Two-layer-atmospheric blocking in a medium with high nonlinearity and lateral dispersion

M.S. Osman *, H.I. Abdel-Gawad, M.A. El Mahdy

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

ABSTRACT

Herein, the extended coupled Kadomtsev–Petviashvili equation (CKPE) with lateral dispersion is investigated for studying the atmospheric blocking in two layers. A variety of new types of polynomial solutions for the CKPE is obtained using the unified method. Furthermore, we use the Hamiltonian systems with two degrees of freedom to discuss the stability of the obtained solutions through the bifurcation diagrams.

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INTRODUCTION

In the past few years, studies on two layer-fluid systems (in the ocean or in the atmosphere) have attracted the attention of the research community [1–7]. Most studies deal with the cases when the waves are distinguishable at the critical values of dispersion or nonlinearity.

The dynamics and the stability of two-layer fluids are investigated using the complex Korteweg-de-Vries equation (cKdV) [8–11] and the mechanism for the rogue waves formation is presented.

The competition between the dispersivity and the nonlinearity of the medium on folding (or unfolding) of the two layers is given by the system [12–15]

\[
\begin{align*}
I_1 &= (A_1)_t + d_1 (A_1)_x + \mu_1 A_1 (A_1)_x + l_1 (A_1)_{xxx} + k_1 (A_2)_x = 0, \\
I_2 &= (A_2)_t + d_2 (A_2)_x + \mu_2 A_2 (A_2)_x + l_2 (A_2)_{xxx} + k_2 (A_1)_x = 0
\end{align*}
\]  

(1)

where \(d_1 - d_2\) is the detuning parameter, \(k_1, k_2\) are the unfolding parameters, \(I_1, I_2\) are the normalising integrals and \(\mu_1, l_1, i = 1, 2,\) are the nonlinear and the dispersive coefficients respectively.

It worth mentioning that the coupling between the two layers occurs via the last two terms in (1) that designate the effect of translational motion.

In the linear long-wave limit the dispersion relation for waves with speed \(c\) is

\[
(c - d_1)(c - d_2) = k_1 k_2.
\]

(2)

Eq. (1) was studied in [8] and it was found that there is instability when \(k_1 k_2 < 0\) and stability when \(k_1 k_2 > 0\).

Here, we can anticipate the presence of terms \((A_1)_{xxx}\) and \((A_2)_{xxx}\) in the Eqs. (1)\(_1\), (1)\(_2\) and a full suite of nonlinear terms \(A_1 (A_1)_x, A_2 (A_2)_x\) in both equations.

The dynamical behavior and the stability of the solutions of Eq. (1) are analyzed when a transverse dispersion is presented. In this case, Eq. (1) is called the extended Kadomtsev–Petviashvili equation, denoted by (CKPE), and is given by

* Corresponding author.
E-mail address: mofatzi@sci.cu.edu.eg (M.S. Osman).

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\[
\begin{align*}
\left( u_t + d_1 u_x + \mu_1 u u_x + \lambda_1 u_{xxx} \right) + k_1 \nu_x + \delta^2 u_{yy} &= 0, \\
\left( v_t + d_2 v_x + \mu_2 v v_x + \lambda_2 v_{xxx} \right) + k_2 u_x + \delta^2 v_{yy} &= 0,
\end{align*}
\]
where \( \delta \) is the coefficient of lateral dispersion.

Eq. (3) will be studied using the unified method (UM) [16–28]. The UM is based on writing the solutions in the form of polynomials or rational functions that contain an auxiliary function satisfying an appropriate auxiliary equation. Here, we confine ourselves to finding only the polynomial solutions. Then we find the quadratic invariant and introduce the phase portrait for these solutions.

The description the UM

In this section, we present the outline of the UM [16–18].

