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To cite this article: H. I. Abdel-Gawad & M. Tantawy (2019): Coupled self-similar-traveling optical wave tunneling induced by an injected light beam, Waves in Random and Complex Media

To link to this article: https://doi.org/10.1080/17455030.2019.1687961
Coupled self-similar-traveling optical wave tunneling induced by an injected light beam

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ABSTRACT

In a recent experimental and theoretical study, the effects of an external light (laser) beam injected in a medium of Kerr-type nonlinearity were studied. It was shown that the system under consideration may undergo to turbulence. In that work, it is claimed that the injected beam may lead to ‘random’ waves (weak turbulence) and in this case the waves generated are called ‘optical wave turbulence’. In this paper, the deterministic model equation proposed is considered. The modification in the proposed equation accounts for the nonlinearity coefficient and the beam wave function to depend on the space variable \( z \) is taken. The results found here include various arbitrary functions in \( z \), thus these solutions may lead to randomness in ‘some sense’. Also, it has been found that the solutions exhibit a train of incoherent waves, coupled lumps and soliton waves, with gaps. That is, the medium is found to act as a barrier wall that induces optical wave tunneling. This may be argued to the fact that the potential of the injected beam is free. Furthermore, the direction of the reorientation angle is found to be almost everywhere along the direction of light (laser) beam. In other words, a final state is dominated by a single strong soliton whenever a soliton beam is considered. It is shown that the intensity of the wave exhibits an inverse cascade. The results obtained here are completely new. The solutions of the (2 + 1) dimensional equation are derived by using the extended unified method. An algorithm for computing the exact solutions by using Mathematica is presented.

1. Introduction

The effects of an injected beam in a medium are great interest in the study of nonlinear optical wave phenomena \([1–4]\). These studies focused on liquid crystals (LCs) which might be classified into fluid or solid (LCs). The first type might be branched to smectic A and C \([5–7]\), and nematic (LCs) \([8, 9]\). In smectic-A (LCs), the molecules have the same local average direction parallel to the local layer with directors parallel to the normal. While in smectic-C (LCs), this is not the case \([10]\). Nematic (LCs) provide a high level of tunable optical nonlinearity and has been widely studied in the literature \([11–13]\). Recent works in this area are...
Based on the Hirota method and Darboux transformation, there have been some studies on lump-soliton and rogue-wave solutions for the nonlinear evolution equations (NLEEs) [19–22]. In a recent work, experimental and theoretical studies were studied for optical wave that were induced by an injected beam [23]. Attention was focused to investigate the behavior of optical waves when a light (or laser) beam is injected in the Kerr medium where a metal plate is located at \( z = 0 \). The medium is self-focusing as the wave function of the injected beam is positive. It was found that the molecules are reoriented to produce self-focusing nonlinearity [2], this is due to the increasing of Kerr effect via nonlinear coupling of waves. This beam produces incoherent waves that may lead to weak turbulence. The study of weak turbulence was carried via nonlinear interaction of cubic and quadratic waves in the phase space \( k - \omega \). In the case of strong wave turbulence, the mean of the solution is evaluated by taking into account the fluctuations about the plasma trajectories that depend on those of the electrostatic or magnetic fields. In [23], the effects of the one-dimensional (1D) optical wave turbulence were studied. It provides different stationary solutions to both long and short waves. It was shown that the long-wave system is governed by 1D nonlinear Schrödinger equation (NLS) which results in molecular rotation that are confined to \( x - z \) plane. The wave function was taken to vary along \( z \) only [24–32]. It was found that this leads to a final condensate state dominated by a single strong soliton relevant to beam injected. Here, it is observed that the beam acts as a reflector barrier so optical wave tunneling occurs. Also we show that the following result holds, namely, randomization of the optical wave does produce which may be argued to the free potential.

The coherence properties of partially incoherent optical waves propagating in nonlinear media have been studied in nonlinear optics [24]. They are produced due to the propagation of inherent waves in the medium. But the model equations (1a) and (1b) in [23], shown here as Equation (11), are deterministic. Here, we consider two types of the beam geometric structures, harmonic and compressed soliton. They depend on the space–influence time \( z \). As the results found here are general, they hold for arbitrary mean distributions, namely soliton, Gaussian, or otherwise. For the reorientation angle, we find that the principal direction of the light propagation \( z \) is merely along the beam. Here the solutions are obtained by using the extended unified method [33–39].

