Communications in Numerical Analysis 2017 No.1 (2017) 50-79

ISPACS International Scientific Publications and Consulting Services

Available online at www.ispacs.com/cna
Volume 2017, Issue 1, Year 2017 Article ID cna-00299, 30 Pages
doi:10.5899/2017/cna-00299
Research Article

Criterion of Existence of Realistic Permanent Travelling Wave Solutions in Reaction Diffusion Systems

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Abstract

In this paper, we aim to postulate the conditions of existence of permanent travelling wave solution in reaction diffusion systems subjected to initial, boundary or initial-boundary conditions. In a concomitant way we present for a method that allows us to treat initial, boundary or initial-boundary value problems. It is based on finding approximate analytical solutions starting from the formal exact ones. That is by constructing the Picard iterative sequence of solutions and proving a theorem for,the uniform convergence of this sequence. This sequence is then truncated at first, second or higher approximations. The relative error estimate between approximate analytical solutions and some known exact solutions are of the same order as the error between numerical solutions and the exact ones. We should mention that numerical schemes treat only initial-boundary value problem. It is found that the necessary conditions for the presence of travelling wave in the form of $u = u(x-ct)$ is that the initial (or boundary) conditions at the extreme points of domain of definition of the problem have to be different. It is also shown that the sufficient condition is the presence of an advection term, with coefficient which is a constant or a function in the dependent variable, in the reaction diffusion equation.

Keywords: Permanent travelling wave, Picard iterative, Relative error, Fisher equation, Nagumo equation.

1 Introduction

Travelling wave fronts are elegant forms and much studied solutions for reaction-diffusion equations. Their relevant applications to chemistry ,physics and biological processes [1, 2, 3, 4, 5, 6, 8, 9, 10] are currently invoked. The classic and simplest case of the nonlinear reaction diffusion equation is

$$u_t = Du_{xx} + f(u), \tag{1.1}$$

where D is the diffusion coefficient. When $f(u)$ is quadratic or cubic in u, the equation (1.1) was suggested in [11, 12] as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population (see also [2]). It is also the natural extension of the logistic growth population model when the population disperses via linear diffusion. This equation and its travelling wave solutions have been widely studied, as has been the more general form with an

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appropriate class of functions $f(u)$ [13, 14, 15, 16, 17]. The discovery, investigation and analysis of travelling waves in chemical reactions were first reported by Luther [18, 19]. This recently re-discovered paper has been translated by Arnold [20]. Showalter and Tyson [19] put Luther's [18] remarkable discovery and analysis of chemical waves in a modern context. Luther obtained the wave speed in terms of parameters associated with the reactions he was studying. The analytical form is the same as that found by Fisher [11] for (1.1) with $f(u) = \lambda u(1-u)$.

2 Travelling wave formation in systems of infinite size

We proceed by investigating the terminology which will be used hereafter. By a travelling wave, we mean that the location of maximum (minimum)or the wave fronts vary with time [21, 22, 23, 24]. While a permanent travelling wave means that the wave front travels equal distances in equal time periods. A realistic permanent travelling waves solution (RPTWS) is obtained from the partial differential equation (PDE) governing the reaction diffusion system. While a hypothetical permanent travelling wave solution (HPTWS) which is derived after the reduced form of the PDE.

Here, attention is focused to systems of infinite size. We start by the linear diffusion equation [25]

$$u_t = Du_{xx} \quad ; D = const,\ (x,t) \in \mathbb{R} \times (0,T] = \Omega. \tag{2.2}$$

It is well known that this diffusion equation does not admit a realistic permanent travelling wave solution RPTWS because of the unbounded speed of propagation. It does admit a hypothetical permanent travelling wave solution HPTWS as

$$u = u(z),\ z = x - ct,\ u = A + Be^{-\frac{c}{D}z}, \tag{2.3}$$

where A and B are arbitrary constants. We remark that for $c > 0$ the solution (2.3) describes a wave travelling from the left (minus infinity) to the right (plus infinity). But as $z \to -\infty$ the solution is unbounded while it is bounded as $z \to \infty$. A similar discussion holds when $c < 0$. This suggests the following proposition:

Proposition 2.1. *A HPTWS $u \equiv u(x-ct)$ for (1.1) is RPTWS if there exists $c \neq 0$ such that the solution u is finite as $|z| \to \infty$.*

2.1 Conditions for the existence of travelling wave solution

It is worth noting that the equation (1.1) is invariant under translation in space and time namely $x \to x + D_1$ and $t \to t + D_2$ and under space reflection $x \to -x$. Here, we confine ourselves to the case of biological or chemical reactions in systems of infinite size where u is the concentration of the chemical species, it is required that $u \to A$ as $|x| \to \infty$. The equation (1.1) admits a HPTWS which satisfies

$$u'' + cu' + f(u) = 0, \quad u \to A \ as \ |z| \to \infty \tag{2.4}$$

We remark that although (1.1) admits a space-reflection symmetry but the equation (2.4) breaks this symmetry when $z \to -z$. Consequently the equation (2.4) can not admit a solution satisfying the condition $u(z) = u(-z)$. In what follows, we show that the solution of (2.4) can not be finite as $|z| \to \infty$.

Theorem 2.1. *The solution of (2.4) is not a RPTWS of (1.1) subjected to the condition $u \to A$ as $|x| \to \infty$.*

Proof. To prove this theorem,we use the prescribed proposition. Now, we assume that

$$f(u) = au + f^*(u) \tag{2.5}$$

where $f^*(u)$ has zero at least of degree 2 at $u = 0$. We confine ourselves to the case when $f^*(u)$ is a polynomial in u, namely

$$f^*(u) = \sum_{i=2}^{m} a_i u^i. \tag{2.6}$$

From (2.5) into (2.4), we get

$$Du'' + cu + au + f^*(u) = 0, \quad u' = \frac{d\upsilon}{dz}. \tag{2.7}$$

Now assuming that we use the regular perturbation expansion $u = \varepsilon u^{(0)} + \varepsilon^2 u^{(1)} + \cdots$ [7]. By bearing in mind the equation (2.6), we have

$$\varepsilon D \sum_{n=0}^{\infty} \varepsilon^n (u^{(n)})'' + c\,\varepsilon \sum_{n=0}^{\infty} \varepsilon^n (u^{(n)})' + a\varepsilon \sum_{n=0}^{\infty} \varepsilon^n u^{(n)} + \sum_{i=2}^{m} a_i \varepsilon^i (\sum_{n=0}^{\infty} \varepsilon^n u^{(n)})^i = 0. \tag{2.8}$$

Terms of order ε gives rise to

$$D(u^{(o)})'' + c(u^{(o)})' + au^{(o)} = 0. \tag{2.9}$$

This equation solves to

$$u^{(o)} = Ae^{-\frac{r_+ z}{2D}} + Be^{-\frac{r_- z}{2D}}, r_\pm = \frac{1}{2}(c \pm \sqrt{c^2 - 4aD}), \quad c^2 > 4\,a\,D. \tag{2.10}$$

We remark that the solution in (2.10) is finite as $z \to \infty$ by taking $A = 0$ and is finite as $z \to -\infty$ by taking $B = 0$. But the value of $u^{(o)}$ is finite as $\mid z \mid \to \infty$ only when $c = 0$ and $a = 0$. Thus, in this case, no finite solution of $u^{(0)}$ as $\mid z \mid \to \infty$ exists unless $c = 0$. Terms of order ε^2 give rise to the equation

$$D(u^{(1)})'' + c(u^{(1)})' + au^{(1)} + a_2(u^{(o)})^2 = 0, \tag{2.11}$$

which solves

$$u^{(1)} = a_2(\frac{A^2 e^{2r_+ z}}{2cr_+ + 3a} - \frac{2ABe^{-cz}}{c^2(D-1)+a} + \frac{B^2 e^{2r_- z}}{2cr_- + 3a}). \tag{2.12}$$

In (2.12), we have dropped the solution of the homogeneous part of (2.11). When analyzing $u^{(1)}$ given by (2.12), we refined the same conclusion, as for $u^{(o)}$. That is, no finite value of $u^{(1)}$ exists as $\mid z \mid \to \infty$ unless $c = 0$ and $a = 0$. By repeating the analysis to higher order solutions $u^{(2)}$, $u^{(3)}$, ..., we refined the same conclusion that no RPTWS of (2.8) (or(2.4)) which satisfies the prescribed boundary condition in this theorem. □

Our aim now is to predict the necessary and sufficient conditions for the existence of RPTWS in a reaction-diffusion system governed by the equations (1.1) by analyzing the initial-boundary conditions and the evolution equation. First, we determine the conditions on the governing equations. To this end, we consider a more general evolution equation than (1.1) as

$$u_t = F(u, u_x, u_{xx}), \tag{2.13}$$

where F is analytic in its arguments. If the independent variables x and t are missing in (2.5), then it admits a HPTWS in the form $u(x,t) = u(z)$, $z = x - c\,t$. We mention that is the presence of an advection term is due to the presence of an external electric field or drag force in the medium (viscosity of the medium). In this case, we prove the following theorem.

Theorem 2.2. *The sufficient condition for the existence of RPTWS of a reaction diffusion equation is that it contains an advection term with coefficient which is a constant or a function in the dependent variable.*

Proof. To prove this theorem we show that there exists $c \neq 0$ such that the solution is finite as $|z| \to \infty$, first we assume that the advection term is with a constant coefficient. To this end, we assume that in (2.13)

$$F = u_{xx} + au - \nu u_x + f^*(u). \tag{2.14}$$

Again, if we use the regular perturbation expansion , we obtain an equation for $u^{(o)}$ as

$$(u^{(0)})'' + (c - \nu)(u^{(0)})' - au^{(0)} = 0. \tag{2.15}$$

The equation (2.15) solves to

$$u^{(0)} = Ae^{-\frac{s_+ z}{2}} + Be^{-\frac{s_- z}{2}}, s_\pm = (c - \nu) \pm \sqrt{(c-\nu)^2 + 4a}. \tag{2.16}$$

After (2.16), we find that, $u^{(0)}$ is finite as $|z| \to \infty$ if $c = \nu \neq 0$ and $a = 0$ Thus there exists $c \neq 0$ where $u^{(0)}$ is finite . By a similar analysis, we can show that the solutions $u^{(1)}, u^{(2)}, \ldots$ are finite as $|z| \to \infty$ if $c = \nu \neq 0$ and $a = 0$.
Second we consider the case where the coefficient of the advection term is a function in the dependent variable. For simplicity , in (2.13), we assume that

$$F = u_{xx} - \nu u^m u_x, \quad m > 0. \tag{2.17}$$

In this case the HTWS of (2.13) and (2.17) satisfies the equation

$$u^{''} + cu^{'} - \nu u^m u^{'} = 0, \tag{2.18}$$

which integrates to

$$u' = -cu + \frac{\nu u^{m+1}}{m+1}, \tag{2.19}$$

where the constant of integration is taken equal to zero. The equation (2.19) solves to

$$u = \left(\frac{e^{-mcz}}{1 + \frac{\nu}{c(m+1)} e^{-mcz}}\right)^{\frac{1}{m}}. \tag{2.20}$$

We find that u is finite as $|z| \to \infty$ and $\nu \neq 0$. This completes the proof. □

It is worth noting that if the coefficient of the advection term in (2.17) depends on x or t, then no HPTWS exists. Consequently no RPTWS exists. In the absence of the advection term we return to the equation (1.1) and investigate the role of initial-boundary condition and nonlinear source term $f(u)$ on the formation of travelling waves.

