		(5, 5, 5)	0.0142	0.0019
LINEX (2)			0.0201	0.0011			0.0137	0.0007
LINEX (−2)			0.0177	0.0008			0.0129	0.0008
SE		(5, 5, 6)	0.0301	0.0041		(5, 5, 6)	0.014	0.0045
LINEX (2)			0.0288	0.0043			0.0112	0.0034
LINEX (−2)			0.0211	0.0033			0.0103	0.0032
SE		(6, 5, 5)	0.0471	0.0047		(6, 5, 5)	0.0131	0.004
LINEX (2)			0.0376	0.0041			0.0129	0.0038
LINEX (−2)			0.0478	0.0043			0.0111	0.004
SE		(7, 7, 7)	0.0271	0.0018		(7, 7, 7)	0.0135	0.0013
LINEX (2)			0.0187	0.0011			0.0132	0.0007
LINEX (−2)			0.0165	0.0007			0.0115	0.0005
SE		(7, 8, 7)	0.0132	0.0023		(7, 8, 7)	0.0321	0.0078
LINEX (2)			0.0214	0.0011			0.0137	0.0028
LINEX (−2)			0.0139	0.0025			0.0197	0.0041

Table 2. (*Continued*).

$(q_1, q_2, p_1 p_2) = (2, 2, 3, 3)$					$(q_1, q_2, p_1 p_2) = (3, 3, 3, 3)$			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**
SE	0.5636	(3, 3, 3)	0.0514	0.0067	0.25	(3, 3, 3)	0.0654	0.0071
LINEX (2)			0.0501	0.0059			0.0512	0.0068
LINEX (−2)			0.0439	0.006			0.0539	0.0064
SE		(3, 3, 5)	0.0475	0.006		(3, 3, 5)	0.0499	0.0065
LINEX (2)			0.0361	0.0034			0.0487	0.0048
LINEX (−2)			0.035	0.0035			0.0412	0.0061
SE		(3, 5, 3)	0.0314	0.0038		(3, 5, 3)	0.0367	0.0044
LINEX (2)			0.0411	0.0047			0.0497	0.0058
LINEX (−2)			0.0377	0.0062			0.0391	0.0055
SE		(5, 5, 5)	0.0531	0.0062		(5, 5, 5)	0.0614	0.006
LINEX (2)			0.0477	0.0043			0.0507	0.0063
LINEX (−2)			0.0433	0.0038			0.0529	0.0057
SE		(5, 5, 6)	0.0399	0.0054		(5, 5, 6)	0.0415	0.0061
LINEX (2)			0.0378	0.0042			0.0412	0.0052
LINEX (−2)			0.0279	0.0049			0.03	0.0069
SE		(6, 5, 5)	0.0192	0.0028		(6, 5, 5)	0.0267	0.0037
LINEX (2)			0.0149	0.0054			0.0139	0.0064
LINEX (−2)			0.0113	0.0008			0.0412	0.0037
SE		(7, 7, 7)	0.0478	0.003		(7, 7, 7)	0.0601	0.0051
LINEX (2)			0.0314	0.004			0.0502	0.0049
LINEX (−2)			0.0217	0.0032			0.0431	0.0049
SE		(7, 8, 7)	0.0285	0.0046		(7, 8, 7)	0.0357	0.0058
LINEX (2)			0.0312	0.0032			0.0345	0.006
LINEX (−2)			0.0147	0.0068			0.0214	0.0037

Table 3. Numerical results of BEs for prior II.

$(q_1, q_2, p_1 p_2) = (1, 2, 3, 3)$					$(q_1, q_2, p_1 p_2) = (2, 1, 3, 3)$			
Loss function	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**	**Real $\Re_{q_1,q_2,p_1,p_2}$**	(n, m, w)	**AB**	**MSE**
SE	0.725	(3, 3, 3)	0.0137	0.0039	0.610	(3, 3, 3)	0.0144	0.005
LINEX (2)			0.0277	0.005			0.028	0.005
LINEX (−2)			0.0101	0.0024			0.0112	0.003
SE		(3, 3, 5)	0.0198	0.0041		(3, 3, 5)	0.02	0.004
LINEX (2)			0.0231	0.0058			0.024	0.006
LINEX (−2)			0.0181	0.004			0.019	0.005
SE		(3, 5, 3)	0.0327	0.0067		(3, 5, 3)	0.033	0.007
LINEX (2)			0.0258	0.0054			0.026	0.007
LINEX (−2)			0.0209	0.0043			0.021	0.005
SE		(5, 5, 5)	0.013	0.0034		(5, 5, 5)	0.0142	0.005
LINEX (2)			0.0241	0.0048			0.0245	0.006
LINEX (−2)			0.0097	0.0009			0.0107	0.002

Table 3. (*Continued*).

