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Entropy Bayesian Estimation for Lomax Distribution Based on Record

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Abstract

In this paper, estimation of entropy for Lomax distribution based on upper record values is considered. Bayesian estimator of Shannon entropy is discussed under informative and non-informative priors. The entropy Bayesian estimator and the corresponding credible interval on the basis of a linear exponential, squared error and precautionary loss functions are derived. The Metropolis-Hastings algorithm is used to generate random variables. Monte Carlo simulations based on Gibbs sampling are conducted to implement the accuracy of estimates for different number of records. Real data example is analyzed for illustration purposes. In general, based on the outcomes of study, the Bayesian estimates of entropy tend to the true value as the number of record increases. Further, Bayesian estimate of entropy under LINEX loss function is preferable than the other estimates in most of situations.

Keywords: Shannon entropy, Bayesian estimators, loss function, Metropolis-Hastings algorithm.

1. Introduction

Record values can be considered as order statistics from a sample which its size is determined by the values and the order of occurrence of the observations. Record data are very important in various real-life applications like; weather, economic, sports and reliability. There are two types of record values, the upper record values and the lower record values. An observation is called upper (lower) record value if its value exceed (less than) that all of the previous observations (see Arnold et al. 1998).

The statistical study of record value is started with Chandler (1952) and formulated the theory of record values as a model for successive extremes in a sequence of independently and identically (iid) random variables.

Let x_i, $i \geq 1$ be a sequence of iid random variables with cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. According to Ahsanullah (1995) an observation x_i is called upper record value if its value exceeds all the preceding values, i.e. x_i is an upper record value if $x_i > x_j$ where $i > j$.

Let R_i, i = 1, 2,…, m be the first m upper record arising from any distribution with a certain PDF and CDF. According to Arnold et al. (1998), the joint PDF of R_i is given by

$$f(r_1, r_2, ..., r_m) = f(r_m)\prod_{i=1}^{m-1}\frac{f(r_i)}{1-F(r_i)}, \quad -\infty < r_1 < r_2 < ... < r_m < \infty. \tag{1}$$

Many authors dealt with upper record values (URV), for example, Ahmadi and Doostparast (2006) considered Bayesian estimation and prediction for some life distributions based on record values. Hassan et al. (2015) discussed stress-strength reliability for exponentiated inverted Weibull distribution from record values. Essam (2017) provided some properties and discussed the maximum likelihood and Bayesian estimators of the power Lomax distribution based on URV. Hassan et al. (2018 a) considered reliability estimator of generalized inverted exponential based on records. Hassan et al. (2018 b) discussed Bayesian estimators using squared error (SE) and linear exponential (LINEX) loss functions for generalized inverted exponential distribution based in URV and upper record ranked set sampling.

Entropy is a measure of expected value of information that contained in a random variable. More entropy indicates that the sample has less information. Measuring of entropy is an important issue in many areas such as statistics, economics, information technology, physics and biological phenomenon.

Shannon (1948) introduced the concept of entropy as a measure of information, which provides a quantitative measure of uncertainty. Let X be a random variable with CDF $F(x)$ and PDF $f(x)$. The Shannon entropy, denoted by $H(x)$, of the random variable is defined by

$$H(x) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx. \tag{2}$$

Many researchers discussed the entropy in case of censored data. For example, Cho et al. (2015) studied the Bayesian estimators of entropy of Weibull distribution based on generalized progressive hybrid censoring scheme. Lee (2017) discussed the maximum likelihood and Bayesian estimators of the entropy of an inverse Weibull distribution under generalized progressive hybrid censoring scheme. Almohaimeed (2017) provided an exact expression for entropy information contained in both types of progressively hybrid censored data and applied it in exponential distribution. Hassan and Zaky (2019) obtained the maximum likelihood estimator of Shannon entropy for inverse Weibull distribution under multiple censored data.

Many researchers discussed the entropy in case of ordered data. Wong and Chan (1990) showed that the amount of entropy is reduced when the sample is ordered. Seo et al. (2012) obtained an entropy estimator using URV from the generalized half-logistic distribution. Chacko and Asha (2018) discussed the estimation of entropy for generalized exponential distribution via record values.

In the literature, few works have been done about the estimation of entropy via record values. So, our objective here is to consider the Bayesian estimation of Shannon entropy of a Lomax distribution based on URV. The Bayesian estimator of Shannon entropy is considered using non-informative and informative priors. The Bayesian estimator of entropy is motivated by three loss functions; namely, SE, LINEX and precautionary (PRE). Due to the complicated forms of Bayesian entropy estimator, we employ the Markov Chain Monte Carlo (MCMC) technique.

This paper can be organized as follows. Section 2 gives the Shannon entropy for Lomax distribution. Section 3 provides Bayesian estimators for entropy of Lomax distribution using different loss functions, in case of URV. Simulation issue and application to real data are given in Section 4. The paper ends with some concluding remarks.

2. Entropy of Lomax Distribution

The Lomax distribution is one of the most important lifetime models. It has been useful in reliability and life testing problems, engineering, and in survival analysis. The application of Lomax distribution can be found in many fields like actuarial science, economics. Atkinson and Harrison (1978) and Harris (1968) applied the Lomax distribution to income and wealth data. While Corbellini et al. (2010) used it to model firm size and queuing problems. More details about applications of Lomax distribution can found in Campbell and Ratnaparkhi (1993), Tarko (2018) and Kang et al. (2019).

The Lomax distribution with shape parameter α and scale parameter λ has the following PDF and CDF.

$$f(x;\alpha,\lambda)=\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)} \qquad , \ x,\alpha,\lambda>0, \tag{3}$$

and,

$$F(x;\alpha,\lambda)=1-\lambda^{\alpha}(x+\lambda)^{-\alpha} \qquad , \ x,\alpha,\lambda>0. \tag{4}$$

Various studies about the Lomax distribution can be found in the literature by several authors. Ahsanullah (1991) discussed the record values of Lomax distribution. Balakrishnan and Ahsanullah (1994) discussed some recurrence relations between the moments of record values from Lomax distribution. Howlader and Hossain (2002) obtained Bayesian estimator of survival function for Lomax distribution. Hassan and Al-Ghamdi (2009) studied the optimum step stress accelerated life testing for Lomax distribution using maximum likelihood procedure. El-Din et al. (2013) discussed the parameter estimation of the Lomax distribution under progressive Type-II censoring using maximum likelihood and Bayesian methods. Hassan et al. (2016) discussed the optimal step stress accelerated life tests for Lomax distribution with adaptive Type-II progressive hybrid censoring.

The Shannon entropy of Lomax distribution can be obtained by substituting (3) in (2) as follows

$$H(x)=-\int_0^{\infty}\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)}\log(\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)})\,dx. \tag{5}$$

The integral (5) can be written as follows

$$\begin{aligned} H(x)=&-\int_0^{\infty}\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)}\log(\alpha\lambda^{\alpha})\,dx \\ &+(\alpha+1)\int_0^{\infty}\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)}\log(x+\lambda)\,dx=-I_1+I_2. \end{aligned} \tag{6}$$

To compute the entropy in (6) we need to find I_1 and I_2 as follows

$$I_1=\int_0^{\infty}\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)}\log(\alpha\lambda^{\alpha})\,dx=\log(\alpha\lambda^{\alpha}).$$

To obtain I_2,

$$I_2=(\alpha+1)\int_0^{\infty}\alpha\lambda^{\alpha}(x+\lambda)^{-(\alpha+1)}\log(x+\lambda)\,dx,$$

using integration by parts,

$$I_2=(\alpha+1)(\frac{1}{\alpha}+\log\lambda).$$

Hence the Shannon entropy of Lomax distribution model takes the following form

$$H(x) = -\log(\alpha\lambda^{\alpha}) + (\alpha+1)(\frac{1}{\alpha} + \log\lambda), \quad \alpha, \lambda > 0. \tag{7}$$

This is the required expression of Shannon entropy of Lomax distribution which can be seen as a function of parameters α and λ.

