Electromagnetic Scattering from a Buried Cylinder using T-Matrix and Signal-Flow-Graph Approach

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Abstract—A fast analytical technique based on T-matrix approach is formulated to solve the problem of direct electromagnetic scattering by an infinite circular cylinder buried in a dielectric half-space and illuminated by a normally incident transverse magnetic (TM) plane wave. The technique employs the signal-flow graph (SFG) model to include all multiple reflections that take place during the scattering process. The selection of the truncation order of the obtained T-matrix is also discussed.

Index Terms—scattering, buried cylinder, signal-flow graph.

I. INTRODUCTION

The study of scattering from buried objects has gained a lot of attention in the past few decades; due to its wide and many applications in several fields like detection of landmines, non-invasive biomedical investigations and remote sensing. Among the different approaches proposed to solve this problem [1, and references therein], the T-matrix approach stands out as a powerful and widely used method [2]–[4]. The main idea of the T-Matrix approach is to expand both the incident and scattered field in terms of spherical or cylindrical wave series. Given that the coefficients of incident and scattered field expansions are put in vector forms, the objective is to find a matrix that relates these two vectors to each other, which is called Transition Matrix or T-Matrix.

The process of finding the T-Matrix is based on enforcing boundary conditions on surfaces in the involved scattering system. Many modifications were proposed to the T-matrix method using fast Fourier Transform [5], local shape function [6], recursion based on method of moments [7], extended boundary condition method (EBCM) [8], stabilized EBCM [9] and iterative technique for large aspect ratios [10]. In [11], an iterative analytical approach is introduced to quickly solve the scattering problem of a buried cylinder where the scattered field is measured at the symmetry point on the interface. The proposed solution in [11] has two drawbacks.

In this work, a T-matrix/SFG approach is proposed to address the problems encountered in [11]. In Section II, the problem formulation is revisited for completeness. Section III details the SFG model of the problem, which results in the sought T-matrix. Using the SFG reduction rules allows for summing the multiple reflections to infinity, then the T-matrix formulation is used to determine the truncation order related to the expansion modes as illustrated in Section IV. Results and discussion followed by conclusions are reported in Sections V and VI respectively.

II. PROBLEM FORMULATION

The geometry of the problem under investigation is shown in Fig. 1. A homogenous circular cylinder of complex relative permittivity $\varepsilon_r$, radius $a$ and of infinite length along $z$ direction is buried at depth $d$ in a homogenous dielectric half-space of complex relative permittivity $\varepsilon_r$. The magnetic permeabilities of the dielectric half-space and the cylinder are assumed to be that of free space. A harmonic time dependence $e^{j\omega t}$ is assumed for all fields arising in the next derivation.

For a TM plane wave of electric field amplitude $E_0$ propagating in the positive $x$-direction in free space, the incident electric field can be written as:

$$E_{inc} = E_0 e^{-jk_0x}, \quad x < -d,$$

where $k_0$ is the wave number in free space. The electric field can be written using cylindrical wave expansion [12] as:

$$E_z^i(r, \phi) = \sum_{n=-\infty}^{\infty} u_n J_n(k_0r) e^{jn\phi}, \quad x < -d,$$

where $u_n = E_0 j^{-n}$, $J_n$ is the Bessel function of order $n$. The field reflected from the interface back to free space is given by:

$$E_z^r = R_0 E_0 e^{jk_0x}, \quad x < -d,$$

where

$$R_0 = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} e^{2jk_0d},$$

and the field transmitted through the dielectric is given by:

$$E_z^t(r, \phi) = \sum_{n=-\infty}^{\infty} a_n J_n(k_1r) e^{jn\phi}, \quad x > -d,$$

where

$$a_n = T_0 u_n, \quad T_0 = \frac{2\eta_1}{\eta_1 + \eta_0} e^{jk_0a},$$

$k_1$ is the wave number in dielectric half-space, and $\eta_0$ and $\eta_1$ are the intrinsic impedances of free space and dielectric respectively. The field reflected from cylinder is given by:

$$E_z^c(r, \phi) = \sum_{n=-\infty}^{\infty} b_n J_n(k_1r) e^{jn\phi}, \quad x > -d, r > a,$$

where
To obtain a relation between the scattered field coefficients \( c_n \), calculated for each \( n \), the incident field components obtained using eqn. (10) and the reflected field component obtained using eqn. (3) and the field transmitted from dielectric to free-space is given by:

\[
E_z^T(r, \phi) = \sum_{n=-\infty}^{\infty} d_n J_n(k_1 r) e^{jn\phi}, x > -d, r > a,
\]

(13)

where

\[
d_n = \sum_{m=-\infty}^{\infty} F_{4nm} b_m,
\]

(14)

\[
F_{4nm} = H_{m-n}(2k_1 d),
\]

(15)

\[
\tilde{b}_n = F_{3n} b_n,
\]

(16)

\[
F_{3n} = \left( F_{2n} H_n(k_0 d) - 1 \right) e^{jn\pi}.
\]

(17)

The total scattered field is obtained by summing the first reflected field component obtained using eqn.(3) and the successive scattered field components obtained using eqn.(10). The previous formulation is consistent with what was reported in [11]. In the next section, the SFG will be exploited to expedite the solution and at the same time improve its accuracy.