2.2 Effects of the presence of nonlinear source terms , initial and boundary condition on the generation of travelling waves

In a reaction diffusion equation when the linear term plays a destabilizing role, and nonlinear terms play a stabilizing role, they may balance each other. The top of the solution stops blowing up and it may travel in space .
The same result holds if the linear term in the reaction diffusion equation plays a stabilizing role while nonlinear ones play a destabilizing role and when the balancing condition holds locally.
This suggests us to adopt the following proposition.

Proposition 2.2. *In a reaction diffusion system, if two antagonistic (destabilizing and stabilizing) affects are produced and one affect balances the other one then a RPTW are generated in this system .*

In the reaction diffusion equation

$$u_t = u_{xx} + f(u), \qquad u(x,0) = u_o(x), \ (x,t) \in \Omega. \tag{2.21}$$

where $f(u)$ is a polynomial in u; namely

$$f(u) = \pm|a_m|\, u^m \pm |a_{m-1}|\, u^{m-1} + \cdots \pm |a_2|\, u^2 \pm |a_1|\, u, \ m > 1 \tag{2.22}$$

Now, if the solution of (2.21) satisfies $|u| \ll 1$ or $|u| \gg 1$ then we may conserve only dominant term in $f(u)$ where it can be approximated respectively by $f(u) \approx \pm|a_1|\, u \pm |a_2|\, u^2$ or $f(u) \approx \pm|a_m|\, u^{(m)} \pm |a_{m-1}|\, u^{m-1}$. For instance, we consider the case when $|u| \ll 1$. Thus, we may write the equation (2.21) as

$$u_t = u_{xx} \pm |a_1|\, u \pm |a_2|\, u^2. \tag{2.23}$$

To investigate the role of the initial and boundary conditions, we have the following theorem.

Theorem 2.3. *The solution of the equation (2.23) admits a RPTWS according to the following statements*

(i) *In the plus-plus case if the initial conditions $u_o(x) \leq 0$ and $|\frac{a_1}{a_2}| \ll 1$ then a RPTWS exists . But if $u_o(x) > 0$ then no RPTWS exists.*

(ii) *In the plus-minus case if $u_o(x) \geq 0$ then a RPTWS exists while it does not exist if $u_o(x) < 0$*

Similar statements hold for the minus-minus and minus-plus cases.

Proof. The proof of this theorem is based on the preceded proposition.
(i) If $u_o(x) \leq 0$ and because of the parity of the diffusion operator , we should have $u(x,t) \leq 0$. Thus the linear term in (2.23); $|a_1|u$ produces a stabilizing effects while the nonlinear term $|a_2|u^2$ produces a destabilizing effects. The balancing condition holds locally on $\bar{\Omega} \subset \Omega$ when $-|a_1||u| + |a_2||u^2| \simeq 0$, or $u \simeq |\frac{a_1}{a_2}|$. Thus the balance condition holds if $|\frac{a_1}{a_2}| \ll 1$. Now, if $u_o(x) > 0$ and due to the parity of the diffusion operator $u(x,t) > 0$. Thus the two terms $|a_1|u$ and $|a_2|u^2$ play a destabilizing role. The top of the solution goes to infinity with time and no RPTWS exists. The proof of (ii) is done in a similar way as for (i). □

We notice that the RPTWS found under the initial conditions in the above theorem when they satisfy the boundary conditions $u(-\infty,t) \neq u(\infty,t)$, they can be obtained as HPTWS satisfying $u(z \to -\infty) \neq u(z \to \infty)$, $z = x - ct$ and $u(z)$ satisfies the equation (2.4)

$$u^{''} + cu' + f(u) = 0. \tag{2.24}$$

But if the same initial conditions hold and the boundary conditions are taken as $u(-\infty,t) = u(\infty,t)$, then a RPTWS exists but it can not be obtained from the solution of (2.4). In this case the RPTWS is considered as two travelling waves. We shall discuss this point in section 4.

3 Formulation of the method

Now, we present for the method to find explicit (approximate analytical) solution for the diffusion equation (1.1) which is rewritten

$$u_t = u_{xx} + f(u), \qquad u(x,0) = u(x),\ (x,t) \in \Omega. \tag{3.25}$$

This method is based on deriving the formal exact solution of (3.25). After the formal exact solution the Picard iterative sequence of solutions is constructed. Theorems of uniform convergency, uniqueness and stability of the solutions will be proved in the Appendix. Truncation of the Picard iterative sequence of solution is done. This applies in the case where the solution of the linear part of (3.25) tends to zero as $t \to \infty$. Otherwise, a rational function approximation to the solution of (3.25) is done. The motivations for presenting this method are the following

(i) The numerical methods; finite difference, finite elements ,$\cdots$ etc for treating reaction diffusion equations [37, 30, 38] work with initial-boundary value problems only. Numerical solutions may not be stable for large value of time . Unless working time is taken sufficiently great.

(ii) The method presented here applies also to systems of coupled diffusion equations [26].

(iii) The relative error estimate between the approximate analytical solution and exact ones is stable for large values of t.

We mention that the Picard iterative sequence of solutions as proposed in the Appendix converges rapidly to the exact solution if the solution of the linear part of the diffusion equation tends to zero (exponentially)as $t \to \infty$. In this case the error between the first (or second) approximation and the exact one is sufficiently small for large values of T. But if the solution of the linear part does not satisfy the above condition as $t \to \infty$, then the error between the truncated solution and the exact one is sufficiently small only for small values of T. In the later case and for the purpose minimizing the error between the truncated solution and the exact one, we give an alternative treatment to that done in the Appendix. That is by using the rational function approximation.
We proceed by distinguishing two cases namely when

(I) The equation (3.26) solves explicitly for u in term of t .

(II) When it solves implicitly.

The algorithm presented here for numerical calculation is formulated adequately in the two cases (I) and (II). In the first case (I), the algorithm consists of the following steps.

(I_1) Solving the homogeneous equation

$$u_t = f(u). \tag{3.26}$$

When this equation has an explicit solution, we have $u(t) = h(u(0),t)$ with $u(0) = h(u(0),0)$.

(I_2) Exploiting the step (I_1) to write the solution of the equation $\bar{u}_t(x,t) = f(\bar{u}(x,t))$ as

$$\bar{u}(x,t) = h(\bar{u}(x,0),t), h(u(\bar{x},0),0) \equiv \bar{u}(x,0). \tag{3.27}$$

We mention that $\bar{u}(x,0)$ replaces $u(0)$ in the solution of (3.25.)

(I_3) By using the variation of parameter in (3.27), we consider the transformation

$$u(x,t) = h(v(x,t),t), u(x,0) \equiv h(v(x,0),0) \equiv v(x,0). \tag{3.28}$$

Now, we construct the iteration scheme as

$$u^{(n)}(x,t) = h(v^{(n-1)}(x,t),t), \quad n \geq 1. \tag{3.29}$$

The zero-approximation is taken appropriately.

(II) In the case where the equation $u_t = f(u)$ does not solve explicitly for u in terms of t , we distinguish two cases

(II_1) The case where the above equation has m fixed points $m > 2$.

We determine among them the relevant sss.

(II_2) Assume that the initial condition satisfies max $u_o(x) = u_2, \ -\infty < x < \infty$.

Now if u_1 and u_2 are the dominant limiting value, we rewrite the equation (3.25) as follows

$$\frac{u_t}{(u-u_1)(u-u_2)} = \frac{u_{xx}}{(u-u_1)(u-u_2)} + \frac{f(u)}{(u-u_1)(u-u_2)} \tag{3.30}$$

By integrating (3.30) formally, we get

$$u(x,t) = \frac{u_2(u(x,0)-u_1) + u_1(u_2 - u(x,0))R(x,t;u)}{(u(x,0)-u_1) + (u_2 - u(x,0))R(x,t;u)}, \tag{3.31}$$

where $R(x,t) = e^{(u_2-u_1)\int_0^t \frac{u_{xx}+f(u)}{(u-u_1)(u-u_2)} dt_1}$. Now, we construct an iteration scheme as

$$u^{(n)}(x,t) = RHS\ of\ (3.31)\ (u \to u^{(n-1)}), n \geq 1 \tag{3.32}$$

The zero-approximation $u^{(o)}(x,t)$ is taken appropriately.
We develop an approach similar to that proposed in [26, 27, 30, 31, 28, 29] to find approximate analytical solutions to the equations (1.1) for the initial value problems (IVP). The approach is based on finding the formal exact solution for the IVP. After the exact solution is found, the Picard iterative sequence of solutions is constructed. We shall prove that this sequence converges uniformly for some special class of initial functions.
We rewrite the equation (1.1) as

$$u_t = u_{xx} + f(u), \tag{3.33}$$

for the initial condition

$$u(x,0) = u_o(x). \tag{3.34}$$

First we shall assume $u_o(x) \in C^2(D)$ where D may be a finite or an infinite domain, and this space is endowed by the supremum-norm namely ;$\|u_o\| = \sup_{x\in D} |u_0(x)|$. In the equation (3.33), we consider the function $f(u)$ as a source term. As in [26, 27], the formal exact solution of (3.33) for initial value problems is given by

$$u(x,t) = u^{(o)}(x,t) + \int_0^t e^{(t-t_1)\partial_x^2} f(u(x,t_1))dt_1. \tag{3.35}$$

In equation (3.35), $u^{(o)}(x,t)$ satisfies the linear diffusion equation.

$$u_t^{(o)} = u_{xx}^{(o)}, \tag{3.36}$$

The Picard iterative sequence of solutions is constructed as

$$u^{(n)}(x,t) = u^{(o)}(x,t) + \int_0^t e^{(t-t_1)\partial_x^2} f(u^{(n-1)}(x,t_1))dt_1 \tag{3.37}$$

The equation (3.36) solves to

$$u^{(o)}(x,t) = e^{t\partial_x^2} u_o(x). \tag{3.38}$$

The exponential operator in (3.35) and (3.38) will be defined in the following.

Definition 3.1. *For $u_o \in C^2(D)$ the exponential operator acts on u_o is defined by*

$$e^{t\partial_x^2} u_o = \frac{1}{2\pi i} \oint e^{\lambda t} (\lambda \hat{I} - \partial_x^2)^{-1} u_o d\lambda \tag{3.39}$$

where $(\lambda \hat{I} - \partial_x^2)^{-1}$ is the resolvent operator and the contour of integration is taken in a way that it encloses the eigenvalues of the resolvent operator.

In the definition of the exponential operator given by (3.39), we note that the resolvent operator $(\lambda \hat{I} - \partial_x^2)^{-1} u_o(x)$ is equivalent to the Green function formulation as [39]

$$(\lambda \hat{I} - \partial_x^2)^{-1} u_o(x) = -\int_D G(x,y;\lambda) u_o(y) dy, \tag{3.40}$$

where $G(x,y;\lambda)$ satisfies the equation

$$(\partial_x^2 - \lambda) G(x,y;\lambda) = \delta(x-y), \tag{3.41}$$

where $\delta(x-y)$ is the generalized dirac delta function. From the integral representation of the exponential operator, we have the following lemmas.