3. Bayesian Estimation

In this section, Bayesian estimator of the Shannon entropy is obtained based on URV. To compute the Bayesian estimator of entropy, we must obtain firstly the Bayesian estimators of α and λ. Bayesian estimator is considered in case of non-informative and informative priors under SE, LINEX and PRE loss functions.The Bayesian estimators cannot be obtained in explicit forms. Hence the MCMC technique is carried out to generate samples from the posterior distributions and consequently computing the Bayesian estimators and construct the corresponding credible intervals.

3.1 Entropy Bayesian Estimation in Case of Non-Informative Prior

In this subsection, the Bayesian estimator of entropy is obtained under symmetric and asymmetric loss functions in case of non-informative prior.

Let $\underline{r} = (r_1, r_2, ..., r_m)^T_{1x\,m}$ be the first *m* URV observed from Lomax distribution with PDF by (3) and CDF by (4), then the likelihood function of Lomax distribution, based on URV, is obtained by inserting (3) and (4) in (1), as follows

$$L(\alpha,\lambda \mid r_1, r_2, ..., r_m) = \alpha\lambda^{\alpha}(r_m+\lambda)^{-(\alpha+1)} \prod_{i=1}^{m-1} \frac{\alpha\lambda^{\alpha}(r_i+\lambda)^{-(\alpha+1)}}{(\frac{r_i+\lambda}{\lambda})^{-\alpha}}$$

$$= \alpha^m \lambda^{\alpha}(r_m+\lambda)^{-\alpha} \prod_{i=1}^{m}(r_i+\lambda)^{-1}.$$

Assuming that the prior of parameters α and λ, denoted by $\pi_1(\alpha)$ and $\pi_2(\lambda)$ has the following uniform distribution

$$\pi_1(\alpha) = \frac{1}{\alpha}, \pi_2(\lambda) = \frac{1}{\lambda}.$$

So, the joint posterior for parameters, denoted by $\pi^*_{1,2}(\alpha,\lambda \mid \underline{r})$, is

$$\pi^*_{1,2}(\alpha,\lambda \mid \underline{r}) = \frac{L(\alpha,\lambda \mid r_1, r_2, ..., r_m)\pi_1(\alpha)\pi_2(\lambda)}{\int_0^{\infty}\int_0^{\infty} L(\alpha,\lambda \mid r_1, r_2, ..., r_m)\pi_1(\alpha)\pi_2(\lambda)\, d\alpha\, d\lambda},$$

the joint posterior can be written as

$$\pi^*_{1,2}(\alpha,\lambda \mid \underline{r}) = c^{-1}\, \alpha^{m-1} \lambda^{\alpha-1}(r_m+\lambda)^{-\alpha} \prod_{i=1}^{m}(r_i+\lambda)^{-1},$$

where,

$$c = \int_0^{\infty}\int_0^{\infty} \alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha} \prod_{i=1}^{m}(r_i+\lambda)^{-1} d\alpha\, d\lambda.$$

So, the marginal posterior PDF of parameter α is:

$$\pi_1^*(\alpha\,|\,\underline{r})=\int_0^\infty c^{-1}\,\alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\lambda.$$

Also, the marginal posterior PDF of parameter λ is:

$$\pi_2^*(\lambda\,|\,\underline{r})=\int_0^\infty c^{-1}\,\alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha.$$

Therefore, the Bayesian estimators of unkown parameters α and λ under SE loss function, denoted by $\hat{\alpha}_{(SE)_1}$ and $\hat{\lambda}_{(SE)_1}$ can be obtained as posterior mean as follows:

$$\hat{\alpha}_{(SE)_1}=E(\alpha\,|\,\underline{r})=\int_0^\infty \alpha\,\pi_1^*(\alpha\,|\,\underline{r})\,d\alpha$$

$$=\frac{\int_0^\infty\int_0^\infty \alpha^{m}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}. \tag{8}$$

Also,

$$\hat{\lambda}_{(SE)_1}=E(\lambda\,|\,\underline{r})=\int_0^\infty \lambda\,\pi_2^*(\lambda\,|\,\underline{r})\,d\lambda$$

$$=\frac{\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{\alpha}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}. \tag{9}$$

Additionally, the Bayesian estimators of parameters α and λ under LINEX loss function, denoted by $\hat{\alpha}_{(LINEX)_1}$, and $\hat{\lambda}_{(LINEX)_1}$ are given as follows

$$\hat{\alpha}_{(LINEX)_1}=\frac{-1}{\nu}\log E(e^{-\alpha\nu})=\frac{-1}{\nu}\log[\int_0^\infty e^{-\alpha\nu}\,\pi_1^*(\alpha\,|\,\underline{r})\,d\alpha],\ \nu\neq 0$$

$$=\frac{-1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty e^{-\alpha\nu}\,\alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}\right), \tag{10}$$

and,

$$\hat{\lambda}_{(LINEX)_1}=\frac{-1}{\nu}\log E(e^{-\lambda\nu})=\frac{-1}{\nu}\log[\int_0^\infty e^{-\lambda\nu}\,\pi_2^*(\lambda\,|\,\underline{r})\,d\lambda],\ \nu\neq 0$$

$$=\frac{-1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty e^{-\lambda\nu}\,\alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{\alpha-1}(r_m+\lambda)^{-\alpha}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}\right), \tag{11}$$

where, ν is a real number.

Furthermore, the Bayesian estimators of parameters α and λ under PRE loss function, denoted by $\hat{\alpha}_{(PRE)_1}$ and $\hat{\lambda}_{(PRE)_1}$ are given as follows

$$\hat{\alpha}_{(PRE)_1} = \sqrt{E(\alpha^2 \mid \underline{r})} = [\int_0^{\infty} \alpha^2 \pi_1^*(\alpha \mid \underline{r})\, d\alpha]^{\frac{1}{2}}$$

$$= \left(\frac{\int_0^{\infty}\int_0^{\infty} \alpha^{m+1} \lambda^{\alpha-1} (r_m+\lambda)^{-\alpha} \prod_{i=1}^{m} (r_i+\lambda)^{-1} d\alpha\, d\lambda}{\int_0^{\infty}\int_0^{\infty} \alpha^{m-1} \lambda^{\alpha-1} (r_m+\lambda)^{-\alpha} \prod_{i=1}^{m} (r_i+\lambda)^{-1} d\alpha\, d\lambda} \right)^{\frac{1}{2}}, \tag{12}$$

and,

$$\hat{\lambda}_{(PRE)_1} = \sqrt{E(\lambda^2 \mid \underline{r})} = [\int_0^{\infty} \alpha^2\ \pi_2^*(\lambda \mid \underline{r}) d\lambda]^{\frac{1}{2}}$$

$$= \left(\frac{\int_0^{\infty}\int_0^{\infty} \alpha^{m-1} \lambda^{\alpha+1} (r_m+\lambda)^{-\alpha} \prod_{i=1}^{m} (r_i+\lambda)^{-1} d\alpha\, d\lambda}{\int_0^{\infty}\int_0^{\infty} \alpha^{m-1} \lambda^{\alpha-1} (r_m+\lambda)^{-\alpha} \prod_{i=1}^{m} (r_i+\lambda)^{-1} d\alpha\, d\lambda} \right)^{\frac{1}{2}}. \tag{13}$$

The integrals (8) to (13) are very difficult to obtain due to their complicated mathematical form. Therefore, the MCMC technique is used to approximate these integrations. Metropolis-Hastings (M-H) algorithm will be implemented to compute the Bayes estimates and credible intervals width under SEL, LINEX and PRE loss functions.