Consider the nonlinear evolution equations (NLEE) of the type \((q + 1)\)-dimension:

\[
F_i \left( u_1, (u_2)_t, (u_3)_t, \ldots, (u_j)_t, (u_j)_x, (u_j)_tx, \ldots \right) = 0, \quad i, j = 1, 2, \ldots, m,
\]
where \( F_i \) are polynomials in their argument and \( u_j = u_j(t, x_1, \ldots, x_q) \).

When \( x_1, x_2, \ldots, x_q \) and \( t \) are missing in the Eq. (4), then each physical observable \( u_j \) possess \((q + 1)\) basic traveling wave solutions that satisfy the equation

\[
H_i \left( U_j, U'_j, U''_j, \ldots \right) = 0, \quad z = \alpha_0 t + \sum_{j=1}^{q} \beta_j x_j,
\]

where \( U_j = U_j(z) \), \( \alpha_0 \) and \( \beta_j \) are arbitrary constants.

To get the polynomial function solutions of Eq. (5), the UM suggests that

\[
U_j = U_j(z) = \sum_{i=0}^{n} p_i \phi^i(z),
\]

\[
(\phi(z))^p = \sum_{i=0}^{\frac{q}{k}} c_i \phi^i(z), \quad z = \alpha_0 t + \sum_{j=1}^{q} \beta_j x_j, \quad j = 1, 2, \ldots, m, \quad p = 1, 2,
\]

3. Finding the exact solution.
4. We check that the obtained solution satisfies the Eq. (5).

The polynomial function solutions of the CKPE

Here, we find the solutions of (3) in the form of quadratic polynomials.

Let \( u(x, y, t) = U(z), \quad v(x, y, t) = V(z) \), \( z = x + \beta y - ct \) where \( \beta, c \) are the characteristic wave length and the wave speed respectively. Substituting about \( u(x, y, t) = U(z) \) and \( v(x, y, t) = V(z) \) in Eq. (3) yields

\[
U''(z)(\beta^2 \delta^2 - c + d_1 + \mu_1 U(z)) + k_1 V''(z) + \lambda_1 U^{(4)}(z) + \mu_2 U^{(2)}(z) = 0
\]

\[
V''(z)(\beta^2 \delta^2 - c + d_2 + \mu_2 V(z)) + k_2 U''(z) + \lambda_2 V^{(4)}(z) + \mu_2 V^{(2)}(z) = 0.
\]

Under the condition given by (2), we find that

\[
c = \frac{1}{2} \left( d_1 + d_2 + \sqrt{d_1^2 - 2d_1d_2 + d_2^2 + 4k_1k_2} \right). \tag{8}
\]

From Eq. (6) when \( n = 2 \), we have

\[
U(z) = q_0 + q_1 \phi(z) + q_2 \phi^2(z),
\]

\[
V(z) = p_0 + p_1 \phi(z) + p_2 \phi^2(z). \tag{9}
\]

Solitary wave solutions

To obtain these solutions, we put \( p = 1 \) (or \( p = 2 \)) and \( k = 2 \) in the auxiliary equation given by (6)_2.

(i) When \( p = 1 \) and \( k = 2 \).

The auxiliary equation will be in the form

\[
\phi'(z) = c_0^2 - c_2 \phi^2(z). \tag{10}
\]

It is worth to be noticing that these solutions may be considered as results of nonlinear of solitary waves. By a direct calculations using Mathematica (or any other package of calculations), we get

\[
U(z) = q_0 - 12 c_x^2 c_y^2 \mu_1 \tanh^2(c_2 c_0 z),
\]

\[
V(z) = \frac{-\lambda_2 \mu_1 R(R + d_1 - d_2) + k_1 \lambda_1 \mu_2 (R + \beta^2 \delta^2) + c_2^2 c_0^2 k_1 \lambda_1 \left( -8 \mu_2^2 + 12 \mu_2^2 \tanh^2(c_2 c_0 z) \right)}{k_1 \lambda_1 \lambda_2 \mu_2}, \tag{11}
\]

where \( R = q_0 \lambda_1 + \beta^2 \delta^2 - 8 c_x^2 c_y^2 \mu_1 \) and \( q_0, c_0, c_x, c_y, d_1, d_2, k_1, \lambda_1, \lambda_2, \mu_1, \mu_2, \delta, \beta \) are arbitrary constants. Fig. 1 shows the solutions given by (11) for different values of \( \mu_1 \).