We are concerned with the study of the optical soliton waves and we consider a more realistic model than one suggested in [23], which is with constant coefficient. The type of soliton pulses can be controlled by the change of the beam potential.

The equations governing the propagation of optical waves in a nonlinear medium [23] are

\[
2qiw_z + w_{xx} + w_{yy} + n_2k_2^2\omega w = 0,
\]

\[
\theta_{xx} + \theta_{yy} - \frac{1}{l_2^2}\theta + \frac{\epsilon_0n_0^2}{4K} | w |^2 = 0,
\]

(1)

where \( w(x, z) \) is the complex wave function, \( \theta(x, z) \) is the reorientation angle. The coordinate \( z \) plays the role of influence time, \( x \) and \( y \) are the space variables. The other parameters are defined in [23].
Here we suggest to take into account the wave function of the beam, namely \( \delta_0(x, y, z) \) and consider the equations

\[
\begin{align*}
2qiw_2 + w_{xx} + w_{yy} + \delta_0(x, y, z)\theta w &= 0, \\
\theta_{xx} + \theta_{yy} - \delta_0(x, y, z)\theta + \frac{e_0n_0^2}{4K} |w|^2 &= 0.
\end{align*}
\]

In Equation (2), the geometric structure of the beam injected is taken into consideration. We distinguish between two cases (i) \( \delta_0(x, y, z) = \delta(z) \), (ii) \( \delta_0(x, y, z) = \delta_1(x, y) \) or (iii) \( \delta_0(x, y, z) = \delta_2(x, z) \). We mention that the nonlinear \( |w|^2 \) is absent in the first equation in Equation (2), so it represents the reduced Zakharov equation. Also, when taking the linear and nonlinear combination gives rise to Gross–Pitaevskii type equation [40–42]. We confine ourselves to consider harmonic and compressed soliton beams. In this work, we aim to find solutions that are induced by direct and indirect nonlinear waves coupling. Thus the solutions of coupled self-similar-traveling waves are obtained. The results of these solutions are used to study the behavior of waves propagation. It is worth noticing that this work is new and was not covered in the literature. This paper is organized as follows. Section 1 is devoted to an introduction. In Section 2, we find the coupled self-similar-traveling wave solutions via polynomial function. The algorithm for computing the exact solution of extended unified method is produced. In Section 4, solutions are found by indirect nonlinear waves coupling or rational function. The solutions are illustrated by three figures. In Section 4, conclusions are presented.

2. Solutions induced by direct nonlinear wave coupling

The solutions in this case are expressed by polynomial solutions in an auxiliary function that satisfies an auxiliary equation. To find the coupled self-similar-traveling waves solutions, we write \( w(x, y, z) = u(x, y, z) + iv(x, y, z) \) and by using the transformations

\[
\begin{align*}
\xi &= \omega(z)x + y\omega_1(z), \quad u(x, y, z) = U(\xi, z), \\
v(x, y, z) = V(\xi, z), \quad \theta(x, y, z) = \Theta(\xi, z).
\end{align*}
\]

\[
\begin{align*}
U(\xi, z) &= \sum_{i=0}^{n} a_i(z)g^i(\xi), \\
V(\xi, z) &= \sum_{i=0}^{n} b_i(z)g^i(\xi), \\
\Theta(\xi, z) &= \sum_{i=0}^{m} d_i(z)g^i(\xi), \quad g^i(\xi) = \sum_{i=0}^{k} c_ig_i(\xi), \quad p = 1, 2,
\end{align*}
\]

where \( g(\xi) \) is the auxiliary function, \( a_i(z), b_j(z) \) and \( d_j(z) \), \( i, j = 0, 1, 2 \) are unknown functions. When substituting from (4) into (2), we find that when balancing the highest order derivative and the highest nonlinear terms of Equation (1), we get \( n = m = 2(k - 1) \) and \( 1 \leq k \leq 5 \). The last consistency condition holds as (1) is integrable in the sense that it passes the Painleve test. For details, see [43].
By substituting these transformations in (1) leads to

\[-2qV_2(\xi, z) + (\omega(z)^2 + \omega_1(z)^2)U_{\xi\xi}(\xi, z) + U(\xi, z)\delta(z)\Theta(\xi, z) = 0,\]

\[2qU_2(\xi, z) + (\omega(z)^2 + \omega_1(z)^2)V_{\xi\xi}(\xi, z) + V(\xi, z)\delta(z)\Theta(\xi, z) = 0,\]

(5)

\[(\omega(z)^2 + \omega_1(z)^2)\Theta_{\xi\xi}(\xi, z) - \delta(z)\Theta(\xi, z) + (U(\xi, z)^2 + V(\xi, z)^2) = 0.\]

We mention that in (3) \(z\) and \(\xi\) are taken as independent variable. This was currently used in the literature for similariton waves [44–46].