$(q_1, q_2, p_1 p_2) = (1, 2, 3, 3)$					$(q_1, q_2, p_1 p_2) = (2, 1, 3, 3)$			
Loss function	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**
SE		(5, 5, 6)	0.0408	0.0082		(5, 5, 6)	0.0411	0.009
LINEX (2)			0.0378	0.0067			0.039	0.007
LINEX (−2)			0.0214	0.005			0.0218	0.005
SE		(6, 5, 5)	0.0417	0.0086		(6, 5, 5)	0.042	0.009
LINEX (2)			0.04	0.0081			0.0406	0.009
LINEX (−2)			0.0312	0.0047			0.0319	0.005
SE		(7, 7, 7)	0.0127	0.0027		(7, 7, 7)	0.0135	0.003
LINEX (2)			0.0211	0.0041			0.0226	0.004
LINEX (−2)			0.0069	0.0007			0.0098	0.001
SE		(7, 8, 7)	0.0601	0.0074		(7, 8, 7)	0.0613	0.008
LINEX (2)			0.0517	0.0068			0.0523	0.008
LINEX (−2)			0.0411	0.0031			0.042	0.005
$(q_1, q_2, p_1 p_2) = (2, 2, 3, 3)$					$(q_1, q_2, p_1 p_2) = (3, 3, 3, 3)$			
Loss function	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**	**Real** $\Re_{q_1,q_2,p_1,p_2}$	(n, m, w)	**AB**	**MSE**
SE	0.5636	(3, 3, 3)	0.0149	0.0053	0.250	(3, 3, 3)	0.0152	0.006
LINEX (2)			0.0291	0.0059			0.03	0.006
LINEX (−2)			0.013	0.0037			0.0141	0.005
SE		(3, 3, 5)	0.0213	0.0048		(3, 3, 5)	0.0246	0.005
LINEX (2)			0.0248	0.0071			0.0252	0.007
LINEX (−2)			0.0198	0.0054			0.0207	0.006
SE		(3, 5, 3)	0.0358	0.008		(3, 5, 3)	0.037	0.008
LINEX (2)			0.027	0.0075			0.0285	0.008
LINEX (−2)			0.0218	0.0062			0.0224	0.007
SE		(5, 5, 5)	0.0146	0.005		(5, 5, 5)	0.0149	0.005
LINEX (2)			0.0266	0.0059			0.0274	0.006
LINEX (−2)			0.0117	0.0023			0.0119	0.003
SE		(5, 5, 6)	0.0421	0.0092		(5, 5, 6)	0.0432	0.01
LINEX (2)			0.0397	0.0076			0.0411	0.008
LINEX (−2)			0.0224	0.0067			0.023	0.007
SE		(6, 5, 5)	0.0432	0.0094		(6, 5, 5)	0.0444	0.01
LINEX (2)			0.0412	0.0092			0.0439	0.01
LINEX (−2)			0.0321	0.002			0.0338	0.003
SE		(7, 7, 7)	0.0142	0.0037		(7, 7, 7)	0.0144	0.004
LINEX (2)			0.0237	0.0048			0.025	0.005
LINEX (−2)			0.0113	0.0016			0.0111	0.002
SE		(7, 8, 7)	0.0614	0.0085		(7, 8, 7)	0.0632	0.009
LINEX (2)			0.0536	0.0094			0.0539	0.01
LINEX (−2)			0.0431	0.005			0.0439	0.006

5. Actual data implementation

We use a data set on the timing of successive air conditioning system failures in a fleet of Boeing 720 jet aircraft in order to illustrate the approaches suggested in the preceding sections. The information given above is compiled in **Table 4** and can be found in Chen[40].

Table 4. Intervals between failures.