Based on (7), the Bayes estimate of $H(x)$, denoted by $\hat{H}_{(SE)_1}(x),$ under SE loss function is obtained as follows

$$\hat{H}_{(SE)_1}(x) = -\log(\hat{\alpha}_{(SE)_1} \hat{\lambda}_{(SE)_1}^{\hat{\alpha}_{(SE)_1}}) + (\hat{\alpha}_{(SE)_1}+1)(\frac{1}{\hat{\alpha}_{(SE)_1}} + \log \hat{\lambda}_{(SE)_1}).$$

By similar way we can obtain the Bayesian estimators of $H(x)$ under LINEX and PRE loss functions. Furthermore, Bayesian cerdible interval is a useful summary of the posterior distribution which reflects its variation that is used to quantify the statistical uncertainty. The Bayesian analog of a confidence interval is called a credible interval. A credible interval of entropy is the probability that a real value of entropy will fall between an upper and lower bounds of a probability distribution. Therefore, an approximate highest posterior density interval for $H(x)$ is obtained by using the same algorithm of Chen and Shao (1999).

3.2 Entropy Bayesian Estimation in Case of Informative Prior

In this subsection, the Bayesian estimator of entropy is obtained under symmetric and asymmetric loss functions in case of informative prior motivated by gamma priors.

Following Pak and Mahmood (2018), assuming that the prior of parameters α and λ , denoted by $\pi_3(\alpha)$ and $\pi_4(\lambda)$ has a gamma distribution with parameters (a_1, b_1) and (a_2, b_2), respectively.

$$\pi_3(\alpha) = \frac{a_1^{b_1}}{\Gamma(b_1)} \alpha^{b_1-1} e^{-\alpha a_1},\ \pi_4(\lambda) = \frac{a_2^{b_2}}{\Gamma(b_2)} \lambda^{b_2-1} e^{-\lambda a_2},$$

where a_i and b_i, $i=1,2$ are known and non negative. So, the joint posterior for parameters, denoted by $\pi_{3,4}^*(\alpha, \lambda \mid \underline{r})$, is

$$\pi_{3,4}^{*}(\alpha,\lambda\,|\,\underline{r})=\frac{L(\alpha,\lambda\,|\,r_1,r_2,...,r_m)\,\pi_3(\alpha)\,\pi_4(\lambda)}{\int_0^\infty\int_0^\infty L(\alpha,\lambda\,|\,r_1,r_2,...,r_m)\,\pi_3(\alpha)\,\pi_4(\lambda)\,d\alpha\,d\lambda}$$

$$=k^{-1}\,\alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1},$$

where,

$$k=\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda.$$

So, the marginal posterior PDF of parameters α and λ are given respectively by:

$$\pi_3^{*}(\alpha\,|\,\underline{r})=\int_0^\infty k^{-1}\,\alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\lambda,$$

and,

$$\pi_4^{*}(\lambda\,|\,\underline{r})=\int_0^\infty k^{-1}\,\alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha.$$

Therefore, the Bayesian estimators of unkown parameters α and λ under SE loss function, say $\hat{\alpha}_{(SE)_2}$ and $\hat{\lambda}_{(SE)_2}$ can be obtained as posterior mean as follows:

$$\hat{\alpha}_{(SE)_2}=\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}\,d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}\,d\alpha\,d\lambda}, \tag{14}$$

and,

$$\hat{\lambda}_{(SE)_2}=\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}\,d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}\,d\alpha\,d\lambda}. \tag{15}$$

While, the Bayesian estimators of parameters α and λ under LINEX loss function, say $\hat{\alpha}_{(LINEX)_2}$ and $\hat{\lambda}_{(LINEX)_2}$ are obtained as follows

$$\hat{\alpha}_{(LINEX)_2}=\frac{-1}{\nu}\log[\int_0^\infty e^{-\alpha\nu}\,\pi_3^{*}(\alpha\,|\,\underline{r})d\alpha],\ \nu\neq 0$$

$$=-\frac{1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\nu\alpha+\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\,\lambda^{\alpha+b_2-1}\,(r_m+\lambda)^{-\alpha}\,e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\,d\lambda}\right), \tag{16}$$

and,

$$\hat{\lambda}_{(LINEX)_2} = \frac{-1}{\nu}\log[\int_0^\infty e^{-\lambda\nu}\,\pi_4^*(\lambda\,|\,\underline{r})d\lambda],\ \nu\neq 0$$

$$= \frac{-1}{\nu}\log\left(\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\lambda^{\alpha+b_2-1}(r_m+\lambda)^{-\alpha}e^{-(\nu\lambda+\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\lambda^{\alpha+b_2-1}(r_m+\lambda)^{-\alpha}e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}\right), \tag{17}$$

where, ν is a real number. Additionaly, the Bayesian estimators of parameters α and λ under PRE loss function, say $\hat{\alpha}_{(PRE)_2}$ and $\hat{\lambda}_{(PRE)_2}$ are given as follows

$$\hat{\alpha}_{(PRE)_2} = [\int_0^\infty \alpha^2\,\pi_3^*(\alpha\,|\,\underline{r})d\alpha]^{\frac{1}{2}}$$

$$= \left(\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1+1}\lambda^{\alpha+b_2-1}(r_m+\lambda)^{-\alpha}e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\lambda^{\alpha+b_2-1}(r_m+\lambda)^{-\alpha}e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}\right)^{\frac{1}{2}}, \tag{18}$$

and,

$$\hat{\lambda}_{(PRE)_2} = [\int_0^\infty \lambda^2\,\pi_4^*(\lambda\,|\,\underline{r})d\lambda]^{\frac{1}{2}}$$

$$= \left(\frac{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\lambda^{\alpha+b_2+1}(r_m+\lambda)^{-\alpha}e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}{\int_0^\infty\int_0^\infty \alpha^{m+b_1-1}\lambda^{\alpha+b_2-1}(r_m+\lambda)^{-\alpha}e^{-(\alpha a_1+\lambda a_2)}\prod_{i=1}^{m}(r_i+\lambda)^{-1}d\alpha\, d\lambda}\right)^{\frac{1}{2}}. \tag{19}$$

MCMC technique is used to approximate the integrals (14) - (19). The M-H algorithm will be implemented to compute the Bayes estimate and credible interval width under SEL, LINEX and PRE loss functions.

Based on (7), the Bayesian estimator of $H(x)$ denoted by $\hat{H}_{(SE)_2}(x)$, under SE loss function is obtained as follows

$$\hat{H}_{(SE)_2}(x) = -\log(\hat{\alpha}_{(SE)_2}\hat{\lambda}_{(SE)_2}^{\hat{\alpha}_{(SE)_2}}) + (\hat{\alpha}_{(SE)_2}+1)(\frac{1}{\hat{\alpha}_{(SE)_2}} + \log\hat{\lambda}_{(SE)_2}).$$

By similar way we can obtain the Bayesian estimator of $H(x)$ under LINEX and PRE loss functions. Furthermore, the Bayesian credible interval is obtained as described in the previous subsection.