Lemma 3.1. *If $u_o \in C^2(D), D \equiv [-\ell, \ell]$, then $e^{t\partial_x^2}$ acting on this space is bounded.*

Proof. First, we note that the norm here is taken as $\|u_o\| = \sup_{|x|\le \ell} \mid u_o(x) \mid$ and $\|e^{t\partial_x^2}\| = \sup_{u_o \in C^2} \frac{\|e^{t\partial_x^2} u_o\|}{\|u_o\|}$. From the equation (3.34) we have

$$e^{t\partial_x^2} u_o(x) = \frac{-1}{2\pi i} \oint e^{t\lambda} \left(\int_{-\ell}^{\ell} G(x,y;\lambda) u_o(y)\, dy \right) d\lambda. \tag{3.42}$$

From (3.41), we have

$$G(x,y;\lambda) = \int_{-\ell}^{x} \left(\int_{-\ell}^{x_1} e^{\sqrt{\lambda}(x-2x_1+x_2)} \delta(x_2 - y) dx_2 \right) dx_1. \tag{3.43}$$

The equation (3.42) gives rise to

$$G(x,y;\lambda) = \begin{cases} \int_{-\ell}^{x} e^{\sqrt{\lambda}(x-2x_1+y)} dx_1, & -\ell < y < x_1 \\ 0, & x_1 < y < \ell \end{cases} \tag{3.44}$$

Substituting (3.44) into (3.42), we get

$$e^{t\partial_x^2}u_o(x)=\frac{-1}{2\pi i}\oint e^{t\lambda}\left[\int_{-\ell}^{x}\left(\int_{-\ell}^{x_1}e^{\sqrt{\lambda}(x-2x_1+y)}u_o(y)dy\right)dx_1\right]d\lambda. \tag{3.45}$$

By bearing in mind the convergence theorem for the complex Fourier series, we expand $u_o(y)$ as

$$u_o(y)=\sum_{n=-\infty}^{\infty}a_n e^{\frac{in\pi y}{\ell}}. \tag{3.46}$$

Substituting (3.46) into (3.45), carrying out the inner integrals and carrying the integral in the complex $\lambda-$plane by the method of residues, we get

$$e^{t\partial_x^2}u_o(x)=\sum_{n=-\infty}^{\infty}e^{\frac{-n^2\pi^2 y}{\ell^2}}a_n e^{\frac{in\pi y}{\ell}}. \tag{3.47}$$

Now, from the assumption, we have $\| u_o \|=\sup_{|x|\le\ell} | u_o(x) |= M$. Thus $\sum_{n=-\infty}^{\infty}| a_n |\le M$. Consequently $| a_n |\le K$ for all n.
Now

$$\begin{aligned}\| e^{t\partial_x^2} \| &= \sup_{u_o}\frac{\| e^{t\partial_x^2}u_o \|}{\| u_o \|}\\ &\le \frac{1}{M}\sum_{-\infty}^{\infty}| a_n | e^{-\frac{n^2\pi^2 t}{\ell}}\\ &\le \frac{2K}{M}\frac{1}{1-e^{-\frac{\pi^2 t}{\ell^2}}} \quad t>0.\end{aligned}$$

this proves the lemma. □

We remark that, the series in the right hand side of (3.47) converges uniformly for $-\ell\le x\le\ell$ and $t>0$. Thus it is infinitely differentiable with respect to x.
Notice that if u_o is piecewisely continuous on D, then the proof of Lemma 3.1 also holds .

Lemma 3.2. *If $u_o(x)$ is piecewisely continuous on $\mathbb{R}$ and $\int_{-\infty}^{\infty}| u_o(x) | dx<\infty$ or $u_o\in L_1(\mathbb{R})$,then $\|e^{t\partial_x^2}\|<\infty$*

Proof. Similarly as in Lemma 3.1, the formula (3.45) becomes

$$e^{t\partial_x^2}u_o(x)=\frac{-1}{2\pi i}\oint e^{t\lambda}\left[\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}e^{\sqrt{\lambda}(x-2x_1+y)}u_o(y)dy\right)dx_1\right]d\lambda. \tag{3.48}$$

As $u_o(y)$ is absolutely integrable on $\mathbb{R}$, we have

$$u_o(y)=\int_{-\infty}^{\infty}\tilde{u}_o(p)e^{ipx}dp=lim_{\varepsilon\to 0^-}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\tilde{u}_o(p)e^{ipx}dp, \tag{3.49}$$

when substituting from (3.49) into (3.48), we get

$$\begin{aligned}e^{t\partial_x^2}u_o(y) &= \frac{1}{2\pi \mathrm{i}}\lim_{\mathrm{c}\to 0^-}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\Bigg[\tilde{u}_o(p)\\ &= \oint e^{t\lambda}\left(\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}e^{\sqrt{\lambda}(x-2x_1+y)+ipy}dy\right)dx_1\right)dp\Bigg]d\lambda.\end{aligned} \tag{3.50}$$

By changing the order of the two inner integral, it becomes

$$I = \int_{-\infty}^{x} \left(\int_{-\infty}^{x_1} e^{\sqrt{\lambda}(x-2x_1+y)+ipy} dy \right) dx_1$$
$$= \int_{-\infty}^{x} e^{(\sqrt{\lambda}+ip)y} \left(\int_{y}^{x} e^{\sqrt{\lambda}x-2x_1)} dx_1 \right) dy. \tag{3.51}$$

Integration by parts and writing $\lambda = Re^{i\theta}$, we find that $|e^{\sqrt{\lambda}y}| = |e^{R^{\frac{1}{2}}(cos\frac{\theta}{2}}$
$^{+i\, sin\frac{\theta}{2})y}| \to 0$ as $y \to -\infty$, by choosing the closed contour for the integral lies in the upper-half of the complex λ-plane where $cos\frac{\theta}{2} > 0$.
Thus the integral in (3.51) gives rise to

$$I = \frac{e^{\sqrt{\lambda}x}}{\sqrt{\lambda}+ip} \int_{-\infty}^{x} e^{(-\sqrt{\lambda}+ip)y} dy = -\frac{e^{ipx}}{\lambda+p^2}, \tag{3.52}$$

because $|e^{ipy}| \to 0$ as $y \to \infty$ as $Im\ p < 0$. Substituting (3.52) into (3.50) and carrying out the integral over λ, we have

$$e^{t\partial_x^2} u_o(y) = lim_{\varepsilon\to 0^-} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \tilde{u}_o(p) e^{-tp^2+ipx}. \tag{3.53}$$

When substituting for the Fourier transform $\tilde{u}_o(p)$, performing the integral over p, we get

$$e^{t\partial_x^2} u_o(x) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x_o)^2}{4t}}}{\sqrt{4\pi t}} u_o(x_o) dx_o. \tag{3.54}$$

We remark that, the integral in the right hand side of (3.54) for $-\ell \le x \le \ell$ when $t > 0$ converges uniformly for $-\infty < x < \infty,\ t > 0$, then it is infinitely differentiable with respect to x.
Now,

$$\| e^{t\ \partial_x^2} \| = \underset{u_0}{sup} \frac{\| e^{t\partial_x^2} u_o \|}{\| u_o \|} = \underset{u_0}{sup} \frac{\underset{x\in\mathbb{R}}{sup} |\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x_o)^2}{4t}}}{\sqrt{4\pi t}} u_o(x_o) dx_o|}{\| u_o \|} \le 1,\ t > 0 \tag{3.55}$$

The equation (3.55) holds because $|u_o(x_o)| < M = \underset{x\in\mathbb{R}}{sup} |u_o(x)| = \|u_o\|$ and $\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x_o)^2}{4t}}}{\sqrt{4\pi t}} dx_o = 1$. This completes the proof of the lemma. □

Lemma 3.3. *If u_o is piecewise of constant functions, then $\| e^{t\partial_x^2} u_o(x) \|$ is bounded.*

Here, we shall apply the following theorem on Fourier transforms [40]

Theorem 3.1.

(a) *If u_1 and u_2 have Fourier transforms, then $u_1 + u_2$ and $u_1 u_2$ have Fourier transforms.*

(b) *If u has a Fourier transform and $au + b$, a and b are constants, does not vanishes on R, then $\frac{u}{au+b}$ also has a Fourier transform.*

Consequently if u has a Fourier transform and $f(u)$ is a polynomial in u, then $f(u)$ and $f(\frac{u}{au+b})$ also have a Fourier transforms.

In what follows, we shall prove that the Picard sequence of solutions u_n converges to the exact solution of equations (3.33) and (3.34). Also, it can be shown that this solution is unique and stable. To this end, we prove the following lemma.

Lemma 3.4. *If $f(u)$ is algebraic in u(a polynomial in u), then*

$$|f(u^{(n)}) - f(u^{(n-1)})| < H|u^{(n)} - u^{(n-1)}|$$

Proof. The proof of this lemma is done by induction on n.
We assume that $f(u)$ is a polynomial of degree m in u.
Now we show that $\|u^{(o)}\|$ is bounded where $u^{(o)}$ satisfies the equation

$$u_t^{(o)} = u_{xx}^{(o)}, \ t > 0, \ -\infty < x < \infty, \quad with \quad u(x,0) = u_0(x)$$

is given by

$$u^{(o)}(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\,\pi\,t}}\, u_0(y)dy, \tag{3.56}$$

from (3.56), we find $\|u^{(o)}\| = \underset{x\in\mathbb{R}}{Sup}|u^{(o)}(x,t)| \leq \int_{-\infty}^{\infty} \frac{|u_o(y)|}{\sqrt{4\,\pi\,t}}dy \leq \|u_o\|_{L_1}$ for $t > \frac{1}{4\,\pi}$ and for $t < \frac{1}{4\,\pi}$ we have $\|u^{(o)}\| < \|u_o\|$.
Thus $\|u^{(o)}\| \leq K = Max(\|u_o\|\ ,\ \|u_o\|_{L_1})$, then $\|(u^{(o)})^m\|$ is bounded; namely $\|(u^{(o)})^m\| < K^m$.
In a similar way, we can show that $\|(u^{(1)})^m\|$ is bounded. By induction $\|(u^{(n)})^m\|$ is bounded for all integers n and m.
Thus $\|f\|$ is bounded.
As $f(u)$ is differentiable in its arguments, then $\|f'(u^{(n)})\|$ is also bounded bearing in mind that the norm is taken as the supremum norm.
Now by using the mean value theorem, we have

$$\begin{aligned} f(u^{(n)}) - f(u^{(n-1)}) &= f'(\theta)(u^{(n)} - u^{(n-1)}), \quad u^{(n-1)} < \theta < u^{(n)}. \\ \Rightarrow |f(u^{(n)}) - f(u^{(n-1)})| &= |f'(\theta)||u^{(n)} - u^{(n-1)}| < H|u^{(n)} - u^{(n-1)}|. \end{aligned} \tag{3.57}$$

□

From the previous lemma, we can prove the following convergence theorem

Theorem 3.2. *If $u_o(x)$ is a bounded and piecewisely continuous function on $\mathbb{R}$, then the Picard sequence of solutions of (3.35)–(3.37) converges uniformly to the exact solution.*

Proof. From the Picard Iteration of (3.35), we have

$$u^{(n+1)} - u^{(n)} = \int_0^t e^{(t-t_1)\partial_x^2}(f(u^{(n)},t_1) - f(u^{(n-1)},t_1))dt_1. \tag{3.58}$$

Thus, we have

$$\begin{aligned} \|u^{(n+1)} - u^{(n)}\| &= \|\int_0^t e^{(t-t_1)\partial_x^2}(f(u^{(n)},t_1) - f(u^{(n-1)},t_1))dt_1\| \\ &\leq \int_0^t \|\, e^{(t-t_1)\partial_x^2}\|\ \ \|(f(u^{(n)},t_1) - f(u^{(n-1)},t_1))\|dt_1. \end{aligned} \tag{3.59}$$