Fig. 1(a) and (b) shows two layers of soliton waves. In the upper layer, the soliton wave corresponds to the function \( v(x, y, t) \) while the soliton wave in the lower layer corresponds to the function \( u(x, y, t) \). We remark that the wave in the upper layer is of odd amplitude which is higher than that one in the lower layer. The waves in Fig. 1(a) exhibit separated soliton waves while Fig. 1(b) so overlapping.

(ii) When \( p = 2 \) and \( k = 2 \).

The auxiliary equation will be in the form

\[
\phi'(z) = (c_3 + c_4 \phi(z)) \sqrt{c_2 \phi^2(z) + c_1 \phi(z) + c_0}. \tag{12}
\]
Elliptic wave solutions

To obtain these solutions, we put \( p = 2 \) and \( k = 2 \) in the auxiliary equation given by (6). To this end, we assume that

\[
(\phi(z))' = \sum_{j=0}^{4} c_j \phi^j(z).
\]  

We mention that \( p_i \) and \( q_i, i = 0, 1, 2 \) in the obtained solutions are represented by the free parameters \( c_j, j = 0, 1, \ldots, 4 \). For particular values of \( c_j, j = 0, 1, \ldots, 4 \), we get different solutions in Jacobi elliptic functions. Here, if we take (according to the classification in [29])

\[
c_0 = -2m^2 + m^4 + m^2, c_1 = 0, c_2 = 6m - m^2 - 1, c_3 = 0, c_4 = -\frac{4}{m},
\]

and substituting into (14), we get

\[
\phi(z) = \frac{m \text{dn}(z, m^2) \text{cn}(z, m^2)}{1 + m \text{sn}^2(z, m^2)}
\]  

where \( 0 < m < 1 \) is called the modulus of the Jacobi elliptic functions. When \( m \rightarrow 0 \), \( \text{sn}(z) \), \( \text{cn}(z) \) and \( \text{dn}(z) \) degenerate to \( \text{sin}(z) \), \( \text{cos}(z) \) and 1, respectively; while, when \( m \rightarrow 1 \), \( \text{sn}(z) \), \( \text{cn}(z) \) and \( \text{dn}(z) \) degenerate to \( \text{tanh}(z) \), \( \text{sech}(z) \) and \( \text{sech}(z) \) respectively.

Finally, the general solutions of (7) in terms of the Jacobi elliptic functions are given by

\[
U(z) = q_0 + \frac{48m \mu_1 \text{cn}^2(z, m^2) \text{dn}^2(z, m^2)}{\lambda_1(1 + m \text{sn}(z, m^2))^2},
\]

\[
V(z) = \frac{1}{k_1 k_2 \lambda_2 \mu_2} \left(-k_2^3 \lambda_2^2 \mu_2^2 \mu_1 - k_1 \lambda_2 \mu_2 \left(d_1 \lambda_2 \mu_1 - d_2 \lambda_2 \mu_1 + q_0 \lambda_1 \lambda_2 \mu_1 - 4k_1^2 \lambda_2^2 + 24m \mu_1 \lambda_2 \mu_1 + 4k_1 \lambda_1 \mu_2 + 4 \lambda_2 \mu_1 \mu_2 - 24m \lambda_2 \mu_1 \mu_2 + 4m^2 \lambda_2 \mu_1 \mu_2\right) + \frac{48m \mu_2 \text{cn}^2(z, m^2) \text{dn}^2(z, m^2)}{\lambda_2(1 + m \text{sn}(z, m^2))^2}\right).
\]  