4. Simulation and Application

This section assesses the performance of the estimators and provides a real data example to illustrate the theoretical results.

4.1 Numerical Study

In this section, a numerical study is performed in order to study the behavior of the Bayesian estimators for entropy of the Lomax distribution based on URV. The Bayesian estimators are discussed using non-informative and informative priors, under SEL, LINEX and PRE loss functions. The MCMC technique is used to generate samples from the posterior distributions. The M-H algorithm is one of the most famous subclasses of MCMC method in Bayesian literature to simulate the deviates from the posterior density and produce the good approximate results. Here, M-H algorithm will be used via R 3.1.2 program.

The M-H algorithm procedure as follows:
Let $g(.)$ be the density of subject distribution
Initialize a starting value x_0 and the number of samples N
for $i = 2$ to N
    set $x = x_{i-1}$
    generate u from U(0,1)
    generate y from $g(.)$
    if $u \le \frac{\pi_\alpha(y)\, g(x)}{\pi_\alpha(\text{x})\, g(\text{y})}$ then
        set $x_i = y$
    else
        set $x_i = x$
    end if
end for

To compare the entropy estimators, MCMC simulations are performed for different record values under SE, LINEX and PRE loss functions. The number of records are selected as m = 5, 6, …,10 and true values of entropy measure are selected as $H(x)$ = 0.1137, 0.5681, 3.0 and 4.3863 (the parameter values are selected as (α, λ) = (2.0,0.5), (1.5,0.5), (0.5,0.5) and (0.5,2.0). The hyper-parameters for gamma prior are selected as $a_1 = a_2 = 1$ and $b_1 = b_2 = 4$. Also, we take ($\nu = -2, 2$) for LINEX loss function. The number of replications = 5,000. The relative absolute biases (RABs), estimated risks (ERs) and the width of credible interval are computed to evaluate the behavior of the Bayesian estimates.

4.2 Numerical Results

Simulation results are given in Tables 1 to 8 and explored in some figures. We provide some observations about the behavior of the entropy estimates.

- The estimated entropy value decreases as the value of scale parameter λ decreases at $\alpha = 0.5$.
- At $\lambda = 0.5$, the estimated entropy value decreases as the value of α increases.
- In non-informative prior the ERs for exact value $H(x)$ = 0.1137 take the smallest values at m = 5 for all selected loss functions (see Figure 1).

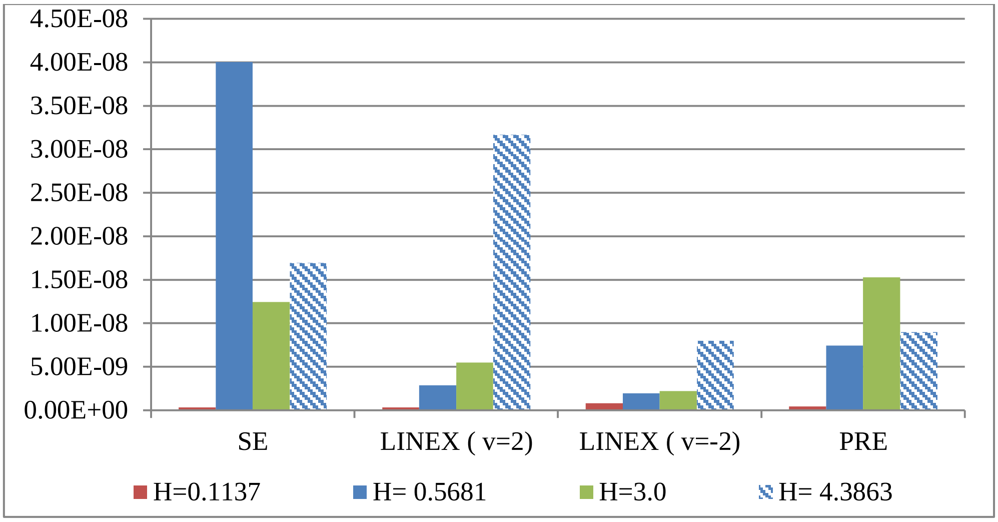

Figure 1 ERs of $\hat{H}(x)$ under SE, LINEX and PRE loss functions for different values of parameters at *m*=5 under non-informative prior

- Figure 2 shows that the ERs for exact value *H*(*x*) = 0.1137 under non-informative prior, take the smallest values at *m*=10.

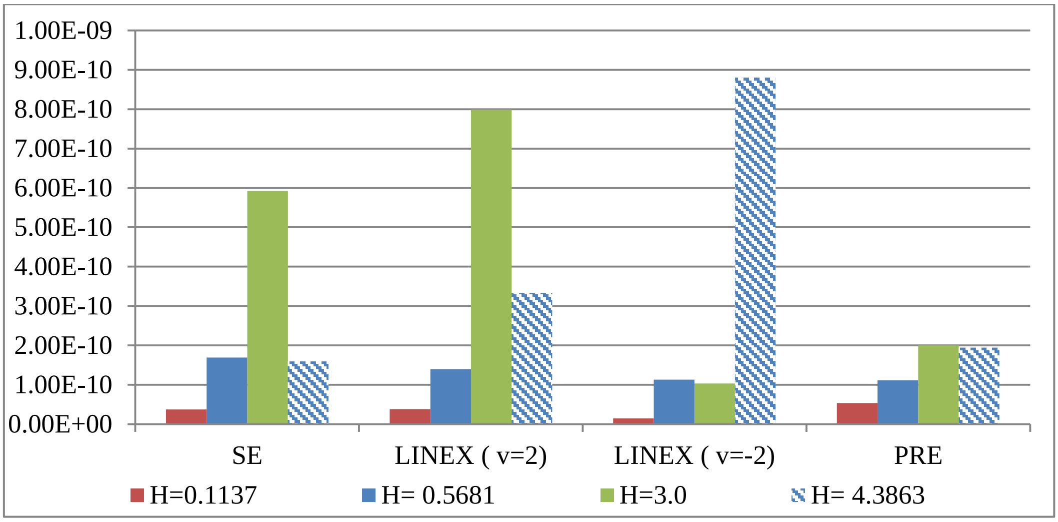

Figure 2 ERs of $\hat{H}(x)$ under SE, LINEX and PRE loss functions for different values of parameters at *m*=10 under non-informative prior

- Figure 3 shows that the ERs for exact value *H*(*x*) = 0.1137 take the smallest value at *m* = 5 in case of informative prior.