From (3.55), we get

$$\|u^{(n+1)} - u^{(n)}\| \leq \int_0^t \|(f^{(n)} - f^{(n-1)}\|dt_1. \tag{3.60}$$

By using (3.57), we get

$$\|u^{(n+1)} - u^{(n)}\| \leq H\int_0^t \|\,(u^{(n)} - u^{(n-1)}\|dt_1,\ n \geq 1. \tag{3.61}$$

for $n = 0$, we have

$$\|u^{(1)} - u^{(o)}\| \leq H\int_0^t \|f_o\|dt_1 \leq R\,Ht, \tag{3.62}$$

where $\|f_o\|$ is bounded. For $n=1$ from (3.60), we have

$$\|u^{(2)}-u^{(1)}\| \leq R\,H^2 \int_0^t t_1 dt_1 = R\frac{H^2t^2}{2!}. \tag{3.63}$$

Similarly, for $n=2$

$$\|u^{(3)}-u^{(2)}\| \leq R\frac{H^3t^3}{3!}. \tag{3.64}$$

By induction, we get

$$\|u^{(n)}-u^{(n-1)}\| \leq R\frac{(Ht)^n}{n!}, n=1,2,\ldots,0<t<T, \tag{3.65}$$

and since

$$u(x,t)=u^{(o)}(x,t)+\sum_{i=1}^{\infty}(u^{(i+1)}(x,t)-u^{(i)}(x,t)), \tag{3.66}$$

it follows

$$\|u\| \leq K+\sum_{i=1}^{\infty}\|u^{(i+1)}-u^{(i)}\| \tag{3.67}$$

with the account of inequality (3.62)

$$\|u\| \leq K+R\,e^{H\,t},\ 0<t<T. \tag{3.68}$$

□

An alternative proof for Theorem 3.2 can be found in [32, 33]. From this theorem, we have the following corollary.

Corollary 3.1. *If $h^{(n)}=\frac{u^{(n)}}{u^{(n)}+(1-u^{(n)})e^{at}}$ and $u^{(n)}$ converges uniformly to u then $h^{(n)}$ converges uniformly to $\frac{u}{u+(1-u)e^{at}}$*

Theorem 3.3. *If $u_o(x)$ is a bounded and piecewisely continuous function, then the solution of the initial value problem (3.33–3.34) is unique and stable.*

Proof. **(I) Uniqueness:**

We assume that $u(x,t)$ and $w(x,t)$ are two solutions of the equations (3.33–3.34) namely

$$u(x,t)=u^{(o)}(x,t)+\int_0^t e^{(t-t_1)\partial_x^2}f(u(x,t_1))dt_1, \tag{3.69}$$

$$w(x,t)=w^{(o)}(x,t)+\int_0^t e^{(t-t_1)\partial_x^2}f(w(x,t_1))dt_1. \tag{3.70}$$

By using the iteration scheme for (3.69,A38) and after manipulations, we have

$$\begin{aligned}\|u^{(n+1)}-w^{(n+1)}\| &= \|\int_0^t e^{(t-t_1)\partial_x^2}(f(u^{(n)},t_1)-f(w^{(n)},t_1))dt_1\| \\ &\leq H\int_0^t \|(u^{(n)}-w^{(n)}\|dt_1\ .\end{aligned} \tag{3.71}$$

By using the inequality
$\|u^{(n)}-w^{(n)}\| \leq \|u^{(n)}-u^{(n-1)}\|+\|w^{(n)}-w^{(n-1)}\|+\|u^{(n-1)}-w^{(n-1)}\|$ and using the previous theorem, we can show that

$$\|u^{(1)}-w^{(1)}\| \leq R\,Ht. \tag{3.72}$$

By induction, we get

$$\|u^{(n)}-w^{(n)}\| \leq R\frac{(H+H^*)^nt^n}{n!}, \tag{3.73}$$

where H^* is given as H. As $n\to\infty$ we find that $u\to w$.

(II) stability:
We assume that $u(x,0)$ is given by (3.34) and $w(x,0) = u(x,0) + \delta(x)$ are two initial conditions for the equation (3.33), where $\delta(x)$ is in L_1 and $\|\delta(x)\| << 1$ for all $x \in \mathbb{R}$. We assume also that $u(x,t)$ and $w(x,t)$ defined in (3.69,3.70) are two solutions of the equations (3.33,3.34) corresponding the first and second initial conditions respectively.
We use the iterations scheme (3.58) to get

$$u^{(n+1)}(x,t) - u^{(o)}(x,t) = \int_0^t e^{(t-t_1)\partial_x^2} f(u^{(n)}(x,t_1))dt_1. \tag{3.74}$$

$$w^{(n+1)}(x,t) - w^{(o)}(x,t) = \int_0^t e^{(t-t_1)\partial_x^2} f(w^{(n)}(x,t_1))dt_1. \tag{3.75}$$

Where $u^{(o)}$ and $w^{(o)}$ are two solutions of the linear problem (3.36) corresponding the first and second initial conditions respectively. Then from (3.38), we have

$$u^{(o)}(x,t) = e^{\partial_x^2} u_o(x). \tag{3.76}$$

$$w^{(o)}(x,t) = e^{\partial_x^2} w_o(x). \tag{3.77}$$

Substituting (3.76,3.77) into (3.74,3.75), we get

$$\begin{aligned} &\|(u^{(n+1)} - e^{\partial_x^2} u_o) - (w^{(n+1)} - e^{\partial_x^2} w_o)\| = \\ &= \| \int_0^t e^{(t-t_1)\partial_x^2} (f(u^{(n)},t_1) - f(w^{(n)},t_1))dt_1 \| \\ &\leq H \int_0^t \|(u^{(n)} - w^{(n)}\| dt_1. \end{aligned} \tag{3.78}$$

Similarly, we can show that

$$\|u^{(n+1)} - w^{(n+1)}\| = H\ \|\delta(x)\| + R\ \frac{(H+H^*)^n t^n}{n!} \tag{3.79}$$

By taking the limit as $n \to \infty$, we get

$$\|u - w\| < H\ \|\delta(x)\|. \tag{3.80}$$

The last inequality shows that the two solutions $u(x,t)$ and $w(x,t)$ depend continuously on the initial conditions. This proves the stability.

□

4 Solutions of Fisher-equation for initial-boundary or initial value problems

4.1 The initial-boundary value problems

The seminal and now classical paper on Fisher equation is that by Kolmogoroff, Petrovsky and Piscounoff [34]. The books by Fife [12] and Britton [3] mentioned above give a full discussion of this equation and an extensive bibliography. We consider the Fisher equation where $f(u) = \lambda u(1-u)$ and (1.1) becomes

$$u_t = u_{xx} + \lambda u(1-u); \quad (x,t) \in \Omega \tag{4.81}$$

subjected to the initial- boundary value problem

$$u(x,0) = u_0(x), u(-\infty,t) \neq u(\infty,t). \tag{4.82}$$

We remark that if $\lambda < 0$, then the solution of the linear part of (4.81), namely $u_t = u_{xx} + \lambda u$, tends to zero (exponentially) as $t \to \infty$. So that if the Picard sequence, is truncated at the first (or second) term, it will give a good

approximation for high values of T.
But for $\lambda > 0$, we find that the solution of the linear problem, tends to infinity as $t \to \infty$. In this case we apply the algorithm proposed in above as follows
The step (I_1) leads to solve the equation

$$u_t = \lambda u(1-u), \qquad \lambda \neq 0, \tag{4.83}$$

to obtain

$$u(t) = \frac{u_o}{u_o + (1-u_o)e^{-\lambda t}}, \tag{4.84}$$

where $u_o = u(0)$ is a constant. We notice that if $u_o = 0$ (or 1) then $u(t) \equiv 0$ or $u(t) \equiv 1$. Also, as $t \to \infty$, we find that $u \to 0$ for $\lambda < 0$ and $u \to 1$ for $\lambda > 0$. These are the homogeneous steady state solutions of (4.83).
In the step (I_2) we solve the equation

$$\bar{u}_t(x,t) = \lambda \bar{u}(x,t)(1-\bar{u}(x,t)), \bar{u}(x,0) = u(x,0) = u_o(x), \tag{4.85}$$

and have

$$\bar{u}(x,t) = \frac{u_o(x)}{u_o(x) + (1-u_o(x))e^{-\lambda t}}. \tag{4.86}$$

In the step (I_3), we use the transformation

$$u(x,t) = \frac{v(x,t)}{v(x,t) + (1-v(x,t))e^{-\lambda t}}. \tag{4.87}$$

Substituting (4.87) into (4.81), we get

$$v_t = v_{xx} + S(v, v_x, t), \tag{4.88}$$

$$S(v, v_x, t) = \frac{-2(v_x(x,t))^2(1-e^{-\lambda t})}{v(x,t) + (1-v(x,t))e^{-\lambda t}}, \tag{4.89}$$

where $v(x,0) = u(x,0) = u_o(x)$.
The formal exact solution of (4.88) is

$$v(x,t) = e^{t\partial_x^2} u_o(x) + \int_0^t e^{(t-t_1)\partial_x^2} S(v(x,t_1), v_x(x,t_1), t_1) dt_1, \tag{4.90}$$

The Picard iterative sequence of solutions is as follows

$$v^{(n)}(x,t) = v^{(o)}(x,t) + \int_0^t e^{(t-t_1)\partial_x^2} S(v^{(n-1)}(x,t_1), v_x^{(n-1)}(x,t_1), t_1) dt_1, \tag{4.91}$$

and

$$u^{(n)}(x,t) = \frac{v^{(n)}(x,t)}{v^{(n)}(x,t) + (1-v^{(n)}(x,t))e^{-\lambda t}}, \tag{4.92}$$

where from the Appendix $v^{(o)}(x,t) \equiv e^{t\,\partial_x^2} u_o(x) = \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-y)^2}{4t}}}{\sqrt{4\pi t}} u(y,0) dy$.
The first approximation of the sequence of solutions in (4.91) is

$$v^{(1)}(x,t) = v^{(o)}(x,t) + \int_0^t e^{(t-t_1)\partial_x^2} S(v^{(o)}(x,t_1), v_x^{(o)}(x,t_1), t_1) dt_1, \tag{4.93}$$

where $S(v^{(o)}, v_x^{(o)}, t_1)$ is given as by (4.89) with $v^{(o)}$ replacing v. Using the results found in the Appendix, (4.93) is written as

$$v^{(1)} = v^{(o)} + \int_0^t \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-y)^2}{4(t-t_1)}}}{\sqrt{4\pi(t-t_1)}} S(v^{(o)}(y,t_1), v_x^{(o)}(y,t_1)t_1) dy dt_1 \tag{4.94}$$

In (4.89), we find that the function $S(v, v_x, t_1)$ tend to zero as t for $t \to 0$ and as t^{-3} for $t \to \infty$ for all $\lambda \neq 0$. So that, we expect that for $t = O(1)$ or $t \gg 1$, the correction arises from taking into account the fact that the second term in (4.94) is negligible. This will be verified in the following example.