When \(p = 1, n_1 = n_2 = m = 2\) and \(k = 2\), we obtain the solitary solution of Equation (3) has the formula

\[U(\xi, z) = a_0(z) + a_1(z)g(\xi) + a_2(z)g^2(\xi),\]

\[V(\xi, z) = b_0(z) + b_1(z)g(\xi) + b_2(z)g^2(\xi),\]

\[\Theta(\xi, z) = d_0(z) + d_1(z)g(\xi) + d_2(z)g^2(\xi),\]

(6)

\[g'(\xi) = c_0 + c_1g(\xi) + c_2g^2(\xi).\]

Substituting Equation (6) into Equation (5), we get

\[a_2(z) = \frac{(6c_2^2(k(z))h(z)}{\sqrt{\delta(z)}}, \quad b_2(z) = \frac{(6c_2^2k(z)\sqrt{1 - h(z)^2})}{\sqrt{\delta(z)}}, \quad d_2(z) = -\frac{(a_2(z)^2 + b_2(z)^2)}{6c_2^2k(z)}),\]

\[a_1(z) := \frac{3h(z)(d_1(z)\delta(z) + 10c_1c_2k(z))}{2\sqrt{\delta(z)}}, \quad b_1(z) = \frac{3\sqrt{1 - h(z)^2}d_1(z)\delta(z) - 10c_1c_2k(z)}{2\sqrt{\delta(z)}},\]

\[d_1(z) = \frac{(-6c_1c_2k(z))}{\sqrt{\delta(z)}}, \quad a_0(z) = \frac{(-2b_0(z)\sqrt{1 - h(z)^2}d_1(z)\delta(z) - (2a^2 - 3c_1^2)k(z))}{2k(z)\sqrt{\delta(z)}},\]

\[d_0(z) = -\left[\frac{1}{(4h^2(z)\delta^2(z))}\left(-\frac{4b_0^2(z)\delta(z) - \delta'(z) - 2(2a^2 - 3c_1^2)k(z)}{\delta(z)k(z)}\right)\right] - \frac{(2a^2 - 3c_1^2)^2\delta(z)}{2k(z)\delta(z)}\]

\[+ 3(a^4 - 4a^2c_1^2 + 3c_1^4)h^2(z)(k^2(z)) - 4b_0(z)\sqrt{1 - h^2(z)}\sqrt{\delta(z)}d(z)\]

\[+ (2a^2 - 3c_1^2)k(z), \quad b_0(z) = \frac{1}{2\delta^{3/2}(z)k(z)}\left((-\sqrt{1 - h^2(z)}\delta(z)k(z))\right)\]

\[-2qh(z)(k(z)\delta'(z) + \delta(z)((-2a^2 + 3c_1^2)\sqrt{1 - h^2(z)}\omega^4(z)\]

\[+ 2(-2a^2 + 3c_1^2)\sqrt{1 - h^2(z)}\omega^2(z)\omega_1^2(z) + (-2a^2 + 3c_1^2)\sqrt{1 - h^2(z)}\omega_1^4(z)\omega_1^4(z)\]

\[+ 8qh(z)(\omega(z)\omega'(z) + \omega_1(z)\omega_1'(z)))\]

\[k(z) = (\omega(z)^2 + \omega_1(z)^2)^2, \quad c_0 = \frac{(c_1^2 - a^2)}{4c_2},\]

(7)

and

\[h(z) = \sin \int \frac{(a^4A_1^4\delta^2(z) + 3\delta^4(z) + 4q^2\delta^2(z))}{8q\delta^3(z)} dz + B_3,\]

(8)

where \(\omega_1(z) = \sqrt{A_1^2 - \omega^2(z)}, \omega(z)\) is an arbitrary function and \(\delta(z)\) satisfies the equation

\[(a^4A_1^4\delta^2(z) - \delta^4(z) + 12q^2\delta^2(z) - 8q^2\delta(z)\delta''(z)) = 0,\]

(9)
which solves to
\[ \delta(z) = \frac{(a^4A_1^4)}{\sqrt{1 + a^4A_1^4 + \cos \left( \frac{A_2^2x + B_2}{2} \right)}}. \]  

When \( B_2 < 0 \), the function \( \delta(z) \) stands to a potential well. The auxiliary equation becomes
\[ g(\xi) = \frac{-(c_1 + a, \tanh(a\xi + A_2)}}{2c_2}. \]  

Here, the potential wave function of the beam \( \delta(z) \) is not free but it is harmonic equation (10).