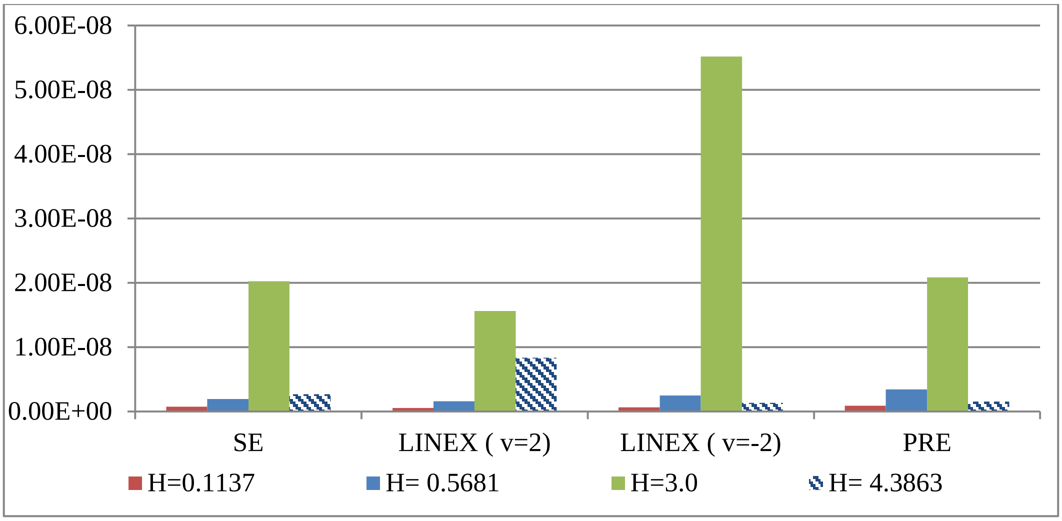

Figure 3 ERs of $\hat{H}(x)$ under SE, LINEX and PRE loss functions for different values of parameters at m = 5 under informative prior

- Figure 4 shows that the ERs for exact value $H(x)$ = 0.1137 and $H(x)$ = 4.3863 in case of informative prior are smaller than the corresponding for another exact value of $H(x)$ for all selected loss functions at m = 10. Also, at $H(x)$ = 0.1137, the ERs of $\hat{H}_{(LINEX)_2}(x)$ at $\nu = 2$ take the smallest value, while, the ERs of $\hat{H}_{(PRE)_2}(x)$ take the largest value.

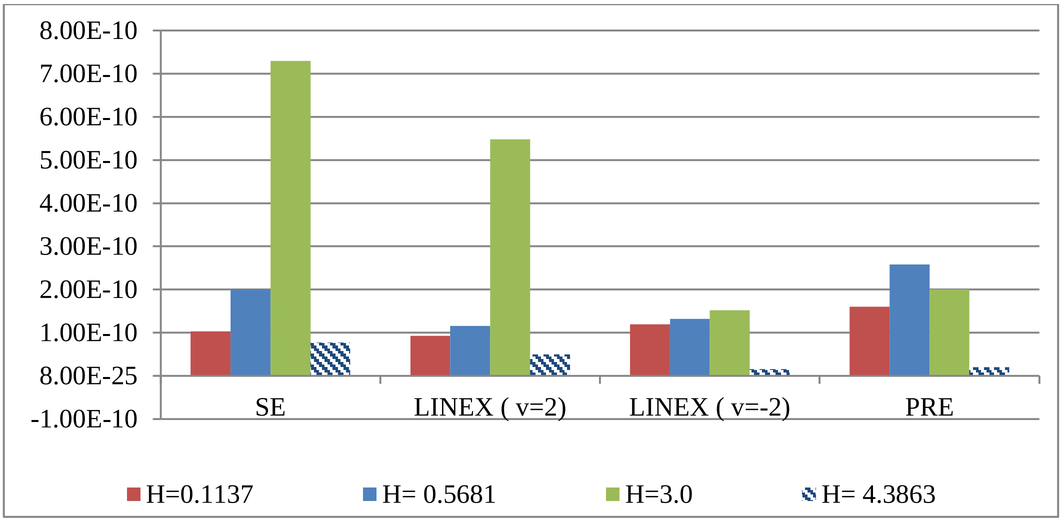

Figure 4 ERs of $\hat{H}(x)$ under SE, LINEX and PRE loss functions for different values of parameters at m=10 under informative prior

- For small true values of entropy, in case of non-informative prior, the width of Bayes credible intervals for $\hat{H}_{(LINEX)_1}(x)$ at $\nu = 2$ is the shortest compared to the width of credible interval in case of $\hat{H}_{(SE)_1}(x)$ and $\hat{H}_{(PRE)_1}(x)$ for most values of m (for example see Figure 5).

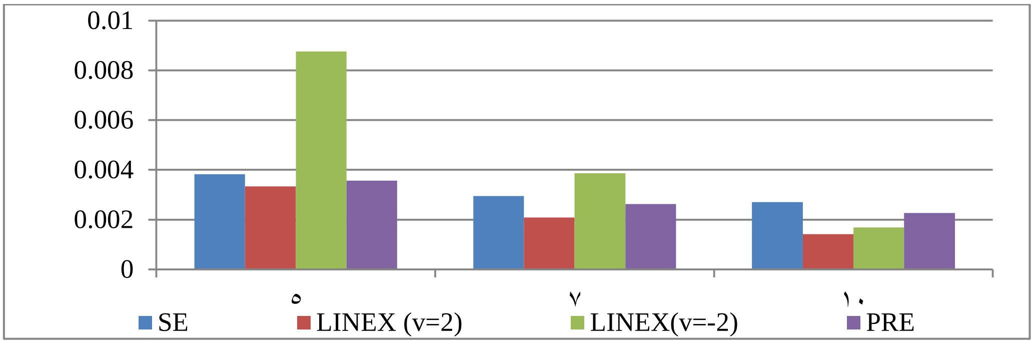

Figure 5 The width of credible interval under SE, LINEX and PRE loss functions for different values of record numbers under non-informative prior for $H(x)$ = 0.1137

- For large true values of entropy, in case of non-informative prior, the width of Bayes credible intervals for $\hat{H}_{(SE)_1}(x)$ is the shortest compared to the width of credible interval $\hat{H}_{(PRE)_1}(x)$ and $\hat{H}_{(LINEX)_1}(x)$ for most values of m (for example see Figure 6).

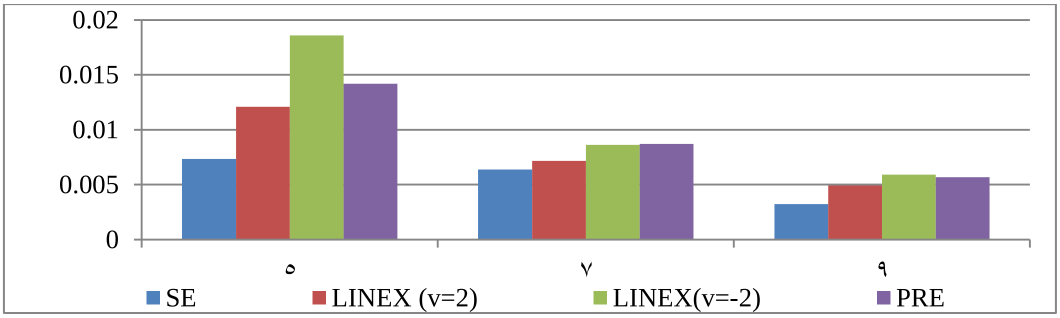

Figure 6 The width of credible interval under SE, LINEX and PRE loss functions for different values of record numbers under non-informative prior for $H(x)$= 4.3863

- Under informative prior, at $H(x)$ = 0.1137, the width of Bayes credible intervals for $\hat{H}_{(SE)_2}(x)$ is the shortest compared to the width of credible interval $\hat{H}_{(PRE)_2}(x)$ and $\hat{H}_{(LINEX)_2}(x)$ for all values of m. The RABs of $\hat{H}_{(SE)_2}(x)$ take the smallest values compared to RABs of $\hat{H}_{(LINEX)_2}(x)$ and $\hat{H}_{(PRE)_2}(x)$ at m = 5, 8 and 10 (see Table 5).