7907	7908	7909	8044	7911	8045	7915	7917	7916
194	413	90	487	55	102	438	130	50
15	14	10	18	320	209	9	493	254
41	58	60	100	56	14	12		5
29	37	186	7	104	57	270		283
33	100	61	98	220	54	63		35
181	65	49	5	239	32	3		12
	9	14	85	47	67	104		
	169	24	91	246	59	2		
	447	56	43	176	134	359		
	184	20	230	182	152			
	36	79	3	33	27			
	201	84	130		14			
	118	44			230			
		59			66			
		29			61			
		118			34			
		25						
		156						
		310						
		76						
		26						
		44						
		23						
		62						

Based on the information in **Table 4** and a separate BShD fit using the Kolmogorov-Smirnov goodness-of-fit test, the URRSS for these groups is calculated as follows (**Table 5**).

Table 5. The selected records from Intervals between failures.

Data group I			Data group II			Data group III		
194			487			438		
413	447		55	320		130	493	
90	186	310	102	209	230	50	254	283

According to the above, the URRSS of groups I, II III are, respectively, as follows

$$(r_{1,1}, r_{2,2}, r_{3,3}) = (194, 447, 310), (t_{1,1}, t_{2,2}, t_{3,3}) = (487, 320, 230), \text{ and } (s_{1,1}, s_{2,2}, s_{3,3}) = (438, 493, 283),$$

then we calculate the estimates of $\Re_{q_1,q_2,p_1,p_2}$ using the ML and Bayesian approaches within SE and LINEX loss functions for $(q_1, q_2, p_1, p_2) = (1, 2, 3, 3)$. Using the above URRSS, the MLEs and BEs of $\Re_{q_1,q_2,p_1,p_2}$, are calculated in **Table 6**.

Table 6. $\Re_{q_1,q_2,p_1,p_2}$ estimates based on real data.

MLE	BE:SE	BE:LINEX (2)	BE:LINEX (−2)
0.4892	0.5603	0.5711	0.5534

6. Concluding remarks

In the current work, the reliability of an MSS system with non-identical component strengths based on upper record ranked set samples is investigated. Both the stress and strength variables are assumed to follow the BShD. This choice was motivated by its ability to model versatile lifetime variables, primarily due to the asymmetric forms of the PDF and bathtub-shaped HF. The reliability of the system was examined using two estimation techniques. The measurements of the strength and stress distribution samples were displayed in the context of URRSS. MCMC techniques were used to assess the validity of the proposed BEs. According to the simulation analysis, based on four choices of (q_1, q_2, p_1, p_2), the MSEs and ABs decrease with the number of records increases. This supports the MLE's consistency characteristic of $\Re_{q_1,q_2,p_1,p_2}$. Additionally, as the true value of $\Re_{q_1,q_2,p_1,p_2}$ increases, the MSEs of $\widehat{\Re}_{q_1,q_2,p_1,p_2}$ decrease. Regarding the MCMC approach, we deduce that the MSEs and ABs of $\breve{\Re}_{q_1,q_2,p_1,p_2}$ via LINEX-LF typically hold the lowest values. The application of real data reveals that the reliability estimates of our model are quite near one, demonstrating its applicability.

Author contributions

Conceptualization, ASH and HFN; methodology, ASH, ME and HFN; software, ASH, ME, CC and HFN; validation, ASH, ME, CC and HFN; formal analysis, ASH and HFN; investigation, ASH, ME, CC and HFN; resources, ASH, ME, CC and HFN; data curation, ASH and HFN; writing—original draft preparation, ASH and HFN; writing—review and editing, ASH, ME, CC and HFN; visualization, ASH and HFN. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