where \( H = (\lambda_2 \mu_1 (c_2 - c_2 c_1 \mu_1 + c_2^2 \beta^2 + c_2^2 \beta^2 q_0) + c_2 k_1 \lambda_2 \mu_1) \) and \( q_0, c_1, c_4, d_1, d_2, k_1, k_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \beta, \delta \) are arbitrary constants. The solutions given by (13) for different values of \( \mu_1 \) are given in Fig. 2.
We remark that in Eqs. (11), (13) and (17), $U(z)$ and $V(z)$ are proportional up to shifting. This may be argued to fact that the coupling in Eq. (3) is expressed by translational terms. The solutions given by (17) for different values of $\mu_1$ are given in Fig. 3.

Fig. 3(a) and (b) shows two layers of elliptic waves. In the upper layer, the elliptic wave corresponds to the function $v(x,y,1)$ while the elliptic wave in the lower layer corresponds to the function $u(x,y,1)$. We remark that the wave in the upper layer is of amplitude which is higher than that one in the lower layer. The waves in Fig. 3(a) exhibit separated elliptic waves while Fig. 3(b) so overlapping.

The quadratic invariant for the CKPE

In this section, we find the quadratic invariant for the solutions given by Eqs. (11), (13), and (17).

Assume we have a quadratic form given by

$$a_2 U^2 + a_1 V^2 + a_0 U' V' + b_2 U(V) U' + b_1 (U, V) V' + b_0 (U, V) = 0.$$  \hspace{1cm} (18)

where $a_0, a_1, a_2$ are constant and $b_0, b_1, b_2$ are function in $U$ and $V$. Differentiating (18) w.r.t. $z$ gives rise to

$$U' b_2 + a_0 U' V' + V' b_1 + 2 a_1 V V' + V^2 \frac{\partial b_1}{\partial U} + V \frac{\partial b_0}{\partial U} + U' \left( V \frac{\partial b_1}{\partial U} + V \frac{\partial b_2}{\partial V} + V \frac{\partial b_0}{\partial U} + 2 a_2 U' + a_0 V V' \right) + U^2 \frac{\partial b_2}{\partial U} = 0.$$  \hspace{1cm} (19)

By using Eqs. (7) and (8) to eliminate $U''$, $V''$ from Eq. (19) and by direct calculations, we get

$$U' = \pm \frac{1}{b_1 \lambda_2} \left( -3 \lambda_2 a_0 V' + \sqrt{3} \left( -\lambda_1 \lambda_2 \left( 4 \frac{a_2}{a_1} \lambda_2 - \lambda_1 \lambda_2 \right) U \\
+ 12 a_2 (2 \lambda_1 a_1 B + A \lambda_2 a_0) V + 6 a_2 (-\lambda_1 \lambda_2 (d_1 + d_2 + G) + a_0 \lambda_1 \lambda_2 + a_0 \lambda_2 \mu_1 V) \\
+ V (4 a_1 \lambda_1 k_2 - a_0 \lambda_2 (-d_1 + d_2 + G) + 2 a_0 B \lambda_1) \\
+ 6 a_2 U' (-d_2 \lambda_2 (-d_1 + d_2 + G) + a_0 \lambda_1 k_2 + a_0 \lambda_2 \mu_1 V) \\
+ 3 \lambda_2 \lambda_2 \left( 4 a_1 a_2 - a_0^2 V^2 + 4 p_1 a_2 \right) \right)^{1/2},$$  \hspace{1cm} (20)

where $G = \sqrt{d_1^2 - 2 d_1 d_2 + d_2^2 + 4 k_1 k_2 - 2 p^2 \delta^2}$ and $A, B$ are constants of the integrations.

Here, we focus our attention to study (graphically) the quadratic invariant for the solutions given by Eqs. (11), (13), and (17) in Figs. 4-6 respectively.

