Bayesian and Non-Bayesian Estimations under Failure-Censored Partially Accelerated Life Tests

ABSTRACT
This article considers the Bayesian and non-Bayesian approaches for estimating the Gompertz distribution parameters and the acceleration factor when the data are obtained under the Type II censoring scheme from a step-stress partially accelerated life test. Both the maximum likelihood and Bayesian estimators of the model parameters are derived. The posterior means and posterior variances are derived under the squared error (SE) loss function using Lindley’s approximation procedure. The advantage of this proposed procedure is shown. Monte Carlo simulations are performed under different samples sizes and different parameter values for investigating and comparing the proposed methods of estimation. A non-informative prior on the model parameters is used to make the comparison more meaningful.

Keywords
reliability, partial acceleration, step-stress testing, Gompertz distribution, maximum likelihood estimation, Bayesian estimation, failure-censoring
Introduction

An efficient way of getting information on the reliability of a manufactured product is to submit its units to higher stress levels than the usual working conditions, giving rise to the accelerated lifetime tests (ALT). However, to predict the performance of a component in the case of its use condition, the data must be extrapolated based on a certain model of acceleration. As indicated by Pathak et al. [1], the model of acceleration is chosen so that the relationship between the parameters of the failure distribution and the accelerated stress conditions is known. These relationships are usually derived from an analysis of the physical mechanisms of failure of the component. The tests performed under accelerated stress conditions are called fully accelerated life tests (FALT or simply ALT). Interested readers can refer to Meeker and Escobar [2] and Nelson [3], which are two comprehensive sources for ALT.

Sometimes, such relationships may not be known or cannot be assumed. So, in this case, ALT cannot be used for reliability prediction. Instead, another type of tests called partially accelerated life tests (PALT) is used according to the proposed model by DeGroot and Goel [4].

As Nelson [3] indicates, the stress can be applied in various ways, commonly used method is step-stress. Under step-stress PALT, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, etc., or some combination of them. The objective of a PALT is to collect more failure data in a limited time without necessarily using high stresses to all test units.

Most of literature performed on PALT considered the classical approach to estimate the parameters of interest; for example, see Goel [5], Bhattacharyya and Soejoeti [6], Bai and Chung [7], Bai et al. [8], Attia et al. [9], Abdel-Ghaly et al. [10], Madi [11], Abdel-Ghani [12], Ismail [13], Aly and Ismail [14], Ismail and Sarhan [15], Ismail and Aly [16], Ismail and Abu-Youssef [17] and Ismail [18–20].

From the Bayesian point of view, few studies have been considered on PALT. Goel [5] used the Bayesian approach for estimating the acceleration factor and the parameters in the case of step-stress PALT (SSPALT), with complete sampling for items having exponential and uniform distributions. DeGroot and Goel [4] investigated the optimal Bayesian design of a PALT in the case of the exponential distribution under complete sampling. Abdel-Ghani [12] considered the Bayesian approach to estimate the parameters of Weibull distribution in SSPALT with censoring. Ismail [13] obtained the Bayesian estimates of the Pareto distribution parameters under SSPALT with censored data. Ismail [20] considered the Bayes approach to estimate the parameters of Gompertz distribution under Type I censoring (time-censoring).

In this paper, the main aim is to perform a Bayesian analysis of step-stress PALT considering the Type II censoring (failure censoring) and the two-parameter Gompertz distribution. The Bayes estimators (BEs) of the acceleration factor and the distribution parameters are derived and compared with the maximum likelihood estimators (MLEs) counterpart by Monte Carlo simulations when the data are Type II censored. The squared error loss function is used, and to make the comparison more meaningful, the non-informative priors on both the shape and scale parameters are assumed.

The rest of this paper is organized as follows. In the next section, the model and test method are described. Approximate BEs for the parameters under consideration are derived in the