An example with known exact solution

We start by an example where an exact solution of (1.1) can be found. To this end, we consider the HTWS of (1.1) as $u = u(z)$, $z = x - c\,t$, where it becomes

$$u'' + cu' + \lambda u(1-u) = 0, u' = \frac{du}{dz} \tag{4.95}$$

It is known in the literature that [25, 41] (4.95) admits a solitary wave solution in the form

$$u = A + B\ \tanh(kz+d) + C\ \tanh^2(kz+d). \tag{4.96}$$

By using Mathematica or by a direct calculation, we get a class of solutions for different values of c given in terms of the parameter λ. Here, we consider the case where $c = 5\sqrt{\frac{\lambda}{6}}$, $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $k = \frac{\sqrt{\lambda}}{2\sqrt{6}}$ and after simplifying (4.96), it becomes

$$u(x,t) \equiv u(z) = \frac{1}{(1 + d\ e^{\sqrt{\frac{\lambda}{6}}z})^2},\ d = const. \tag{4.97}$$

For instance let us take $d = 1$. After (4.97), we mention that $u(-\infty, t) = 1$ and $u(\infty, t) = 0$. The initial condition is taken after (4.97) as

$$u_o(x) = \frac{1}{(1 + e^{\sqrt{\frac{\lambda}{6}}x})^2}. \tag{4.98}$$

Now from (4.92), we find that

$$u^{(o)}(x,t) = \frac{v^{(o)}(x,t)}{v^{(o)}(x,t) + (1 - v^{(o)}(x,t)e^{-\lambda t})}, \tag{4.99}$$

$$v^{(o)}(x,t) = e^{t\partial_x^2} u_o(x) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} u_o(y) dy \tag{4.100}$$

When substituting from (4.98) into (4.100) and (4.99) we get an explicit approximate analytical solution for the Fisher equation. The utility of this solution is that it satisfies the initial condition, preserves the homogeneous steady state solution of (1.1) and contains the effects of the non linear source term and the diffusion process. The first approximation $u^{(1)}(x,t)$ is given by

$$u^{(1)}(x,t) = \frac{v^{(1)}(x,t)}{v^{(1)}(x,t) + (1 - v^{(1)}(x,t))e^{-\lambda\ t}}, \tag{4.101}$$

where $v^{(1)}(x,t)$ is given by (4.94). The results in (4.99– 4.101) for the zero and first approximation compared with the exact solution given by (4.97) are displayed in Figures 1 (a)–(c) for $t = 3$, 7 and 100.

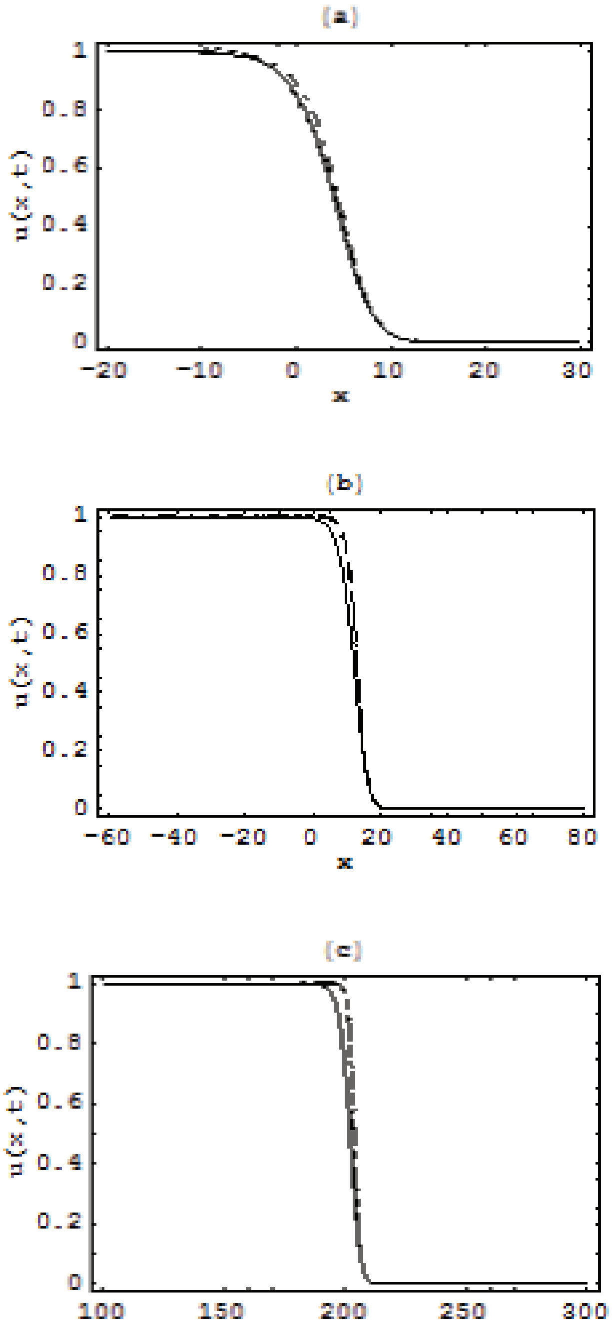


Figure 1: Approximate analytical solutions $u^{(1)}$, $u^{(o)}$ and exact solution of (4.81) are displayed against x for $\lambda = 1$ in (a) $t = 3$, (b) $t = 7$ and (c) $t = 100$ respectively. (- -) $u^{(1)}$, (-.-) $u^{(o)}$, (-) exact solution.

After these figures, we observe that the first approximation and the zero order approximation are identical for $t = 3$ and $t = 100$ while they are different in the case $t = 7$. It is nearer to the exact solution . The relative maximum error estimate is of order 5×10^{-3} even for large values of T (or t).
We should mention that the numerical schemes can be applied to this example if the boundary conditions are given as $u(-\infty, t) = 1$ and $u(\infty, t) = 0$. To carry out numerical calculations, we take $u(-\ell, t) = 1$, $u(m, t) = 0$, where ℓ and m are

taken sufficiently large. We mention that by increasing t , we change the values of ℓ and m adequately. The calculations were repeated with boundary conditions specified for different values of $\triangle x$ and no remarkable differences were found between the results. Comparison between exact and numerical solutions by varying $\triangle x$ for $t = 100$ shows an error of order 10^{-3}. We mention that numerical solutions found here are calculated by using Mathematica. We consider the equation (4.81) with the initial condition

$$u(x,0) = \mu x^2 e^{-(x+2)^2}, \quad -\infty < x < \infty, \tag{4.102}$$

where μ is a positive constant. Here, we will take $\mu = 0.2$. We remark that the initial condition (4.102) is asymmetric. Here, we use the algorithm proposed previously and evaluate the first approximation of (4.81) and (4.102). First, we have

$$v^{(o)}(x,t) = \frac{0.2e^{-\frac{(x+2)^2}{1+4t}}}{(1+4t)^{\frac{5}{2}}}(72t^2 + t(2-16x) + x^2). \tag{4.103}$$

The first approximation is given by $v^{(1)}(x,t)$ (cf. (4.94)) and $u^{(1)}(x,t)$ is given by as (4.101). In Figure 2 (a), (b) the results for the first approximation $u^{(1)}$ are displayed against x for $\lambda = 1$.

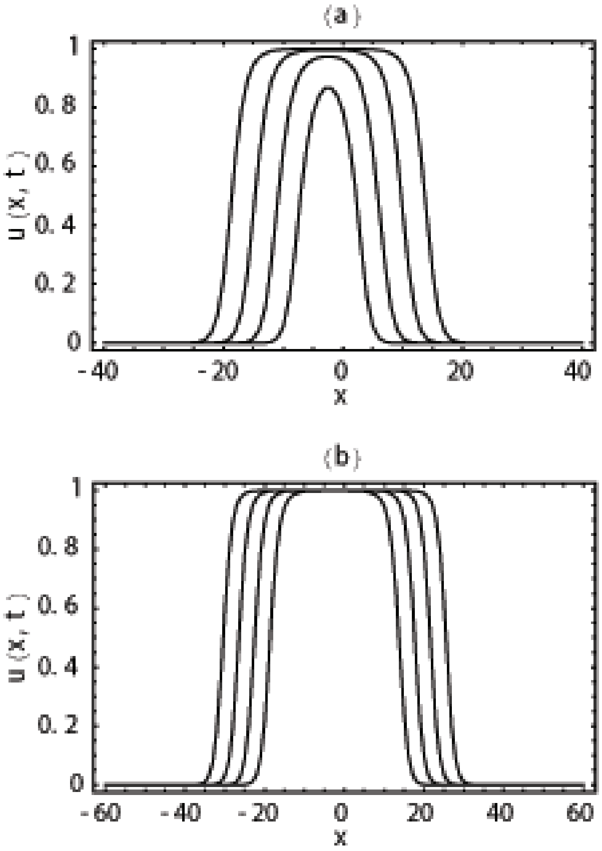


Figure 2: An approximate analytical solution $u(x,t)$ of (4.81) and (4.102) is displayed against x for $\lambda = 1$ in (a) $t = 1$, 3, 5 and 7 arranged from inner to outer respectively. (b) $t = 9$, 11, 13 and 15 arranged from inner to outer respectively.

In Figure 2 (a), the curves are arranged from inner to outer with increasing values of t ; where $t = 1$, 3, 5 and 7. In Figure 2 (b), the values of t are 9, 11, 13, and 15. We shall comment on these figures in section 6.
Here, we make comparison between the results found previously by the approximate analytic solutions and those

found by the numerical scheme built in Mathematica. The approximate analytical and numerical solutions for $u(x,t)$ is shown in Figure 3.

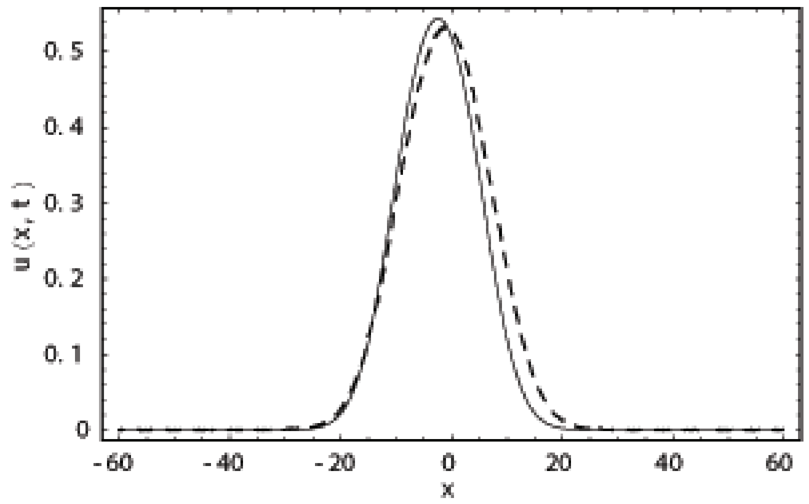


Figure 3: Approximate analytical and numerical solutions of (4.81) and (4.102) are displayed against x for $\lambda = 0.1$ and $t = 25$.