Fig. 4(a) shows that the solution is stable when $0 < \mu_1 < 0.7$.

Fig. 4(b) shows that the solution is unstable when $0.7 < \mu_1 < 2.8$.

Fig. 4(c) shows that the solution is stable when $\mu_1 > 2.8$.

Fig. 5(a) shows that the solution is stable when $0 < \lambda_1 < 0.54$.

Fig. 5(b) shows that the solution is unstable when $0.54 < \lambda_1 < 2$.

Fig. 5(c) shows that the solution is stable when $\lambda_1 \geq 2$.

Fig. 6(a) and (b) show that the solution is stable for all values of $\mu_1$.

Now, we investigate (graphically) the stability and bifurcation to one of the previous solutions, say the solution given by Eq. (11), against the parameters $\mu_1$ and $\lambda_1$ for different values of $\delta$ (the coefficient of the lateral dispersion).
Fig. 4. (a)–(c) The phase portrait of the solution $U$ given by (11). $U$ is displayed against $U'$ with different values of the parameter $\mu_1$. $\beta = 0.2$, $\delta = 0.2$, $q_0 = 3$, $\lambda_1 = \lambda_2 = 1.5$, $d_1 = 1$, $d_2 = -2$, $k_1 = 0.1$, $c_2 = 0.3$, $\mu_2 = 2$, $q_2 = 2$, $a_2 = a_0 = 1$, $A = 1$, $\beta = 3$, $p_1 = 1$, $k_2 = 30.496 + 82.944 \mu_1^1 + \frac{34}{146} + 146.304 \mu_1 + \frac{54}{146}$.

Fig. 5. (a)–(c) The phase portrait of the solution $U$ given by (13). $U$ is displayed against $U'$ with different values of the parameter $\lambda_1$. $\beta = 0.2$, $\delta = 0.2$, $q_0 = 3$, $\mu_1 = 1.5$, $\lambda_2 = 1.5$, $d_1 = 1$, $d_2 = -2$, $k_1 = 0.1$, $c_2 = 0.3$, $\mu_2 = 2$, $q_2 = 2$, $a_2 = a_0 = 1$, $A = 1$, $\beta = 3$, $p_1 = 1$, $k_2 = 11.8568 \mu_1^2 + 13.6817 \lambda_1 - 7.67815$. 
In Fig. 7(a), the bifurcation points are located at \( \mu_* = 0.69 \) and \( \mu_1 = 2.8 \). When \( 0.7 < \mu_* < 2.8 \) the solution is stable and \( 0.7 < \mu_* < 2.8 \) the solution is unstable.

In Fig. 7(b), the bifurcation points are located at \( \lambda_2 = 0.53 \) and \( \lambda_1 = 2.8 \). When \( 0.53 < \lambda_2 < 2.8 \) the solution is stable and \( 0.53 < \lambda_1 < 2.8 \) the solution is unstable.

After Fig. 7(a) and (b), we find that the location of the bifurcation points moved to the left, namely the location of the bifurcation points move at smaller values of \( \mu_* \) and \( \lambda_1 \). This means that the region of instability of the solutions is changed by increasing the value of \( \lambda_2 \).

From Fig. 7(a) we find that the region of instability is obtained when \( 0.7 < \mu_* < 2.8 \). This means that the atmosphere blocking hold in small region of \( \mu_1 \) (nonlinearity).

From Fig. 7(b) we find that the region of instability is obtained when \( \lambda_1 > 2.8 \). This means that the atmosphere blocking hold in small region of \( \lambda_1 \) (dispersion).

**Conclusion**

In this paper, we derived several types of polynomial solutions for the CKPE via the UM to study the atmospheric blocking in two layers. From the bifurcation diagrams, we discussed the stability for these solutions using the Hamiltonian systems with two degrees of freedom. We found that the destabilization of the atmosphere blocking holds significantly when the value of the coefficient of lateral dispersion is increased.

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