Nomenclature

- ALT = accelerated life test
- BE = Bayes estimate/estimator
- GD = Gompertz distribution
- MLE = maximum likelihood estimate
- MSE = mean square error
- n = number of step-stress test units (total sample size)
- n_c = number of censored units (n_c = n - n_u + n_c)
- NIP = non-informative prior
- n_u, n_a = numbers of test units failed at use and accelerated conditions, respectively
- PALT = partially accelerated life test
- pdf = probability density function
- r = n_u + n_a
- SE = squared error
- SSPALT = step-stress PALT
- T = lifetime of an item at normal use condition
- Y = total lifetime of an item in a step PALT
- y_{(i)} \leq \cdots \leq y_{(n_c)} \leq \tau \leq y_{(n_c+1)} \leq \cdots
- y_{(i)} = ordered failure times
- y_i = observed value of the total lifetime Y_i of item i, i = 1, \ldots, n
- y_{(r)} = the time of the rth failure at which the test is terminated
- \alpha = shape parameter (\alpha > 0)
- \beta = acceleration factor (\beta > 1)
- \theta = scale parameter (\theta > 0)
- \tau = stress change-time in a step PALT (\tau < y_{(r)})
- ^\land = implies a maximum likelihood estimator
- \downarrow (\cdot) = evaluated at (\cdot)
section on Bayesian Estimation. Next, the BEs derived are obtained numerically using Lindley’s approximation and compared with the MLEs. Finally, a conclusion and points for future work are given.

The Model and Test Method

THE GOMPERTZ DISTRIBUTION AS A LIFETIME MODEL

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. According to the literature, the Gompertz distribution was formulated by Gompertz [21] to fit mortality tables. Recently, many authors have contributed to the statistical methodology and characterization of this distribution. For example, Read [22], Gordon [23], Makany [24], Rao and Damaraju [25], Franses [26], and Wu and Lee [27].

In this paper, the lifetimes of the test items are assumed to follow a Gompertz distribution with probability density function (pdf) as follows:

\[
f(t; \theta, x) = \theta e^{\theta t} \exp\left(-\frac{(\theta/x)(e^{\theta t} - 1)}{1}\right), \quad t > 0, \quad \theta > 0, \quad x > 0
\]

This distribution does not seem to have received enough attention, possibly because of its complicated form, Garg et al. [28]. It is worth noting that when \(x \to 0\), the Gompertz distribution will tend to an exponential distribution (Wu et al. [30]). The two-parameter Gompertz model is a commonly used survival time distribution in actuarial science and reliability and life testing (Ananda et al. [31]).

The Gompertz distribution is a theoretical distribution of survival times. Gompertz [21] proposed a probability model for human mortality, based on the assumption that the “average exhaustion of a man’s power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals” (Johnson et al. [34]).

Gompertz distribution has many applications, particularly in medical and actuarial studies. However, there has been little recent work on the Gompertz in comparison with its early investigation. Osman [36] derived a compound Gompertz model by assuming that one of the parameters of the Gompertz distribution is a random variable following the gamma distribution. He studied the properties of compound Gompertz distribution and suggested its use for modeling lifetime data and analyzing the survivals in heterogeneous populations.

The reliability function of the Gompertz distribution takes the form:

\[
R(t) = \exp\left(-\frac{(\theta/x)(e^{\theta t} - 1)}{1}\right)
\]

and the corresponding hazard rate is given by

\[
h(t) = \theta e^{\theta t}
\]

Thus, the hazard rate increases exponentially over time.

THE TEST METHOD

Basic Assumptions

1. Two stress levels \(x_1\) and \(x_2\) (design and high) are used.
2. For any level of stress, the life distribution of test unit is Gompertz.
3. The total lifetime \(Y\) of an item is as follows:

\[
Y = \begin{cases} 
T & \text{if } T \leq \tau \\
\tau + \beta^{-1}(T - \tau) & \text{if } T > \tau 
\end{cases}
\]

where:

\(T = \) the lifetime of an item at normal use condition.

This model is called the tampered random variable (TRV) model. It was proposed by DeGroot and Goel [4].

4. The lifetimes \(Y_1, \ldots, Y_n\) of the \(n\) test items are independent and identically distributed random variables (IIDRVs).

Test Procedure

1. Each of the \(n\) test items is first run at normal use condition.
2. If it does not fail at normal use condition by a pre-specified time \(\tau\), then it is put on accelerated use condition and run until it fails or the censoring time is reached.

Bayesian Estimation

In this section, the SE loss function is considered. Under SE loss function, the Bayes estimator of a parameter is its posterior expectation. The Bayes estimators cannot be expressed in explicit forms. Approximate Bayes estimators will be obtained under the assumption of non-informative priors using Lindley’s approximation.