In Figure 3 $t = 25$, $\lambda = 0.1$. From this Figure we observe the error between the two solutions is only relevant in a small domain and the maximum relative error there is of order 10^{-4}. Now, the important question is with what kind of initial conditions $u(x,0)$ the solution of the Fisher equation evolves to a travelling wave
Kolmogoroff [34] proved that if $u(x,0)$ has the form

$$u(x,0) = \begin{cases} 1, & x \leq x_1 \\ g(x), & x_1 < x < x_2 \\ 0, & x \geq x_2 \end{cases} \tag{4.104}$$

$$u(-\infty,t) = 1, \;\; u(\infty,t) = 0, \tag{4.105}$$

where $g(x)$ is continuous in $x_1 < x < x_2$, then the solution $u(x,t)$ of (4.81) evolves to a travelling wave front solution. For initial data other than (4.104) the solution depends critically on the behavior of $u(x,0)$ as $|x| \to \infty$. Now, we consider the equation (4.81) for the initial (4.104) but $g(x) = 0$ so that (4.104) becomes

$$u(x,0) = \begin{cases} 1, & x \leq x_1 \\ 0, & x \geq x_1 \end{cases} \tag{4.106}$$

The first approximation for this problem is given by (4.94, 4.101) where $v^{(o)}(x,t)$ is given by

$$v^{(o)}(x,t) = \frac{1}{2} erfc\left(\frac{x}{2\sqrt{t}}\right). \tag{4.107}$$

In Figure 4 the results for the first approximation $u^{(1)}(x,t)$ are displayed against x for $\lambda = 1$ and $t = 1, 5, 10, 15, 20$. The curves are arranged from inner to outer ones with increasing the value of t.

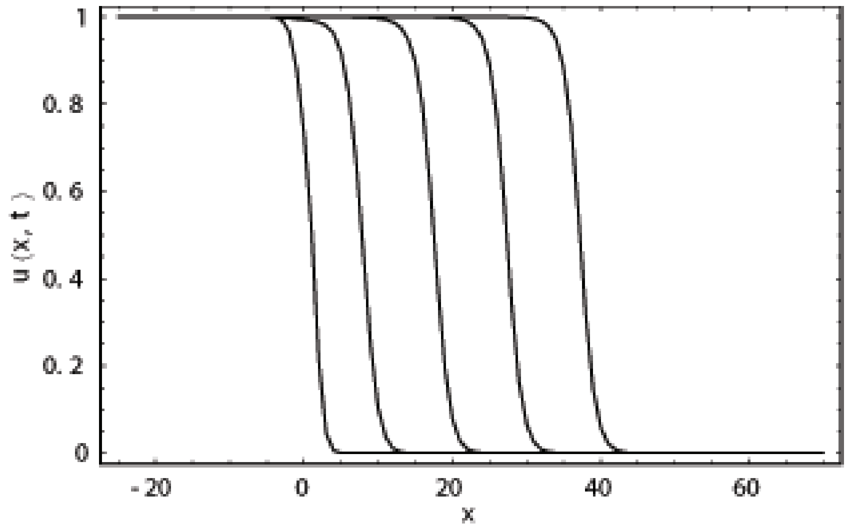


Figure 4: An approximate analytical solution $u(x,t)$ of (4.81) and (4.106) is displayed against x for $\lambda = 1$ and $t = 1$, 5, 10, 15, 20 arranged from left to right respectively.

From this figure, we can see that a travelling wave is generated in the region $x > 0$ where wave fronts travel from the left to the right. This wave formation is a transient state to the homogeneous steady state $u = 1$. This travelling wave solution is a RPTWS because the wave front travels equal distances in equal time periods. This agrees with the work of Kolmogoroff et al. [34]. Now, if we take the initial-boundary conditions as

$$u(x,0) = \begin{cases} 0, & x \leq x_1 \\ 1, & x \geq x_1 \end{cases} \tag{4.108}$$

$$u(\infty,t) = 1, u(-\infty,t) = 0. \tag{4.109}$$

The results are displayed in Figure 5 for the same caption as in Figure 4.

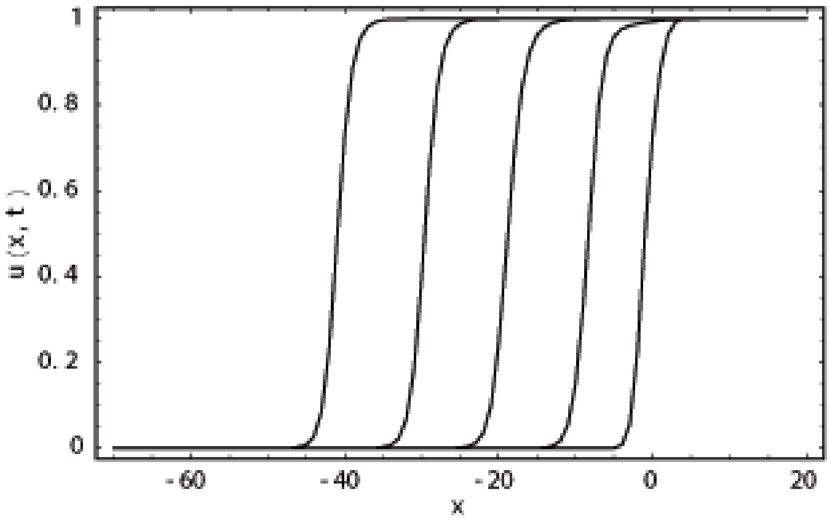


Figure 5: An approximate analytical solution $u(x,t)$ of (4.81) and (4.108) is displayed against x for $\lambda = 1$ and $t = 1$, 5, 10, 15, 20 arranged from right to left respectively.

We find that a travelling wave is generated in the region $x < 0$ and travels from the right to the left.
We evaluate the speed of propagation of these two waves by evaluating the distance $\triangle x$ travelled in a time period $\triangle t$. In both two cases, we find that $c = \frac{\triangle x}{\triangle t} = 2$. This agrees with the expected value from the work of [4, 35], namely, $c = 2\sqrt{\lambda}$ where $\lambda = 1$.
Now, we investigate the formation of travelling waves in the solution of the Fisher equation when $\lambda < 0$. If we take the initial conditions (4.105) and (4.106) and use the equations (4.94, 4.101), the first approximation $u^{(1)}$ is shown in Figure 6.

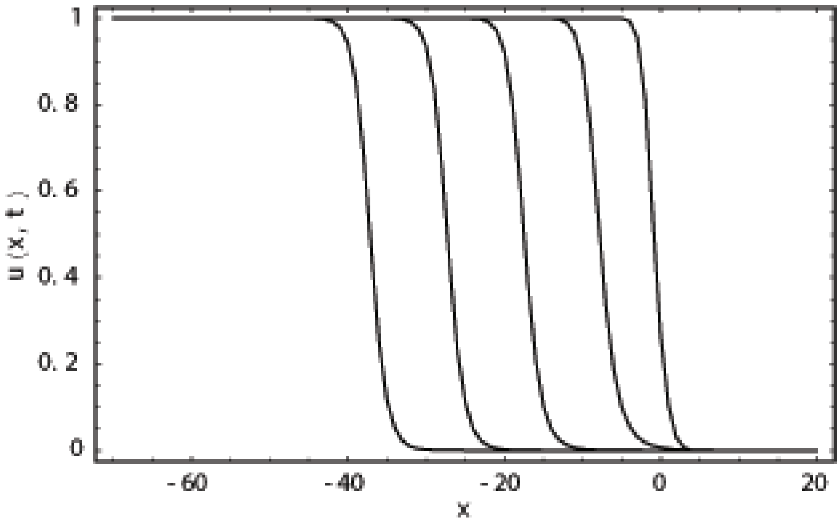


Figure 6: An approximate analytical solution $u(x,t)$ of (4.81) and (4.106) is displayed against x for $\lambda = -1$ and $t = 1$, 5, 10, 15, 20 arranged from right to left respectively.

From this figure, we find that a travelling wave is generated in region $x < 0$ and it travels from the right to the left.

But if we take the conditions (4.108), we find that the wave is generated $x > 0$ and travels from the left to the right. In this case the travelling wave formation is a transient state to the homogeneous steady state $u = 0$. The speed of propagation is evaluated for after Figure 6 as $c = 2$. This last result is in contrast to what predicated by Rinzel [42] after the analysis of the linear part.
Finally, we determine the values of λ in the equation (4.81) where a travelling wave solution exists. We remark that for $\lambda = 0$, no travelling wave solution exists. By the previous results, we find it exists for $\lambda \neq 0$

5 Systems of finite size

Now, we apply the algorithm proposed in section 1.3, to a system of finite size. To this end, we consider the following examples. We consider the following initial-boundary value problem for $(x,t) \in (0,\ell) \times (0,T)$

Example 5.1.

$$u_t = u_{xx} + \lambda u(1-u), \tag{5.110}$$

$$u(x,0) = 1 - sin(\frac{\pi}{\ell}x), \tag{5.111}$$

$$u(\ell,t) = u(0,t) = 1. \tag{5.112}$$

Here, we should mention that numerical methods apply to this case.
By a similar calculation as in section 1.3, we obtain the zero-order solution as

$$v^{(o)}(x,t) = 1 - e^{-(\frac{\pi}{\ell})^2 t} sin(\frac{\pi}{\ell}x) \tag{5.113}$$

The first approximation is given by

$$v^{(1)}(x,t) = v^{(o)}(x,t) - \int_0^t \sum_{n=1}^{\infty} a_n\, sin(\frac{n\pi}{\ell}x) e^{-(t-t_1)(\frac{n\pi}{\ell})^2} dt_1, \tag{5.114}$$

where

$$a_n = \frac{2}{\ell}\int_0^{\ell} S(x,t_1) sin(\frac{n\pi}{\ell}x) dx, \tag{5.115}$$

$$S(x,t_1) = \frac{\frac{2\pi^2}{\ell^2}(1-e^{-t_1\lambda}) e^{-\frac{\pi^2}{\ell^2}t_1} cos^2(\frac{\pi}{\ell}x)}{e^{-t_1\lambda} + (1-e^{-t_1\lambda})(1-e^{-\frac{\pi^2}{\ell^2}t_1} sin(\frac{\pi}{\ell}x))}, \tag{5.116}$$

Now the first approximation $u^{(1)}(x,t)$ of (5.110– 5.112) is evaluated from (4.101) and (5.114–5.116). In figures 7 (a) and (b), the first approximation $u^{(1)}(x,t)$ and the numerical solution are displayed against x.

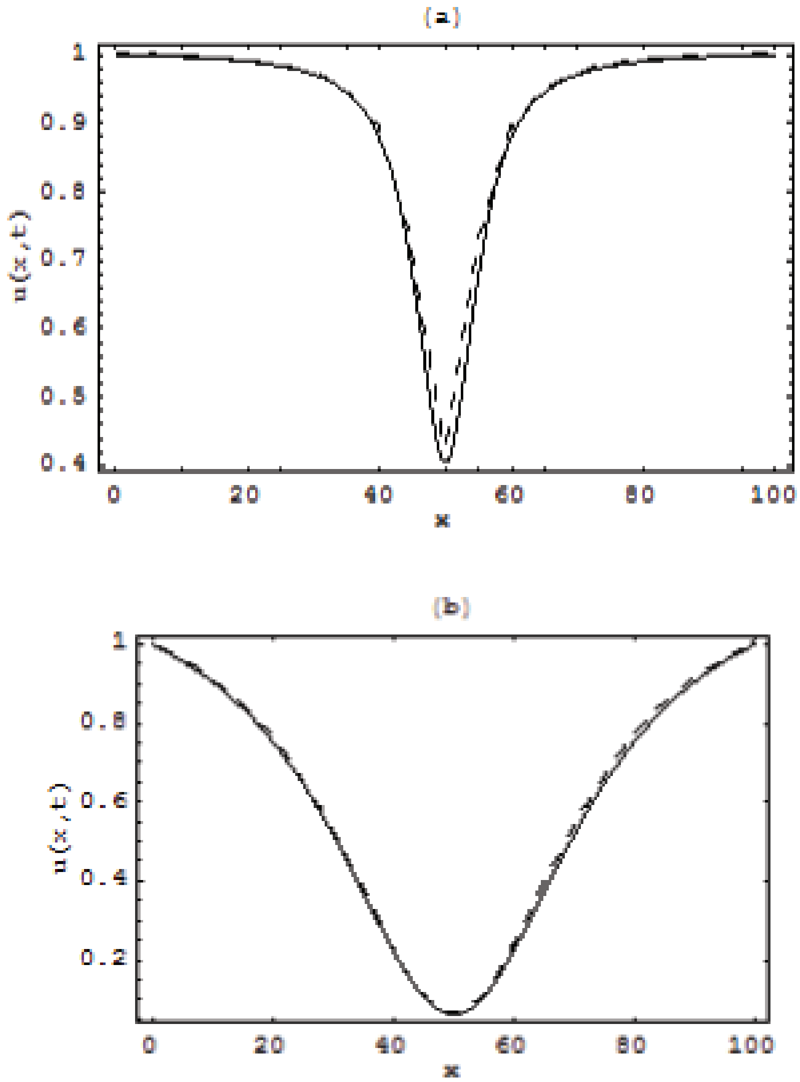


Figure 7: Approximate analytical solutions $u(x,t)$ and numerical solution of (5.110) and (5.112) are displayed against x in (a) $\lambda = 1$ and $t = 5$. (b) $\lambda = 0.1$ and $t = 15$. $(--)$ $u(x,t)$, $(-)$ numerical solution.