Substituting Equations (7)–(11) into Equation (6), we get the solutions of \( U(\xi, z), V(\xi, z) \) and \( \Theta(\xi, z) \) where \( \xi = \omega(z)x + y\omega_1(z) \). We find the solutions \( u, v \) and \( \theta \) in \( x, y, z \). The results are too lengthy to be produced here. These results are shown in Figure 1(a,b), plotted \(|w(x, y, z)|\) against \( x \) and \( z \) in 3D and contour plots in Figure 1(a,b) respectively.

Figure 1(a) shows complex incoherent waves. Tunneling optical waves are propagating longitudinal which split to some transversal waves. An inverse cascade is remarked for high value of \( x \).

Figure 1(b) shows the density plot with gaps and tunneling. Where that is occur when \( z \) is taken higher value and wave is splitting.

Figure 2(a,b) shows the intensity of the wave function \(|w(x, y, z)|\) and the reorientation angle \( \theta(x, y, z) \) for different values of \( x(-50 \leq x \leq 50) \). The highest curves in these figures correspond to \( x = 0 \). One finds that the angle attains its maximum values almost near \( x = 0 \).

### 2.1. The algorithm

We present an algorithm for computing the exact solution of (2) (or(5)).
Here, the equations in Section 2 are used. The algorithm is

Input 1: Equations (5) and (6)
Simplify (5)
Output 1: \( \sum_{i=0}^{5} a_i(z) g(\xi), \sum_{i=0}^{5} b_i(z) g(\xi), \sum_{i=0}^{5} d_i(z) g(\xi) \)
Input 2: output 1
Coefficient List [Input2, g(\xi)]
Output 2: \( h_0[a_i(z), b_i(z), \ldots, c_i], \ldots, h_5[a_i(z), b_i(z), \ldots, c_i] \)
Input 3: \( h_5(a_i(z), b_i(z), \ldots, c_i) \)
Solve \( h_5(a_i(z), b_i(z), \ldots, c_i) == 0 \)
Simplify [output 3]
Output 3-1 \( a_2(z) = a_2(a_i(z), b_i(z), \ldots, c_i) \)
Output 3-2 \( h_0[a_i, b_i, c_i, \ldots, 0] \)
Repeat:
Output 7-1 \( [0, 0, \ldots, 0] \)
Output 7-2 \( [a_i(z), b_i(z), d_i(z), \ldots, c_i] \)
Simplify [(6)]

By the same way, an algorithm applies to the results of Section 3 in the case of rational solutions.

3. Solutions induced by indirect nonlinear waves coupling

In this case, the solutions are found via rational function that corresponds to \( n = m = 0, k = 1 \), we write

\[
U(\xi, z) = \frac{a_0(z) + a_1(z)g(\xi)}{s_0(z) + s_1(z)g(\xi)}, \quad V(\xi, z) = \frac{b_0(z) + b_1(z)g(\xi)}{s_0(z) + s_1(z)g(\xi)},
\]

\[
\Theta(\xi, z) = \frac{d_0(z) + d_1(z)g(\xi)}{s_0(z) + s_1(z)g(\xi)}, \quad g'(\xi) = c_1 g(\xi) + c_0,
\]

(12)
where $a_i(z), b_i(z), d_i(z)$ are arbitrary functions and $c_i$ are constants. For the solutions (12) to exist the balance condition must hold in the present case. To this end, we use the transformations $u(x,y,z) = u_0(x,y,z), v(x,y,z) = v_0(x,y,z), \theta(x,y,z) = \theta_0(x,y,z)$ and $u_0(x,y,z) = U_0(\xi,z)$. Similar equations hold to the other variables. In this case $n = m = (k - 1)$ and $1 \leq k \leq 4$. Equation (5) becomes

\[-2\omega(z)q U_{\xi\xi}(\xi,z) + \omega(z)(\omega(z)^2 + \omega_1(z)^2) U_{\xi\xi\xi\xi}(\xi,z) + \omega(z)^2 U_{\xi\xi}(\xi,z) \delta(z) \Theta_{\xi\xi}(\xi,z) = 0,\]