In many practical situations, the information about the parameters are available in an independent manner, see Basu et al.
Thus, here it is assumed that the parameters are independent a priori, and let the non-informative prior (NIP) for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

It follows that a NIP for the acceleration factor \( \beta \) is given by:

\[
\pi_1(\beta) \propto \beta^{-1}, \quad \beta > 1
\]

Also, the NIPs for the scale parameter \( \theta \) and the shape parameter \( \alpha \) are, respectively,

\[
\pi_2(\theta) \propto \theta^{-1}, \quad \theta > 0 \quad \text{and} \quad \pi_3(\alpha) \propto \alpha^{-1}, \quad \alpha > 0
\]

Therefore, the joint NIP of the three parameters can be expressed by:

\[
\pi(\beta, \theta, \alpha) \propto (\beta \theta \alpha)^{-1}, \quad \beta > 1, \quad \theta > 0, \quad \alpha > 0
\]

Assuming that the lifetime of test unit is to follow GD(\( \theta, \alpha \)) with pdf in Eq (1). Therefore, the pdf of total lifetime \( Y \) of a unit tested under step-stress PALT is given by

\[
f(y) = \begin{cases} 
  f_1(y) & \text{if } 0 < y \leq \tau \\
  f_2(y) & \text{if } y > \tau 
\end{cases}
\]

where:

\[
f_1(y) = \theta \exp\{z(y - (\theta/\alpha) \exp(y) - 1)\}
\]

given in Eq (1) and

\[
f_2(y) = \beta \theta \exp\{z[\beta(y - \tau) + \tau] - (\theta/\alpha) \exp(z[\beta(y - \tau) + \tau]) - 1\}
\]

which is obtained by the transformation-variable technique using \( f_1(y) \) and the model given in Eq (4)

The observed values of the total lifetime \( Y \) are given by

\[
Y_{(1)} \leq \cdots \leq Y_{(n_u)} \leq \tau \leq Y_{(n_{u+1})} \leq \cdots \leq Y_{(n)}
\]

Because the total lifetimes \( Y_1, \ldots, Y_n \) of \( n \) units are IID RVs, then the general form of the total likelihood function for them can be obtained as follows:

\[
L(\beta, \theta, \alpha) \propto \prod_{i=1}^{n_u} \theta \exp\{z y_i - (\theta/\alpha) \exp(y_i) - 1)\} \\
\times \prod_{i=1}^{n} \beta \theta \exp\{z[\beta y_i - \tau + \tau] - (\theta/\alpha) \exp(z[\beta y_i - \tau + \tau]) - 1\} \\
\times \prod_{i=1}^{n} \exp\{-(\theta/\alpha) \exp(z[\beta y_i - \tau + \tau]) - 1\}
\]

Forming the product of Eqs (5) and (6), the joint posterior density function of \( \beta, \theta, \) and \( \alpha \), given the data, can be written as:

\[
\hat{P}(\beta, \theta, \alpha | \text{data}) \propto L(\beta, \theta, \alpha) \cdot \pi(\beta, \theta, \alpha)
\]

\[
\propto \beta^{\nu_u - 1} \theta^{\nu_u} \alpha^{\nu_u - 1} \exp^{-\frac{\nu_u}{\nu_0} - 1} \\
\times \exp\{z y_i - (\theta/\alpha) \exp(y_i) - 1\} \\
\times \prod_{i=1}^{n_u} \exp\{z[\beta y_i - \tau + \tau] - (\theta/\alpha) \exp(z[\beta y_i - \tau + \tau]) - 1\}
\]

\[
\times \prod_{i=1}^{n} \exp\{-(\theta/\alpha) \exp(z[\beta y_i - \tau + \tau]) - 1\}
\]

(7)

As mentioned earlier, under a squared error loss function, the Bayes estimator of a parameter is its posterior expectation. To obtain the posterior means and posterior variances of \( \beta, \theta, \) and \( \alpha \), non-tractable integrals will be confronted. It is not possible to compute them analytically. The marginal posteriors are somewhat unwieldy and require a numerical integration that may not converge. Instead, an approximation because of Lindley [38] via an asymptotic expansion of the ratio of two non-tractable integrals is used to obtain the approximate Bayes estimators. Lindley’s approximation is evaluated at the ML estimates of the model parameters.