- Under informative prior, the width of Bayes credible intervals for $\hat{H}_{(LINEX)_2}(x)$ is the shortest compared to the width of credible interval $\hat{H}_{(PRE)_2}(x)$ and $\hat{H}_{(SE)_2}(x)$ for most values of *m* (for example see Figure 7). The RABs of $\hat{H}_{(LINEX)_2}(x)$ at $\nu = 2$ have the smallest values compared to RABs of $\hat{H}_{(SE)_2}(x)$ and $\hat{H}_{(PRE)_2}(x)$ at *m*=5,6, 8 and 9 (see Table 7).

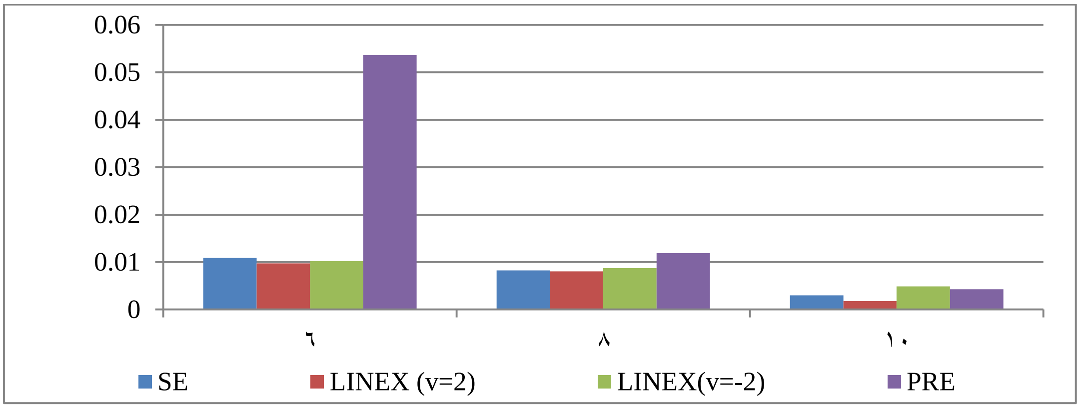

Figure 7 The width of credible interval under SE, LINEX and PRE loss functions for different values of record numbers under non-informative prior for *H*(*x*) = 4.3863

Table 1 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha, \lambda) = (2.0,\ 0.5)$ under non-informative prior

number of records (*m*)		5	6	7	8	9	10
Exact value		0.1137					
SE	Estimate	0.1140	0.1153	0.1161	0.1141	0.1140	0.1141
	RAB	2.39E-03	1.37E-02	2.13E-02	3.28E-03	1.50E-02	6.79E-03
	ER	3.20E-10	2.48E-10	1.58E-10	1.40E-10	5.71E-11	3.72E-11
	Width	3.83E-03	3.40E-03	2.95E-03	2.74E-03	2.72E-03	2.70E-03
LINEX $(\nu = 2)$	Estimate	0.1126	0.1115	0.1144	0.1134	0.1137	0.1132
	RAB	9.56E-03	1.98E-02	5.67E-03	2.86E-03	1.75E-02	4.60E-03
	ER	3.30E-10	2.69E-10	2.43E-10	1.65E-10	9.15E-11	3.79E-11
	Width	3.34E-03	2.99E-03	2.09E-03	1.55E-03	1.51E-03	1.42E-03
LINEX $(\nu = -2)$	Estimate	0.1075	0.1109	0.1141	0.1127	0.1147	0.1138
	RAB	5.49E-02	1.52E-03	3.61E-03	9.03E-03	8.55E-03	1.80E-02
	ER	8.16E-10	6.05E-10	3.03E-10	1.62E-10	8.70E-11	1.42E-11
	Width	8.76E-03	4.12E-03	3.87E-03	1.78E-03	1.73E-03	1.69E-03
PRE	Estimate	0.1128	0.1138	0.1140	0.1133	0.1144	0.1142
	RAB	8.06E-03	8.20E-04	1.15E-02	4.00E-02	5.84E-03	4.55E-03
	ER	4.24E-10	2.42E-10	1.53E-10	1.08E-10	6.81E-11	5.36E-11
	Width	3.57E-03	2.92E-03	2.63E-03	2.52E-03	2.35E-03	2.27E-03

Note: E-a is stands for 10^-a

Table 2 Bayes estimates, RAB, ER and width of Entropy based on

URV for $(\alpha,\lambda)=(1.5,0.5)$ under non-informative prior

number of records (m)		5	6	7	8	9	10
Exact value		0.5681					
SE	Estimate	0.5674	0.5663	0.5653	0.5705	0.5653	0.5667
	RAB	1.18E-03	3.15E-03	4.82E-03	4.37E-03	4.84E-03	2.34E-03
	ER	4.00E-08	8.83E-09	4.28E-09	3.07E-10	2.28E-10	1.69E-10
	Width	5.53E-03	4.91E-03	4.52E-03	3.74E-03	3.15E-03	3.13E-03
LINEX $(\nu=2)$	Estimate	0.5699	0.5693	0.5670	0.5680	0.5677	0.5673
	RAB	3.19E-03	2.24E-03	1.89E-03	1.63E-04	6.33E-04	4.72E-03
	ER	2.85E-09	2.22E-09	1.05E-09	8.61E-10	3.01E-10	1.40E-10
	Width	3.42E-03	3.31E-03	3.15E-03	3.12E-03	2.86E-03	2.01E-03
LINEX $(\nu=-2)$	Estimate	0.5694	0.5691	0.5695	0.5673	0.5687	0.5681
	RAB	6.29E-04	1.13E-03	6.21E-03	1.29E-03	2.86E-03	2.26E-03
	ER	1.93E-09	1.75E-09	1.29E-09	6.44E-10	4.47E-10	1.13E-10
	Width	4.04E-03	3.72E-03	3.36E-03	2.98E-03	2.61E-03	2.15E-03
PRE	Estimate	0.5687	0.5684	0.5680	0.5670	0.5683	0.5681
	RAB	1.21E-03	6.84E-04	1.51E-04	1.85E-03	2.20E-03	8.10E-04
	ER	7.43E-09	1.75E-09	9.46E-10	9.31E-10	2.56E-10	1.11E-10
	Width	4.88E-03	3.87E-03	3.29E-03	2.82E-03	2.29E-03	2.19E-03

Note: E-a is stands for $10^{\wedge -a}$

Table 3 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha,\lambda)=(0.5,0.5)$ under non-informative prior

number of records (m)		5	6	7	8	9	10
Exact value		3.0					
SE	Estimate	2.9970	3.0050	2.9950	2.9940	3.0150	2.9980
	RAB	1.13E-03	1.66E-03	9.20E-04	1.86E-03	5.07E-03	1.57E-03
	ER	1.24E-08	6.23E-09	5.26E-09	7.71E-10	7.27E-10	5.93E-10
	Width	6.91E-03	6.23E-03	5.85E-03	4.53E-03	4.25E-03	3.88E-03
LINEX $(\nu=2)$	Estimate	3.0050	3.0030	2.9990	2.9970	3.0040	3.0010
	RAB	1.77E-03	1.08E-03	2.51E-04	9.33E-04	1.30E-03	3.17E-04
	ER	5.47E-09	3.37E-09	2.19E-09	1.96E-09	1.70E-09	7.99E-10
	Width	1.07E-02	7.36E-03	6.32E-03	5.87E-03	5.21E-03	2.93E-03
LINEX $(\nu=-2)$	Estimate	2.9972	2.9943	2.9943	2.9997	3.0006	3.0002
	RAB	9.44E-04	1.91E-03	1.91E-03	8.81E-05	1.90E-04	3.57E-03
	ER	2.19E-09	1.29E-09	3.26E-10	2.54E-10	5.28E-10	1.03E-10
	Width	1.54E-02	1.08E-02	1.05E-02	7.85E-03	7.74E-03	3.52E-03
PRE	Estimate	3.0040	2.9930	2.9950	2.9950	3.0070	2.9990
	RAB	1.19E-03	2.17E-03	1.51E-03	1.51E-03	2.31E-03	1.11E-03
	ER	1.53E-08	4.29E-09	3.65E-09	2.95E-09	1.44E-09	2.00E-10
	Width	7.06E-03	6.91E-03	6.73E-03	5.52E-03	5.25E-03	3.75E-03