In Figure 7 (a) $\lambda = 1, t = 5$ and in Figure 7 (b) $\lambda = 0.1, t = 15$. From these figures, we find that the relative maximum error between the first approximation $u^{(1)}(x,t)$ and numerical solution of order 2×10^{-3}. We have investigated the behavior of the solution of (5.110-5.112) for large values of time and found that, in the cases of equal boundary values at the ends of the domain, no formation of travelling wave occurs.

Example 5.2. *We consider the Fisher equation with initial and boundary conditions as*

$$u(x,0) = cos(\frac{\pi}{2\ell}x), \tag{5.117}$$

$$u(0,t) = 1 \text{ and } u(\ell,t) = 0. \tag{5.118}$$

Here, we remark that $u(0,t) \neq u(\ell,t)$.

Now, we expand the initial condition as

$$cos(\frac{\pi}{2\ell}x) - \frac{\ell - x}{\ell} = \sum_{n=1}^{\infty} \frac{2}{\pi(4n^3 - n)} sin(\frac{n\pi}{\ell}x), \tag{5.119}$$

Similar argument as in example 1, the zero solution $v^{(o)}$ is given by

$$v^{(o)}(x,t) = \frac{\ell - x}{\ell} + \sum_{n=1}^{\infty} \frac{2}{\pi(4n^3 - n)} sin(\frac{n\pi}{\ell}x) e^{-(\frac{n\pi}{\ell})^2 t} \tag{5.120}$$

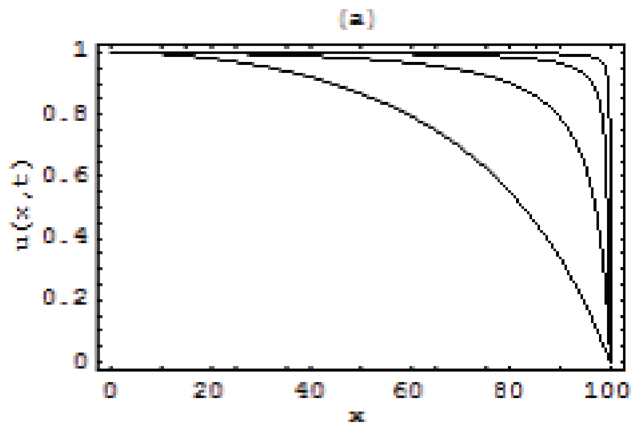
and the first approximation $v^{(o)}(x,t)$ is

$$v^{(1)}(x,t) = v^{(o)}(x,t) - \int_0^t \sum_{m=1}^{\infty} a_m \, sin(\frac{m\pi}{\ell}x) e^{-(t-t_1)\frac{m^2\pi^2}{\ell^2}} dt_1, \tag{5.121}$$

where

$$a_m = \frac{2}{\ell} \int_0^{\ell} S(v^{(o)}(x,t_1),t_1) sin(\frac{m\pi}{\ell}x) dx. \tag{5.122}$$

The results (5.120–5.122) for $u^{(1)}(x,t)$ are displayed in figures 8 (a) and 8 (b).



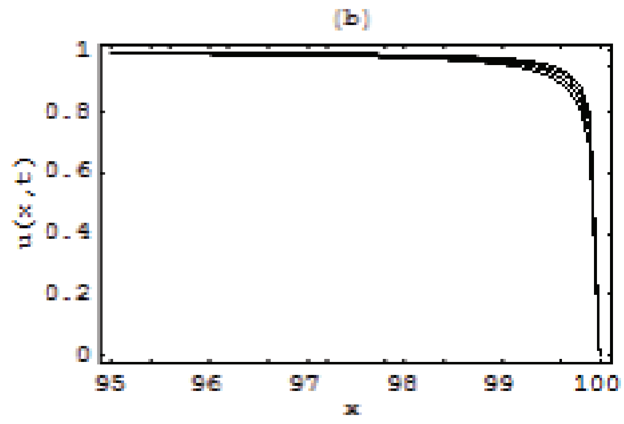


Figure 8: An approximate analytical solution of (5.110)–(5.112) is displayed against x in (a) $\lambda = 1$, $t = 1$, 3, 5 and 7 from left to right respectively. (b) $\lambda = 1$, $t = 7.2$, 7.4, 7.6 and 7.8 from left to right respectively.

In Figure 8 (a) the solution is displayed against x for $\lambda = 1$ and $t = 1, 3, 5, 7$. In Figure 8 (b) the solution is displayed against x for $\lambda = 1$ and $t = 7.2, 7.4, 7.6, 7.8$. We find that, as $u(0,t) \neq u(\ell,t)$, the solution of the Fisher equation evolves towards the hsss $u_s = 1$.

6 Solutions of The Nagumo-equation for initial and boundary value problems

Second we consider the Nagumo-equation, where $f(u) = u(1-u)(u-a)$. Now equation (1.1) becomes

$$u_t = u_{xx} + u(1-u)(u-a), \quad (x,t) \in \Omega, \tag{6.123}$$

where $-1 < a < 1$. Here, we shall consider the initial-boundary value problem with $u(-\infty,t) = A$ and $u(\infty,t) = B$ and $u(x,0) = u_0(x)$. We remark that for $t \to \infty$, the solution of the linear part, namely $u_t = u_{xx} - au$, tends to zero exponentially for $0 < a < 1$ and tends to infinity for $-1 < a < 0$. Now,we carry out the steps of the algorithm for the equation (6.123). In the first step, we solve the equation (6.123) in the absence of the diffusion term namely

$$u_t = -u(u-1)(u-a), \tag{6.124}$$

We notice that the number of fixed points is 3 namely, $u = 0$, $u = a$ and 1. Also, the equation (6.124) solves implicitly to

$$\left(\frac{u}{u_o}\right)^{\frac{1}{a}} \left(\frac{1-u}{1-u_o}\right)^{\frac{1}{1-a}} \left(\frac{u-a}{u_o-a}\right)^{\frac{-1}{(1-a)a}} = e^{-t}; \quad a \neq 0 \tag{6.125}$$

where $u_o = u(0)$, $u \equiv u(t)$.
We notice that if $u(0) = 0$ (a or 1) then $u(t) \equiv 0$ (a or 1). The steady state sss of (6.124) are either $u_s = 0$ or $u_s = 1$. They depend on the initial conditions and the value of the parameter a. If $0 < a < 1$, $u_o < 0$ or $0 < u_o < a$ then $u_s = 0$. But if $a < u_o < 1$ or $u_o > 1$ then $u_s = 1$. If $a < 0$, $u_o < a$ or $a < u_o < 0$ then $u_s = a$. But if $0 < u_o < 1$ or $u_o > 1$ then $u_s = 1$ The importance of these results are that they unable us to determine the sss of the equation (6.123) for a given non homogeneous initial conditions $u_o(x)$ in a similar way. For example, if $a < 0$ and $a \leq u_o(x) \leq 1$, then the dominant hsss are a and 1. By the hsss a and 1, we mean that the solution of (6.123) satisfies the inequality $a \leq u(x,t) \leq 1$ for large values of t. In what follows, we assume that the hsss are a and 1.
If we use a transformation after (6.125) and replace u_0 by $v(x,t)$, the derivation of the results will be cumbersome. Also,the technique presented in section 1.3 can be simpler as follows. By bearing in mind that the hsss are taken as a and 1, the equation (6.123) is rewritten as

$$\frac{u_t}{(u-a)(u-1)} = \frac{u_{xx}}{(u-a)(u-1)} - u. \tag{6.126}$$

Integrating (6.126) gives

$$u = \frac{u(x,0) - a + a(1-u(x,0))e^{\int_0^t P(u(x,t_1))dt_1}}{u(x,0) - a + (1-u(x,0))e^{\int_0^t P(u(x,t_1))dt_1}} \tag{6.127}$$

where $P(u) = (1-a)(\frac{u_{xx}}{(u-a)(u-1)} - u)$. If $u_o \in C^\infty(\mathbb{R})$ and $u_o(x) \equiv u(x,0) \neq a$ or 1 for all $x \in \mathbb{R}$ then, we construct the iterative sequence of solutions, after equation (6.127) as follows

$$u^{(o)} = \frac{u_o - a + a(1-u_o)e^{tP(u_o)}}{u_o - a + (1-u_o)e^{tP(u_o)}}, \tag{6.128}$$

$$u^{(n)} = \frac{u_o - a + a(1-u_o)e^{\int_0^t P(u^{(n-1)}(x,t_1))dt_1}}{u_o - a + (1-u_o)e^{\int_0^t P(u^{(n-1)}(x,t_1))dt_1}}, \quad n \geq 1. \tag{6.129}$$

For $n = 0$, we take

$$u_t^{(o)} = u_{xx}^{(o)}, \tag{6.130}$$

We remark that the iterative sequence of solutions given by (6.128) and (6.129) satisfies the following conditions

(i) The denominator does not vanish for $(x,t) \in \Omega$

(ii) When $t = 0$, $u^{(n)}(x,0) = u(x,0)$

(iii) As $t \to \infty$, we find that the exponential function in the right-hand-side of (6.128) and (6.129) tends to either ∞ or 0. If it tends to infinity or zero, we find that $u^{(n)} \to a$ or 1. That is $u^{(n)}$ tends to the hsss for all n.