Now, let \( \Theta \) be a set of parameters \( \{\Theta_1, \Theta_2, \ldots, \Theta_m\} \), where \( m \) is the number of parameters, then the posterior expectation of an arbitrary function \( u(\Theta) \) can be asymptotically estimated by:

\[
E(u(\Theta)) = \frac{\int_{\Theta} u(\Theta) \pi(\Theta) e^{\ln u(\Theta)} d\Theta}{\int_{\Theta} \pi(\Theta) e^{\ln u(\Theta)} d\Theta}
\]

\[
\approx u + (1/2) \sum_{ij} (u^{(2)}_{ij} + 2u^{(1)}_{ij} \rho^{(1)}_{ij}) \sigma_{ij} + (1/2)
\]

\[
\times \sum_{ijk} L^{(3)}_{ijk} \sigma_{ijk} u^{(1)}_{ij}
\]

(8)

which is the Bayes estimator of \( u(\Theta) \) under a squared error loss function, where \( \pi(\Theta) \) is the prior distribution of \( \Theta, u \equiv u(\Theta), \)

\( L \equiv L(\Theta) \) is the likelihood function, \( \rho \equiv \rho(\Theta) = \log \pi(\Theta), \)

\( \sigma_{ij} \) are the elements of the inverse of the asymptotic Fisher’s information matrix of \( \beta, \theta, \) and \( \alpha, \)

\[
u^{(1)}_{ij} = \frac{\partial u}{\partial \Theta_i}, \quad u^{(2)}_{ij} = \frac{\partial^2 u}{\partial \Theta_i \partial \Theta_j}, \quad \rho^{(1)}_{ij} = \frac{\partial \log \pi(\Theta)}{\partial \Theta_i}, \quad \text{and}
\]

\[
L^{(3)}_{ijk} = \frac{\partial^3 \ln L(\Theta)}{\partial \Theta_i \partial \Theta_j \partial \Theta_k}
\]

Such an approximation is easy to use and does not require innovative programming and extensive computer time. According to Green [39], the linear Bayes estimator in Eq (8) is a “very good and operational approximation for the ratio of
multi-dimension integrals.” As indicated by Sinha [40], it has led to many useful applications. However, if the domain of the parameters is a function of the parameters, Bayes estimators using Lindley’s rule are not obtainable unless the MLEs exist. The derivation of posterior means and posterior variances is shown in the Appendix.

Monte Carlo Simulation Study

A Monte Carlo simulation study is performed for investigating and comparing the methods of ML and Bayes estimators, under SE loss function. The posterior means and posterior variances of the three parameters $\beta$, $\theta$, and $\alpha$ are derived assuming the NIP for each parameter under a SE loss function using Type II censored data. Because the BEs of the model parameters cannot be obtained analytically, approximate BEs are obtained numerically using the method of Lindley. The behavior sampling of the approximate BEs is investigated and compared with the MLEs in terms of their variances and mean squared errors (MSEs) for different sample sizes and for different parameter values. The process is replicated 1000 times for each sample size and the average of estimates is computed. The results are listed in Tables 1 and 2.

Some of the points are quite clear from the numerical results. As expected it is observed that the performances of both BEs and MLEs become better when the sample size increases. Also, it is observed that the approximate BEs approach the true values with increasing the sample size. When we compare the MLEs with the approximate BEs using Lindley’s technique in terms of their variances and MSEs, it is noted that the approximate BEs perform better than the MLEs. That is, the approximate BEs become with smaller variances and smaller MSEs as the sample size increases. These results coincide with the note of Achcar [41]. He said that the use of approximate Bayesian methods could be a good alternative for the usual asymptotically classical methods in accelerated life testing.

Conclusion

In this article the ML and Bayes estimations of the parameters of Gompertz distribution and the acceleration factor were considered. The Bayes estimators were obtained under the assumptions of squared error loss functions and non-informative priors. It was observed that the Bayes estimators cannot be obtained in explicit forms. Instead, Lindley’s approximation was used to obtain the Bayesian estimates numerically. It was seen

### Table 1

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<td>$\alpha$</td>
<td>ML</td>
<td>1.4104</td>
<td>0.0006</td>
<td>0.0011</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Bayes</td>
<td>1.4003</td>
<td>0.0002</td>
<td>0.0006</td>
</tr>
</tbody>
</table>
that the approximation works very well even for small sample sizes. It was observed that Lindley’s method usually provides posterior variances smaller than the variances of the maximum likelihood estimators. That is, it gives better estimates, which is an advantage of this method. It can be said that the intrinsic appeal of that method can be expressed in is being a sort of adjustment to the maximum likelihood approach to reduce variability. However, it was also observed for very large sample sizes that the Bayesian estimates and the MLEs become closer in terms of MSES and variances. That is, for very large sample sizes, the performances are so far similar as expected. But if we consider informative priors, then the performances of BEs will be much better than those of MLEs and then there is no need for comparisons. As a future work, a Bayesian analysis via a Laplace approximation method and via a Markov Chain Monte Carlo (MCMC) algorithm will be considered.