References

1. Birnbaum ZW. On a use of the Mann-Whitney statistic. In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability; December 1954–July–August 1955; California, USA. pp. 13–17.
2. Birnbaum ZW, McCarty RC. A distribution-free upper confidence bound for Pr(Y<X), based on independent samples of X and Y. The Annals of Mathematical Statistics 1958; 29(2): 558–562. doi: 10.1214/aoms/1177706631
3. Sadeghpour A, Salehi M, Nezakati A. Estimation of the stress-strength reliability using lower record ranked set sampling scheme under the generalized exponential distribution. Journal of Statistical Computation and Simulation 2020; 90(1): 51–74. doi: 10.1080/00949655.2019.1672694
4. Hassan AS, Al-Omari A, Nagy HF. Stress-strength reliability for the generalized inverted exponential distribution using MRSS. Iranian Journal of Science and Technology, Transactions A: Science 2021; 45(2): 641–659. doi: 10.1007/s40995-020-01033-9
5. Alsadat N, Hassan AS, Elgarhy M, et al. An efficient stress-strength reliability estimate of the unit Gompertz distribution using ranked set sampling. Symmetry 2023; 15(5): 1121. doi: 10.3390/sym15051121
6. Hassan AS, Alsadat N, Elgarhy M, et al. Analysis of P[Y<X<Z] using ranked set sampling for a generalized inverse exponential model. Axioms 2023; 12(3): 302. doi: 10.3390/axioms12030302
7. Bhattacharyya GK, Johnson RA. Estimation of reliability in a multicomponent stress-strength model. Journal of the American Statistical Association 1974; 69(348): 966–970. doi: 10.1080/01621459.1974.10480238
8. Hassan AS, Basheikh HM. Estimation of reliability in multi-component stress-strength model following exponentiated Pareto distribution. The Egyptian Statistical Journal 2012; 56(2): 82–95.
9. Rao GS, Aslam M, Kundu D. Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. Communications in Statistics-Theory and Methods 2015; 44(23): 4953–4961. doi: 10.1080/03610926.2013.821490
10. Rao GS, Aslam, M, Arif OH. Estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull Distribution. Communications in Statistics—Theory and Methods 2017; 46(15): 7495–7502. doi: 10.1080/03610926.2016.1154155
11. Dey S, Mazucheli J, Anis MZ. Estimation of reliability of multicomponent stress-strength for a Kumaraswamy distribution. Communications in Statistics-Theory and Methods 2017; 46(4): 1560–1572. doi:

10.1080/03610926.2015.1022457
12. Kızılaslan F. Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on a general class of inverse exponentiated distributions. Statistical Papers 2018; 59(3): 1161–1192. doi: 10.1007/s00362-016-0810-7
13. Pak A, Jafari AA, Khoolenjani NB. On reliability in a multicomponent stress-strength generalized Rayleigh model based on record values. Journal of Testing and Evaluation 2019; 48(6): 20170732. doi: 10.1520/jte20170732
14. Kayal T, Tripathi YM, Dey S, Wu SJ. On estimating the reliability in a multicomponent stress-strength model based on Chen distribution. Communications in Statistics-Theory and Methods 2020; 49(10): 2429–2447. doi: 10.1080/03610926.2019.1576886
15. Hassan AS, Nagy HF, Muhammed HZ, Saad MS. Estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values. Journal of Taibah University for Science 2020; 14(1): 244–253. doi: 10.1080/16583655.2020.1721751
16. Kotb MS, Raqab MZ. Estimation of reliability for multi-component stress-strength model based on modified Weibull distribution. Statistical Papers 2021; 62(6): 2763–2797. doi: 10.1007/s00362-020-01213-0
17. Jana N, Bera S. Estimation of parameters of inverse Weibull distribution and application to multi-component stress-strength model. Journal of Applied Statistics 2022; 49(1): 169–194. doi: 10.1080/02664763.2020.1803815
18. Azhad QJ, Arshad M, Khandelwal N. Statistical inference of reliability in multicomponent stress strength model for pareto distribution based on upper record values. International Journal of Modelling and Simulation 2022; 42(2): 319–334. doi: 10.1080/02286203.2021.1891496
19. Hassan AS, Ismail DM, Nagy HF. Reliability Bayesian analysis in multicomponent stress–strength for generalized inverted exponential using upper record data. IAENG International Journal of Applied Mathematics 2022; 52(3). 555–567.
20. Yousef MM, Almetwally EM. Multi stress-strength reliability based on progressive first failure for Kumaraswamy model: Bayesian and non-Bayesian estimation. Symmetry 2021; 13(11): 2120. doi: 10.3390/sym13112120
21. Almetwally EM, Alotaibi R, Mutairi AA, et al. Optimal plan of multi-stress-strength reliability Bayesian and non-Bayesian methods for the alpha power exponential model using progressive first failure. Symmetry 2022; 14(7): 1306. doi: 10.3390/sym14071306
22. Haj Ahmad H, Ramadan DA, Mansour MMM, Aboshady MS. The reliability of stored water behind dams using the multi-component stress-strength system. Symmetry 2023; 15(3): 766. doi: 10.3390/sym15030766
23. Kotz S, Lumelskii Y, Pensky M. The Stress-Strength Model and Its Generalizations: Theory and Applications. World Scientific; 2003.
24. Johnson RA. Stress-strength models for reliability. Handbook of Statistics 1988; 7: 27–54. doi: 10.1016/S0169-7161(88)07005-1
25. Chandler KN. The distribution and frequency of record values. Journal of the Royal Statistical Society: Series B (Methodological) 1952; 14(2): 220–228. doi: 10.1111/j.2517-6161.1952.tb00115.x
26. Wu SF. Interval estimation for the two-parameter exponential distribution based on the upper record values. Symmetry 2022; 14(9): 1906. doi: 10.3390/sym14091906
27. McIntyre G. A method for unbiased selective sampling, using ranked sets. Australian Journal of Agricultural Research 1952; 3(4): 385–390. doi: 10.1071/ar9520385
28. Dell TR, Clutter JL. Ranked set sampling theory with order statistics background. Biometrics 1972; 28(2): 545–555. doi: 10.2307/2556166
29. Jiang H, Gui W. Bayesian inference for the parameters of Kumaraswamy distribution via ranked set sampling. Symmetry 2021; 13(7): 1170. doi: 10.3390/sym13071170
30. Alotaibi N, Elbatal I, Shrahili M, et al. Statistical inference for the Kavya-Manoharan Kumaraswamy model under ranked set sampling with applications. Symmetry 2023; 15(3): 587. doi: 10.3390/sym15030587
31. Salehi M, Ahmadi J. Record ranked set sampling scheme. Metron 2014; 72(3): 351–365. doi: 10.1007/s40300-014-0038-z
32. Pandey M, Uddin MB, Ferdous J. Reliability estimation of an s-out-of-k system with non-identical component strengths: The Weibull case. Reliability Engineering & System Safety 1992; 36(2): 109–116. doi: 10.1016/0951-8320(92)90091-X
33. Paul RK, Uddin MB. Estimation of reliability of stress-strength model with non-identical component strengths. Microelectronics Reliability 1997; 37(6): 923–927. doi: 10.1016/0026-2714(94)00119-7
34. Hassan AS, Basheikh HM. Reliability estimation of stress-strength model with non-identical component strengths: The exponentiated Pareto case. International Journal of Engineering Research and Applications (IJERA) 2012; 2(3): 2774–2781.
35. Karam NS, Yousif SM, Tawfeeq BJ. Multicomponent inverse Lomax stress-strength reliability. Iraqi Journal of Science 2020; 2020: 72–80. doi: 10.24996/ijs.2020.si.1.10
36. Kohansal A, Fernández AJ, Pérez-González CJ. Multi-component stress-strength parameter estimation of a non-identical-component strengths system under the adaptive hybrid progressive censoring samples. Statistics 2021; 55(4): 925–962. doi: 10.1080/02331888.2021.1985499
37. Çetinkaya Ç. Reliability estimation of a stress-strength model with non-identical component strengths under