Thus $u^{(n)}$ preserves the hsss and the initial conditions. For $0 < t < \infty$, the convergence of the sequence of solutions $u^{(n)}$ given by (6.129) is governed by the following theorem

Theorem 6.1. *If u_o is piecewisely continues function which is bounded on $-\infty < x < \infty$, namely $a < u(x,0) \leq 1$, then the sequence of solutions $u^{(n)}$ given by (6.128) and (6.129) converges uniformly to the exact solution u.*

Proof. The initial conditions stated in the theorem are inspired after the work in [33]. In this work, a uniform convergence theorem had been proved for the Picard iteration scheme for Fisher equation. To prove the uniform convergence here, first we show that $a \leq u^{(n)}(x,t) \leq 1$ for all n. From the initial conditions, we find that the denominator of (6.126) or (6.129) is strictly positive. Also, from (6.130), we have $a \leq u^{(o)}(x,t) \leq 1$. While $u^{(1)}(x,t)$ is given by

$$u^{(1)}(x,t) = 1 - \frac{(1-a)(1-u_o)e^{\int_0^t P(u^{(o)}(x,t_1)dt_1}}{(u-a)+(1-u_o)e^{\int_0^t P(u^{(o)}(x,t_1)dt_1}} \tag{6.131}$$

The second term in (6.131) is non-negative and then $u^{(1)}(x,t) \leq 1$. Similarly, we can show that $u^{(1)}(x,t) \geq a$, or $a \leq u^{(1)}(x,t) \leq 1$. By induction the inequality $a \leq u^{(n)}(x,t) \leq 1$ holds. From the Wierestrass theorem $u^{(n)}(x,t)$ converges uniformly to $u(x,t)$. □

In what follows, we shall evaluate the first-order approximation. The results are compared with some known exact solutions.

Example with known exact solution

In what follows, we shall consider an example when exact solution of (6.123) is known as a solitary wave in the form

$$u(x,t) = A + B\ \tanh\ (k(x - c\ t) + d). \tag{6.132}$$

Calculations by using Mathematica or otherwise show that there is a class of eight solutions of the form (6.132). Here consider the case where $c = \frac{1+a}{\sqrt{2}}$, $A = \frac{1+a}{2}$, $B = -\frac{1-a}{2}$, $k = \frac{1+a}{2\sqrt{2}}$ and d is arbitrary, which will be taken to be zero in the following.

We mention that the solution (6.132) satisfies $u(-\infty,t) = 1$ and $u(\infty,t) = a$. The initial condition is then taken by setting $t = 0$ in (6.132), as

$$u(x,0) = \frac{1}{2}(1 + a + (a-1)\tanh(\frac{(1+a)}{2\sqrt{2}}x). \tag{6.133}$$

By (6.133), $u(x,0)$ is in $C^\infty(\mathbb{R})$ and

$$\frac{u_{xx}(x,0)}{(u(x,0)-a)(u(x,0)-1)} = -\frac{1}{2}(a-1)\tanh(\frac{(a-1)x}{2\sqrt{2}})$$

is a smooth function, then, we can use the iteration scheme given by (6.128) and (6.129). The zero-order approximation given by (6.128) is displayed in Figure 9 together with the exact solution given by (6.132) and the numerical.

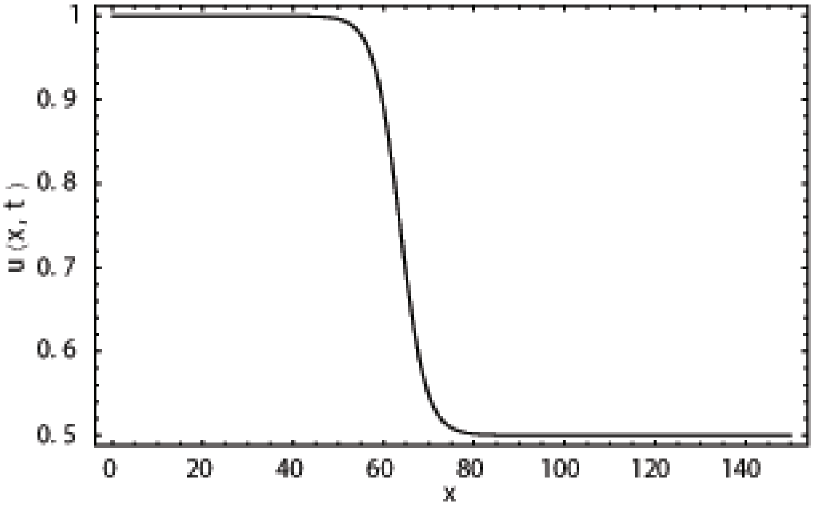


Figure 9: Approximate analytical, numerical and exact solutions of (6.123) and (6.133) are displayed against x for $a = 0.5$ and $t = 60$, the three solutions are identical.

From Figure 9, we can see that the three solutions are practically identical. The relative error estimate is of order 10^{-5}.
To investigate the existence of RPTWS for the Nagumo equation, we consider the initial conditions,

$$u(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases} \tag{6.134}$$

Here, we remark that $u(x,0)$ is not in $C^{\infty}(\mathbb{R})$. Thus, we can not use the iteration scheme given by (6.128). Also we notice that the dominant hsss in this case are 0 and 1. So we have the first approximation as

$$u^{(1)}(x,t) = \frac{u^{(o)}(x,t)}{u^{(o)}(x,t) + \left(1 - u^{(o)}(x,t)\right) e^{\int_0^t P(u^{(o)}(x,t_1))\,dt_1}}, \tag{6.135}$$

where $u^{(o)}$ satisfies (6.130). The results (6.135) and (6.130) are shown in Figure 10.

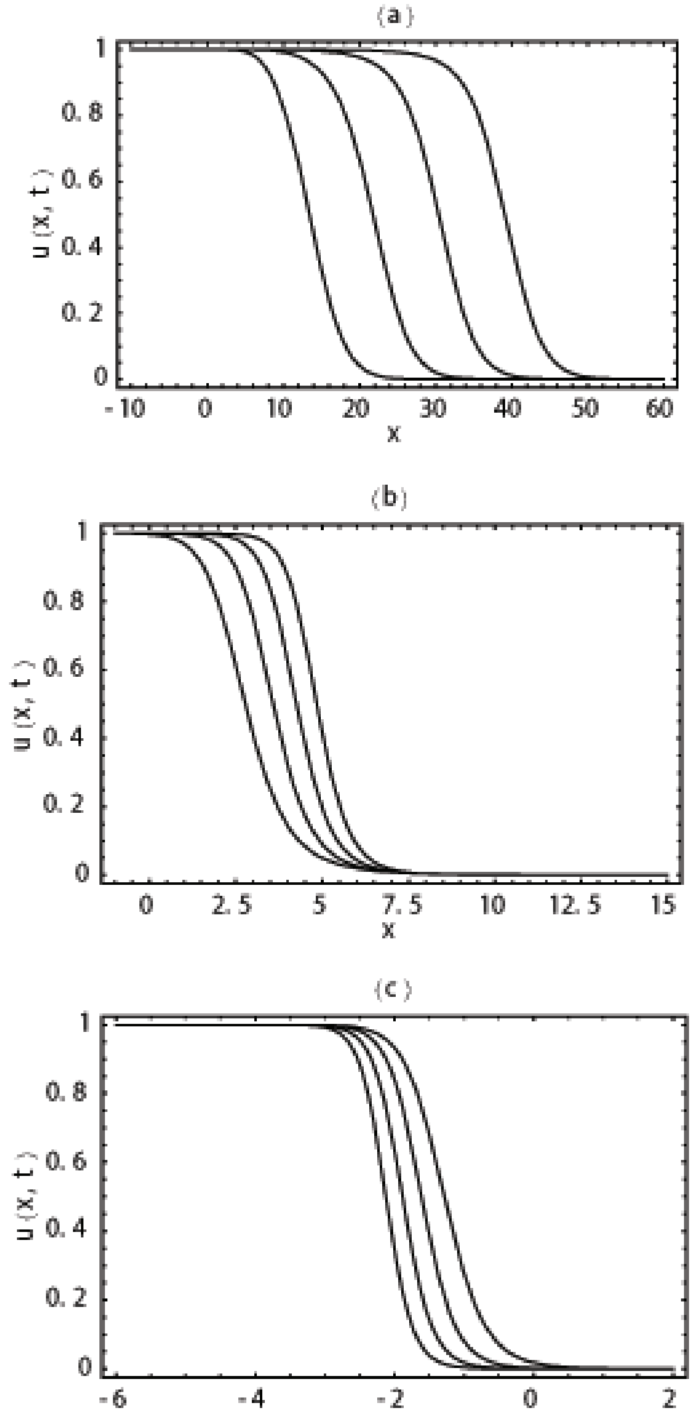


Figure 10: The approximate analytical solution of (6.123) and (6.133) is displayed against x for $t = 20$, 30, 40 and 50 in (a) $a = -0.2$. (b) $a = 0.2$ and (c) $a = 0.7$.

From this figure, we can see that formation of travelling waves occurs for large values of t towards a RPTWS for $a \leq 0$ or $a \geq 0$. We have observed that this RPTWS is a transient solution the hsss $u = 1$ if $a < 0$ or $0 < a < 0.5$. But if

$0.5 < a < 1$ the hsss will be zero.
It is worth noting that one of the boundary conditions on the chemical concentration is taken as $u \to A$ as $\mid x \mid \to \infty$. Thus no travelling wave solution in the sense of equation (2.4) exists. In some problems an additional boundary condition is taken as $u_x(x,t) = 0$ at $x = 0$, that is the condition of zero flux at $x = 0$. By requiring that the initial values $u(x,0)$ satisfying this condition, namely $u_x(x,0) = 0$ at $x = 0$ and because of the reflection symmetry of the diffusion operator (in (1.1) and (2.2)), we shall have the solution $u(x,t)$ symmetric with respect to $x = 0$. Then, we shall have $u_x(0,t) = 0$ for $t > 0$. Thus, we may conjecture that the condition of zero flux at $x = 0$ has no role on the initiation of travelling waves.

7 Conclusions

In this paper we have examined the necessary and sufficient condition for the existence of a RPTWS. We have shown that the sufficient condition for a RPTWS to exist is that the equation governing the reaction diffusion system contains an advection term. If the governing equation does not contain the advection term, a necessary condition that the solution evolves towards a RPTWS is that $u(-\infty,t) \neq u(\infty,t)$. But when $u(-\infty,t) = u(\infty,t)$ two wave fronts exist which travel in opposite directions at the same speed. We mention that such solution corresponds to the classical symmetry of the reaction diffusion (1.1), which is given by the equation (2.4). Here, we point out that solutions of (1.1) which satisfy $u(\infty,t) = u(-\infty,t)$ (or $u(-\infty,t) = u(\infty,t)$) and the condition of permanent waves may be incorporated into the class of RPTWS. But they can not be found after the equation for travelling wave solution (cf.(2.4)). In fact these solutions can be considered as a superposition of two travelling waves, one travels to the right of $x = x_0$ (for some x_0) and the other one travels to its left. This suggests to write the solution of (1.1) as

$$u(x,t) = u(\varphi_1(x-ct) + \varphi_2(x+ct)), \tag{7.1}$$

where $\varphi_2(x+ct) = \varphi_1(-(x-ct))$ as $x - ct \to \infty$
In this case, the equation (1.1) reduces to

$$\frac{d^2u}{d\xi^2}(\frac{d\xi}{dz})^2 + \frac{du}{d\xi}\ \frac{d^2\xi}{dz^2} + c\frac{du}{d\xi}\left(\frac{-d\varphi(z)}{dz} + \frac{d\varphi(-z)}{dz}\right) + f(u(\xi)) = 0$$
$$,\ \ \xi = \varphi_1(z) + \varphi_2(-z),\ \ z = x - c\,t \tag{7.2}$$

We mention that the equation (7.2) preserves the reflection symmetry. This is invariant under the transformation $z \to -z$. This in contrast to the equation (2.4). Solution of (7.2) would agree with nonclassical symmetry reduction of the equation (1.1) [36]. In this paper , nonclassical symmetry solutions are found to be a two waves travelling in two opposite directions (see equations (3.10)-(3.13) in [36]).

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