ACKNOWLEDGMENT

This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

Appendix

Here, there are three parameters in the model. That is, \( m = 3 \). Let the subscripts 1, 2, and 3 refer to \( \beta, \theta, \) and \( \alpha \), respectively. Therefore, the posterior means (BEs) of the three parameters can be expressed by

\[
\beta^* = E(\beta | y) = \left[ \beta - \left( \frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha} \right) + (1/2)(\sigma_{11}\Psi_1 + \sigma_{12}\Psi_2 + \sigma_{13}\Psi_3) \right] \downarrow \delta
\]

\((A1)\)

\[
\theta^* = E(\theta | y) = \left[ \theta - \left( \frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha} \right) + (1/2)(\sigma_{21}\Psi_1 + \sigma_{22}\Psi_2 + \sigma_{23}\Psi_3) \right] \downarrow \delta
\]

\((A2)\)

\[
\alpha^* = E(\alpha | y) = \left[ \alpha - \left( \frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha} \right) + (1/2)(\sigma_{31}\Psi_1 + \sigma_{32}\Psi_2 + \sigma_{33}\Psi_3) \right] \downarrow \delta
\]

\((A3)\)

Thus, the posterior variances can be obtained by

\[
\text{Var}(\beta | y) = E(\beta^2 | y) - (\beta^*)^2 = \sigma_{11} - \left[ \left( \frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha} \right)^2 + (1/2)(\sigma_{11}\Psi_1 + \sigma_{12}\Psi_2 + \sigma_{13}\Psi_3) \right]^2 \downarrow \delta
\]

\((A4)\)

\[
\text{Var}(\theta | y) = E(\theta^2 | y) - (\theta^*)^2 = \sigma_{22} - \left[ \left( \frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha} \right)^2 + (1/2)(\sigma_{21}\Psi_1 + \sigma_{22}\Psi_2 + \sigma_{23}\Psi_3) \right]^2 \downarrow \delta
\]

\((A5)\)

\[
\text{Var}(\alpha | y) = E(\alpha^2 | y) - (\alpha^*)^2 = \sigma_{33} - \left[ \left( \frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha} \right)^2 + (1/2)(\sigma_{31}\Psi_1 + \sigma_{32}\Psi_2 + \sigma_{33}\Psi_3) \right]^2 \downarrow \delta
\]

\((A6)\)

where:

\[
\Psi_1 = \sum_{ij} \sigma_{ij} \Omega_{ij}^{(3)} = \sigma_{11} \Omega_{11}^{(3)} + 2\sigma_{12} \Omega_{12}^{(3)} + 2\sigma_{13} \Omega_{13}^{(3)} + 2\sigma_{22} \Omega_{22}^{(3)} + 2\sigma_{33} \Omega_{33}^{(3)}
\]

\[
\Psi_2 = \sum_{ij} \sigma_{ij} \Omega_{ij}^{(3)} = \sigma_{11} \Omega_{11}^{(3)} + 2\sigma_{12} \Omega_{12}^{(3)} + 2\sigma_{13} \Omega_{13}^{(3)} + 2\sigma_{22} \Omega_{22}^{(3)} + 2\sigma_{33} \Omega_{33}^{(3)}
\]

and

\[
\Psi_3 = \sum_{ij} \sigma_{ij} \Omega_{ij}^{(3)} = \sigma_{11} \Omega_{11}^{(3)} + 2\sigma_{12} \Omega_{12}^{(3)} + 2\sigma_{13} \Omega_{13}^{(3)} + 2\sigma_{22} \Omega_{22}^{(3)} + 2\sigma_{33} \Omega_{33}^{(3)}, \text{ for } i, j = 1, 2, 3
\]

To compute the posterior means and the posterior variances of the three parameters \( \beta, \theta, \) and \( \alpha \), both second and third derivatives of the natural logarithm of the likelihood function must be obtained.

The likelihood function is shown in Eq (6). Its natural logarithm can be written as:

\[
\ln L = (n_{y_1} + n_{a_1}) \ln \theta + n_{y_1} \ln \beta + \alpha \left\{ \sum_{i=1}^{n_{y_1}} [y_i - \tau] + \sum_{j=1}^{n_{a_1}} [\beta(y_j - \tau) + \tau] \right\}
\]

\[
- \left( \theta/\alpha \right) \left\{ \sum_{i=1}^{n_{y_1}} [\exp(y_i) - 1] + \sum_{i=1}^{n_{a_1}} [\exp(\beta(y_i - \tau) + \tau) - 1] + n_{a_1} [\exp(\beta(y_i - \tau) + \tau) - 1] \right\}
\]