generalized progressive hybrid censoring scheme. Statistics 2021; 55(2): 250–275. doi: 10.1080/02331888.2021.1890739
38. Arshad M, Khandelwal N, Azhad QJ. Stress-strength reliability estimation in a system with p-type non-identical multicomponents from PRHR family based on records. Journal of Statistical Computation and Simulation 2022; 92(13): 2740–2771. doi: 10.1080/00949655.2022.2045988
39. Hassan AS, Ismail DM, Nagy HF. Analysis of a non-identical component strengths system based on lower record data. Reliability: Theory & Applications, 2023; 18(2): 513-528.
40. Chen Z. A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Statistics & Probability Letters 2000; 49(2): 155–161. doi: 10.1016/S0167-7152(00)00044-4
41. Wu SJ. Estimation of the two-parameter bathtub-shaped lifetime distribution with progressive censoring. Journal of Applied Statistics 2008; 35(10): 1139–1150. doi: 10.1080/02664760802264996
42. Rastogi MK, Tripathi YM, Wu SJ. Estimating the parameters of a bathtub-shaped distribution under progressive type-II censoring. Journal of Applied Statistics 2012; 39(11): 2389–2411. doi: 10.1080/02664763.2012.710899
43. Sarhan AM, Hamilton DC, Smith B. Parameter estimation for a two-parameter bathtub-shaped lifetime distribution. Applied Mathematical Modelling 2012; 36(11): 5380–5392. doi: 10.1016/j.apm.2011.12.054
44. Ahmed EA. Bayesian estimation based on progressive Type-II censoring from two-parameter bathtub-shaped lifetime model: An Markov chain Monte Carlo approach. Journal of Applied Statistics 2014; 41(4): 752–768. doi: 10.1080/02664763.2013.847907
45. Arnold BC, Balakrishnan N, Nagaraja HN. Records. John Wiley & Sons; 2011.