Note: E-a is stands for $10^{\wedge -a}$

Table 4 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha,\lambda)=(0.5,2.0)$ under non-informative prior

number of records (m)		5	6	7	8	9	10
Exact value		4.3863					
SE	Estimate	4.3900	4.3890	4.3790	4.3960	4.3790	4.3800
	RAB	7.35E-04	6.34E-04	1.59E-03	2.26E-03	1.58E-03	1.35E-03
	ER	1.69E-08	9.72E-09	6.80E-09	5.04E-09	6.78E-10	1.58E-10
	Width	7.35E-03	7.02E-03	6.38E-03	5.19E-03	3.25E-03	3.10E-03
LINEX $(\nu=2)$	Estimate	4.3800	4.3900	4.3920	4.3890	4.3900	4.3850
	RAB	1.46E-03	8.54E-04	1.38E-03	6.27E-04	7.38E-04	1.52E-03
	ER	3.16E-08	3.11E-09	2.02E-09	1.06E-09	5.33E-10	3.33E-10
	Width	1.21E-02	8.71E-03	7.16E-03	5.19E-03	4.92E-03	4.62E-03
LINEX $(\nu=-2)$	Estimate	4.3803	4.3975	4.3907	4.3802	4.3863	4.3851
	RAB	1.37E-03	2.56E-03	1.00E-03	1.38E-03	8.62E-06	2.72E-04
	ER	7.96E-09	6.88E-09	4.11E-09	1.64E-09	9.50E-10	8.80E-10
	Width	1.86E-02	1.50E-02	8.63E-03	6.53E-03	5.92E-03	5.12E-03
PRE	Estimate	4.3820	4.3910	4.3840	4.3900	4.3830	4.3820
	RAB	1.00E-03	1.09E-03	4.33E-04	9.26E-04	8.28E-04	9.13E-04
	ER	8.99E-09	7.31E-09	5.65E-09	5.62E-09	8.10E-10	1.94E-10
	Width	1.42E-02	1.32E-02	8.72E-03	6.07E-03	5.67E-03	4.81E-02

Note: E-a is stands for $10^{\wedge-a}$

Table 5 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha,\lambda)=(2.0,0.5)$ under informative prior

number of records (m)		5	6	7	8	9	10
Exact value		0.1137					
SE	Estimate	0.1143	0.1133	0.1139	0.1139	0.1138	0.1137
	RAB	9.97E-04	1.38E-02	2.43E-02	2.01E-03	1.83E-02	4.56E-03
	ER	7.09E-10	5.98E-10	1.76E-10	1.69E-10	1.37E-10	1.02E-10
	Width	3.03E-03	2.99E-03	2.24E-03	2.15E-03	1.75E-03	1.20E-03
LINEX $(\nu=2)$	Estimate	0.1166	0.1172	0.1143	0.1141	0.1126	0.1144
	RAB	2.57E-02	3.05E-02	2.63E-03	2.27E-02	9.71E-03	6.28E-03
	ER	5.27E-10	4.65E-10	3.03E-10	1.43E-10	1.12E-10	9.20E-11
	Width	5.92E-03	4.93E-03	4.39E-03	4.21E-03	3.86E-03	3.70E-03
LINEX $(\nu=-2)$	Estimate	0.1134	0.1145	0.1144	0.1136	0.1141	0.1137
	RAB	2.77E-03	6.95E-03	6.28E-03	2.72E-02	3.21E-03	1.60E-02
	ER	6.34E-10	4.50E-10	3.77E-10	1.80E-10	1.24E-10	1.19E-10
	Width	3.71E-03	3.37E-03	3.21E-03	2.98-03	2.67E-03	2.21E-03
PRE	Estimate	0.1124	0.1156	0.1149	0.1125	0.1144	0.1130
	RAB	2.05E-02	2.73E-02	1.02E-02	1.03E-02	5.91E-03	5.83E-03
	ER	8.67E-10	3.33E-10	2.91E-10	2.49E-10	1.98E-10	1.60E-10
	Width	4.08E-03	3.84E-03	3.76E-03	3.73E-03	3.72E-03	3.41E-03

Note: E-a is stands for $10^{\wedge-a}$

Table 6 Bayes estimates, RAB, ER and width of Entropy based on

URV for $(\alpha, \lambda) = (1.5, 0.5)$ under informative prior

number of records (m)		5	6	7	8	9	10
Exact value		0.5681					
SE	Estimate	0.5678	0.5664	0.5695	0.5660	0.5657	0.5673
	RAB	4.91E-04	2.93E-03	7.44E-04	3.58E-03	5.83E-03	1.27E-03
	ER	1.93E-09	1.55E-09	6.95E-10	5.04E-10	3.69E-10	2.00E-10
	Width	4.52E-03	4.28E-03	2.91E-03	2.83E-03	2.73E-03	2.34E-03
LINEX $(\nu = 2)$	Estimate	0.5676	0.5710	0.5690	0.5660	0.5684	0.5700
	RAB	8.01E-04	5.48E-03	2.17E-03	3.55E-03	5.48E-04	4.13E-03
	ER	1.56E-09	4.57E-10	4.41E-10	2.13E-10	2.02E-10	1.16E-10
	Width	3.05E-03	2.97E-03	2.51E-03	2.48E-03	2.28E-03	2.01E-03
LINEX $(\nu = -2)$	Estimate	0.5650	0.5660	0.5670	0.5689	0.5690	0.5669
	RAB	5.11E-03	3.40E-03	1.98E-03	1.43E-03	1.73E-03	1.95E-03
	ER	2.48E-09	2.28E-09	6.91E-10	4.92E-10	2.78E-10	1.32E-10
	Width	6.47E-03	5.69E-03	4.86E-03	4.78E-03	3.40E-03	3.00E-03
PRE	Estimate	0.5690	0.5670	0.5690	0.5666	0.5676	0.5680
	RAB	9.21E-04	2.68E-03	2.17E-03	2.47E-03	7.22E-04	3.63E-03
	ER	3.44E-09	6.99E-10	5.64E-10	4.14E-10	3.47E-10	2.58E-10
	Width	4.10E-03	3.85E-03	3.03E-03	2.69E-03	2.35E-03	2.10E-03