The second derivatives of \( \ln L \) with respect to \( \beta, \theta, \) and \( \alpha \) are given by

\[
\frac{\partial^2 \ln L}{\partial \beta^2} = - \frac{n_{a_1}}{\theta^2} - \theta \alpha \left\{ \sum_{i=1}^{n_{y_1}} [(y_i - \tau)^2 \exp(\beta(y_i - \tau) + \tau)] + \sum_{j=1}^{n_{a_1}} [(y_j - \tau)^2 \exp(\beta(y_j - \tau) + \tau)] \right\}
\]

\[
\frac{\partial^2 \ln L}{\partial \theta^2} = - \left( \theta/\alpha^2 \right) \left[ \alpha \omega_1 - \omega_2 + \left( \theta/\alpha^4 \right) \left[ \alpha^4 \omega_2 - 2\alpha \omega_1 \right] \right]
\]

where:

\[
\omega_1 = \sum_{i=1}^{n_{y_1}} [\exp(y_i) - 1] + \sum_{i=1}^{n_{a_1}} [\exp(\beta(y_i - \tau) + \tau)] - 1
\]

\[
+ n_{a_1} [\exp(\beta(y_i - \tau) + \tau) - 1]
\]
\[ \omega_2 = \frac{\partial \omega_1}{\partial x} = \sum_{i=1}^{n} \left[ \sum_{i=1}^{n} \left[ (y_i \exp(xy_i)) \right] + \sum_{i=1}^{n} \left[ \beta(y_i - t) + \tau \right] \right. \\
\times \exp(x[\beta(y_i - t) + \tau]) + n_i[\beta(y_i - t) + \tau] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) \]

\[ \omega_3 = \frac{\partial \omega_2}{\partial x} = \sum_{i=1}^{n} \left[ \frac{\partial^2}{\partial x^2} \exp(xy_i) \right] + \sum_{i=1}^{n} \left[ \beta(y_i - t) + \tau \right] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) + n_i[\beta(y_i - t) + \tau] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) \]

\[ \frac{\partial^2 \ln L}{\partial \beta \partial \theta} = -\sum_{i=1}^{n} \left[ \frac{(y_i - t) \exp(x[\beta(y_i - t) + \tau])}{\theta} \right] + n_i[x(y_i - t) \exp(x[\beta(y_i - t) + \tau])]. \]

Now, the third derivatives of \( \ln L \) with respect to \( \beta, \theta, \) and \( x \) are as follows:

\[ L^{(3)}_{111} = \frac{\partial^3 \ln L}{\partial \beta^3} = 3 \frac{n_x}{\beta^3} \left( 2 \sum_{i=1}^{n} \left[ (y_i - t)^3 \exp(x[\beta(y_i - t) + \tau]) \right] + n_i[x(y_i - t)^3 \exp(x[\beta(y_i - t) + \tau])] \right) \]

\[ \frac{L^{(3)}_{222}}{\partial \theta^3} = \frac{2(n_x + n_a)}{\theta^3} - \frac{4\theta}{\alpha} \left[ \frac{\theta_0}{\alpha} \right] \]

\[ L^{(3)}_{333} = \frac{\partial^3 \ln L}{\partial x^3} = \frac{2\theta}{\alpha^3} \left[ \omega_3 + \frac{\theta_0}{\alpha} \right] - \frac{4\theta}{\alpha^3} \left[ \omega_2 - 2\omega_1 \right] \]

where:

\[ \omega_4 = \frac{\partial \omega_3}{\partial x} = \sum_{i=1}^{n} \left[ \frac{\partial^3}{\partial x^3} \exp(xy_i) \right] + \sum_{i=1}^{n} \left[ \beta(y_i - t) + \tau \right] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) + n_i[\beta(y_i - t) + \tau] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) \]

\[ L^{(3)}_{112} = \frac{\partial^3 \ln L}{\partial \beta \partial \theta} = -2 \sum_{i=1}^{n} \left[ (y_i - t)^2 \right. \exp(x[\beta(y_i - t) + \tau]) + n_i[y_i(y_i - t)^2 \exp(x[\beta(y_i - t) + \tau])] \left. \right] \times \exp(x[\beta(y_i - t) + \tau]) \]

\[ \frac{L^{(3)}_{221}}{\partial \theta^3} = \frac{\partial^3 \ln L}{\partial x^3} = \frac{\partial^3 \ln L}{\partial x^3} = 0 \]

References