### III. SIGNAL-FLOW GRAPH (SFG) APPROACH

If each of the summations involved in Section II is calculated for \( n = -N : N \), then we can describe the coefficients \( u_n, a_n, b_n, \tilde{b}_n, c_n \) and \( d_n \) as vectors each of size \((2N + 1)\). To obtain a relation between the scattered field coefficients \( c_n \)

\[
b_n = F_{1n} u_n,
\]

(8)

and

\[
F_{1n} = \begin{cases} \frac{-J_n(k_1 a)}{H_n(k_1 a)}, & \text{PEC}, \\ \frac{k_2 J_n(k_1 a) J_n'(k_2 a) - k_1 J_n(k_2 a) J_n'(k_1 a)}{k_1 J_n(k_2 a) H_n(k_1 a) - k_2 J_n'(k_2 a) H_n(k_1 a)}, & \text{dielectric}, \end{cases}
\]

(9)

where \( H_n \) is the Hankel function of second type of order \( n \).

The reflection coefficient is defined as:

\[
\Gamma = \frac{E_{\text{scat}}}{E_{\text{inc}}},
\]

(25)

which is calculated at the point of symmetry on the interface between the two media.

Figure 1. Buried perfect electrically conducting (PEC) or dielectric cylinder in a dielectric medium. Direction of the incident plane wave is \( \mathbf{k}' \). \( \phi \) is measured in the standard way with respect to global \((x, y)\).
Figure 2. Signal-Flow Graph representation of the scattering interactions

IV. SERIES TRUNCATION

To utilize the vector equations formulated in Section III, all summations involved should be truncated for numerical calculations. In this section the selection of the value of truncation order $N_{\text{trunc}}$ is discussed. As the truncation order increases, it is expected that the accuracy of field calculation increases. However, the numerical ill-conditioning problem arises from the matrix $F_4$ and all the calculations based on it. It is pointed out in [12] that in case of scattering from a conducting cylinder in free space, the truncation order needed to calculate the scattered field using the series expansion form is proportional to the radius of the cylinder. The truncation order selection was also discussed in [14] where a polynomial is used to describe the truncation order $N_{\text{trunc}}$ as a function of the ratio of the cylinder radius to the used wavelength.

An empirical formula for truncation order selection is used in [5] for the case of a PEC cylinder. In the current problem, the truncation order will be calculated as a function of the input variables of the problem: $k_0 a$, $d a$, $\varepsilon_{r1}$ and $\varepsilon_{r2}$. To obtain the truncation order for each case, the size of vectors is incremented until the change in the value of the calculated reflection coefficient is less than the square root of the value of the machine epsilon. The relation between the truncation order $N_{\text{trunc}}$ and problem variables is shown in Figs. 3 through 6.

From the given figures it is observed that:

- $N_{\text{trunc}}$ increases as $k_0 a$ or $\varepsilon_{r1}$ increases.
- $N_{\text{trunc}}$ decreases as $d a$ increases.
- $N_{\text{trunc}}$ has a minimum value when $\varepsilon_{r1} = \varepsilon_{r2}$, at which case the cylinder and the dielectric half-space act as a single medium. $N_{\text{trunc}}$ increases as the value of $\varepsilon_{r2}$ gets away from the value of $\varepsilon_{r1}$.

From the discussion, the truncation order should be selected based on a function that relates it to the given input variables. To find this function, two Feed-Forward Neural Networks are used; one for the case of dielectric cylinder and the other for the case of PEC cylinder, using Matlab Neural Network Toolbox [15]. The input variables ranges are shown in Table I.

![Figure 3. Truncation Order vs. $k_0 a$ ($\frac{d}{a} = 5, \varepsilon_{r1} = 2, \varepsilon_{r2} = 10$)](image)

![Figure 4. Truncation Order vs. $\frac{d}{a}$ ($k_0 a = 1, \varepsilon_{r1} = 2, \varepsilon_{r2} = 10$)](image)

![Figure 5. Truncation Order vs. $\varepsilon_{r1}$ ($k_0 a = 1, \frac{d}{a} = 5, \varepsilon_{r2} = 10$)](image)
For simplicity, only lossless media are used i.e. the values of relative permittivity are real. The effect of losses on the truncation order can be introduced by adding input variables that account for medium losses. After training the network the output is used as the truncation order of the involved series. Knowledge of the suitable truncation order will prevent the ill-conditioning of the involved matrices.

V. RESULTS AND DISCUSSION

The results for calculating the reflection coefficient versus $k_{0\alpha}$ at the point of symmetry were compared against those reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected reported in [11] and perfect agreement was observed (not given here for space considerations). Tables II and III show selected.

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Table II

DIFFERENT SIMULATION CASES FOR BURIED PEC CYLINDER

VI. CONCLUSIONS

In this paper a fast analytical method is introduced to find the field scattered from a buried cylinder based on T-Matrix method. The approach includes all multiple reflections involved during the scattering process. The results obtained using the proposed approach are compared against results obtained using the finite-difference time-domain (FDTD) technique, and exhibit very good agreement. The significant time saving may be helpful specially in case of inverse scattering problems in which the forward scattering problem is solved several times during optimization processes.

REFERENCES