Note: E-a is stands for 10^{-a}

Table 7 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha, \lambda) = (0.5, 0.5)$ under informative prior

number of records (m)		5	6	7	8	9	10
Exact value		3.0					
SE	Estimate	3.0053	3.0043	2.9956	2.9929	3.0021	3.0003
	RAB	7.60E-04	1.43E-03	8.11E-04	2.38E-03	7.01E-04	1.10E-04
	ER	2.03E-08	1.01E-08	3.66E-09	2.08E-09	8.84E-10	7.30E-10
	Width	1.17E-02	1.09E-02	1.02E-02	8.26E-03	6.29E-03	3.02E-03
LINEX $(\nu = 2)$	Estimate	3.0008	2.9991	3.0057	2.9983	3.0004	2.9998
	RAB	2.73E-04	3.01E-04	1.91E-03	5.72E-04	1.47E-04	3.39E-03
	ER	1.56E-08	1.31E-08	2.88E-09	1.62E-09	1.33E-09	5.48E-10
	Width	9.83E-03	9.72E-03	8.18E-03	8.02E-03	6.56E-03	1.80E-03
LINEX $(\nu = -2)$	Estimate	2.9926	3.0041	2.9918	2.9955	2.9929	2.9991
	RAB	2.47E-03	1.38E-03	2.74E-03	1.51E-03	2.38E-03	2.90E-04
	ER	5.52E-08	1.80E-08	1.99E-09	1.08E-09	8.73E-10	1.52E-10
	Width	1.72E-02	1.02E-02	1.01E-02	8.73E-03	7.43E-03	4.90E-03
PRE	Estimate	2.9987	2.9983	3.0083	3.0022	3.0004	3.0017
	RAB	4.47E-04	5.76E-04	2.75E-03	7.41E-04	1.49E-04	5.70E-04
	ER	2.08E-08	1.28E-08	8.35E-09	7.35E-09	2.85E-10	2.00E-10
	Width	7.87E-02	5.37E-02	2.15E-02	1.19E-02	5.04E-03	4.26E-03

Note: E-a is stands for 10^{-a}

Table 8 Bayes estimates, RAB, ER and width of Entropy based on URV for $(\alpha,\lambda)=(0.5,2.0)$ under informative prior

number of records (m)		5	6	7	8	9	10
Exact value		4.3863					
SE	Estimate	4.3894	4.3784	4.3739	4.3887	4.3828	4.3870
	RAB	7.04E-04	1.79E-03	2.83E-03	5.51E-04	7.89E-04	1.73E-03
	ER	2.67E-09	1.68E-09	8.73E-10	6.39E-10	4.76E-10	7.69E-11
	Width	5.21E-02	3.34E-02	2.07E-02	9.39E-03	6.34E-03	5.59E-03
LINEX $(\nu=2)$	Estimate	4.3787	4.3845	4.3920	4.3832	4.3893	4.3879
	RAB	1.74E-03	4.08E-04	1.30E-03	6.94E-04	6.78E-04	1.06E-03
	ER	8.32E-09	9.59E-10	8.78E-10	7.57E-10	1.24E-10	4.95E-11
	Width	1.21E-02	1.13E-02	9.68E-03	9.35E-03	9.04E-03	8.81E-03
LINEX $(\nu=-2)$	Estimate	4.3923	4.3863	4.3816	4.3845	4.3939	4.3862
	RAB	1.37E-03	1.18E-05	1.08E-03	4.07E-04	1.74E-03	2.12E-05
	ER	1.35E-09	1.28E-09	6.42E-10	2.01E-10	1.36E-10	1.52E-11
	Width	1.23E-02	9.43E-03	9.35E-03	9.18E-03	8.83E-03	4.37E-03
PRE	Estimate	4.3823	4.3847	4.3883	4.3908	4.3855	4.3861
	RAB	9.01E-04	3.74E-04	4.61E-04	1.02E-03	1.84E-04	1.18E-03
	ER	1.51E-09	5.13E-10	3.10E-10	2.70E-10	1.30E-10	2.01E-11
	Width	6.18E-02	6.02E-02	3.02E-02	1.29E-02	5.08E-03	5.01E-03

Note: E-a is stands for 10^-a

4.3 Real Data Analysis

In this subsection, a real data is employed to illustrate the above theoretical results. The data represent the time to break down of an insulating fluid between electrodes at a voltage of 34 K. Volt (Nelson 1982). The data are recorded as follows

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

The validity of the fitted model, has been checked by Abd Ellah (2006). The Kolmogorov-Smirnov goodness of fit test is employed for real data and its p value indicates that the Lomax distribution fits the data. The estimated PDF and CDF are represented in Figure 8.

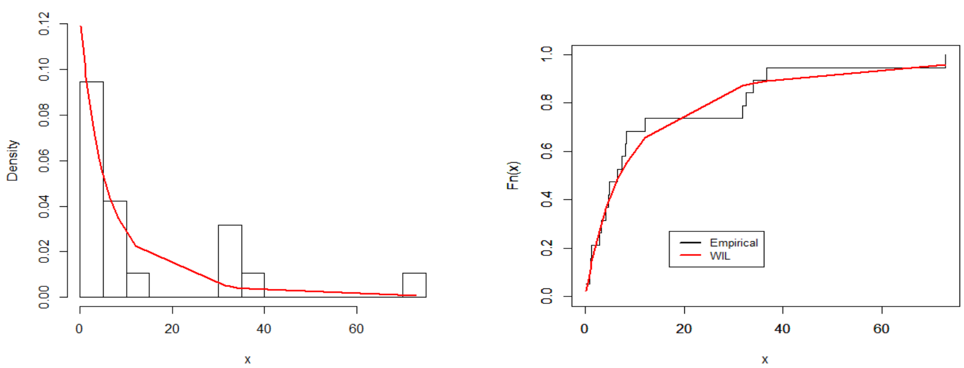

Figure 8 Estimated PDF and CDF of Lomax distribution for real data

Hence, the URV from this data are: 0.96, 4.15, 8.01, 31.75, 33.91, 36.71, 72.89.

Regarding this record data, the Bayesian estimate of entropy at m = 5 and 7 under SE, LINEX and PRE loss functions are obtained and listed in Table 9.

Table 9 Estimated of Shannon entropy under non- informative and informative priors

m	Prior	SEL	LINEX$(\nu=2)$	LINEX$(\nu=-2)$	PRE
5	non-informative	0.1135390	0.1133742	0.1129690	0.1132783
7		0.1134979	0.1114639	0.1113676	0.1110350
5	informative	0.1124057	0.1135005	0.1157789	0.1151257
7		0.1122747	0.1125413	0.1143497	0.1151248

5. Conclusions

This paper provides Bayesian estimation of Shannon entropy for Lomax distribution using upper record values. The Bayesian estimators of entropy are obtained in case of informative and non-informative prior functions for three loss functions. Bayesian estimator of Shannon entropy is discussed under non-informative and informative priors. The Bayesian estimators are computed using the idea of Markov Chain Monte Carlo method based on Gibbs sampling. The performance of the entropy estimates for Lomax distribution is investigated in terms of their relative absolute bias, estimated risk and the width of credible intervals. Application to real data and simulation issue are provided.

From simulation results we conclude that, the Bayesian estimate of entropy approaches the true value as the number of record increases. Generally, the entropy and ERs are directly proportional, that is; if the real value of entropy decreases, the ERs decrease.

Under non-informative prior, for small true values of entropy, the width of Bayes credible intervals for estimated values of entropy under LINEX loss function are smaller than the corresponding estimated values based on SE and PRE loss functions for most values of selected records. But for large true values of entropy, the width of Bayes credible intervals for estimated values of entropy under SE loss function is smaller than the corresponding estimated values based on LINEX and PRE loss functions for most values of m

Under informative prior, the width of Bayes credible intervals for estimated values of entropy under LINEX loss function is smaller than the corresponding estimated values based on SE and PRE loss functions for most values of m.

Regarding simulation results, Bayesian estimates under LINEX loss function at $(\nu=2)$ are more suitable than other selected loss functions for different types of prior functions in most of situations.

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