\[2\omega(z)q U_{\xi\xi}(\xi,z) + \omega(z)(\omega(z)^2 + \omega_1(z)^2) V_{\xi\xi\xi\xi}(\xi,z) + \omega(z)^2 V_{\xi\xi}(\xi,z) \delta(z) \Theta_{\xi\xi}(\xi,z) = 0,\]

\[\omega(z)(\omega(z)^2 + \omega_1(z)^2) \Theta_{\xi\xi\xi\xi}(\xi,z) - \omega(z) \delta(z) \Theta_{\xi\xi}(\xi,z) + \omega(z)^2 (U_{\xi\xi}(\xi,z)^2 + V_{\xi\xi}(\xi,z)^2) = 0.\]

Substituting Equation (12) into Equation (13), we have

\[d_0(z) = \frac{d_1(z)}{A_0} - \frac{6s_0(z)(c_1 - A_0c_0)s_0(z)}{c_1^2 \omega(z)^2}, a_1(z) = (A_0c_1a_0(z) + \sqrt{-c_1^2(b_1(z) - A_0b_0(z))} + 36A_1A_0^2(c_1 - A_0c_0)^2 s_0^2(z)) / c_1, b_0(z) = B_0 \tanh(rz),\]

\[b_1(z) = B_2 t \tanh(rz), \quad \omega_1(z) = \omega(z) \sqrt{A_1 \omega(z) - 1}, \quad \omega(z) = \frac{\delta(z)^{1/4}}{(A_1^{1/4} / \sqrt{c_1})},\]

\[s_0(z) = k(z)(b_1(z) - A_0b_0(z)), \quad s_1(z) = A_0s_0(z),\]

\[k(z) = \frac{c_1 \cosh\left(2 \tanh^{-1}\left(\tan\left(\frac{1}{2} \left(\int \frac{1}{2} - \frac{A_1 c_2 \omega(z)^3}{2q} d_1(z) + B_1(z)\right)\right)\right)\right)}{6 \sqrt{A_1 A_0 (c_1 - A_0 c_0)}},\]

where $b_0(z), b_1(z), d_1(z), a_0(z)$ and $\delta(z)$ are arbitrary functions. We mention that the potential wave function of the beam $\delta(z)$ is free. It is taken as compressed soliton in the form

\[\delta(z) = k_0^2 \sech(rz)^4.\]

**Figure 3.** The solution of Equations (12)–(15) when $c_0 = -0.3, c_1 = 0.08, q = 3, A_0 = 0.09, A_1 = 1.5, B_0 = 0.9, B_1 = -0.4, B_2 = 0.24, r = \frac{1}{4}, k_0 = 1.5$ and $y = 2$. 
Figure 4. The solution of Equations (12)–(15) against $x$-axis and at the same values in Figure 4.

Results are displayed against $x$ and $z$ for the 3D and contour plots against $x$ and $z$ in 3D in 3a and 3b when

\[
\begin{align*}
c_1 &= 0.08, & A_0 &= 0.09, & c_0 &= -0.3, & A_1 &= 1.5, & B_1 &= -0.4, & y &= 2, & r &= 1/4, \\
k_0 &= 1.5, & B_0 &= 0.9, & B_2 &= 0.24, & b_0(z) &= B_0 \tanh(rz), & b_1(z) &= B_2 \tanh(rz), & q &= 3.
\end{align*}
\]

Figure 3(a) shows wave steepening while Figure 3(b) shows the medium behaviors as a barrier with dominant soliton wave, which blocks propagation of other sub-dominant waves. The medium may be considered as reflector.

Figure 4(a and b) shows the behavior of the wave intensity $|w|$ and the reorientation angle $\theta$ for different values of $z$ respectively. The highest curves in intensity correspond to $z = 0$, while vice versa in reorientation angle. Here waves show wide soliton waves but with small intensity values.

4. Conclusion

The coupled self-similar-traveling optical waves, propagating in a medium and subjected to an injected light (laser) beam, were studied. It was found that the medium acts as a barrier and leads to incoherent optical wave tunneling with gaps. When the beam wave function is harmonic (potential well), waves that are propagating in the medium are compressed. The medium behaves as a reflector barrier to the optical waves. These results are of great interest in the application of a suitable semi-self-similar pulse by using appropriate controlling functions for long-distance optical beam waveguides.

Disclosure statement

No potential conflict of interest was reported by the authors.

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