ELC1010

Electric Circuits (I) – Circuit Theory

Part (II) – AC Circuits

Lecture (6)

Sinusoidal Time Domain Analysis

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Outline

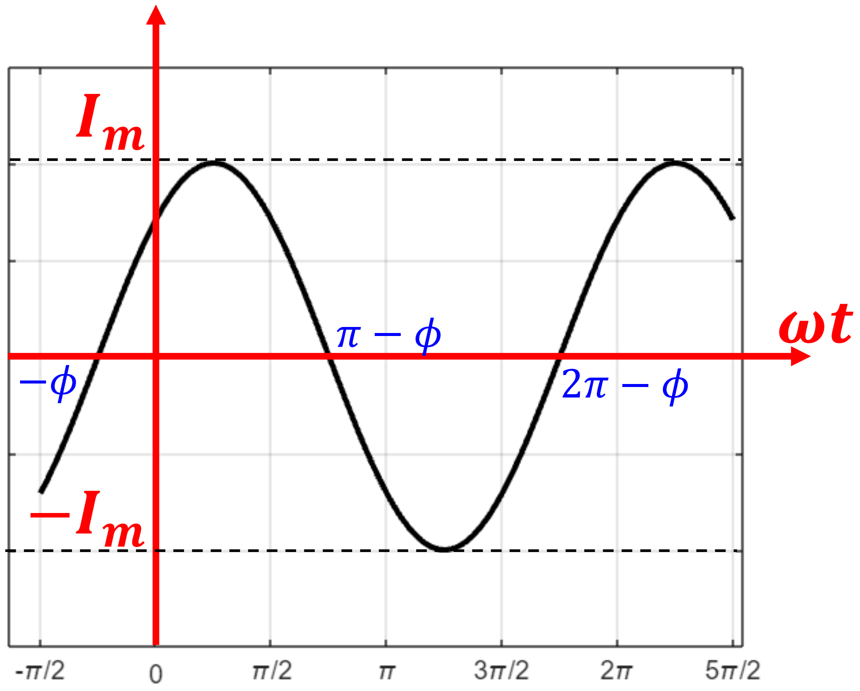
- Sinusoidal Time Domain Analysis
- Review:
  - Sinusoidal, Average and RMS
  - Phase Shift and Phase Relationship (Lead, Lag)
- Element Response (R, L, C)
- Series RLC Network
- Parallel RLC Network
- Example

Introduction

- In most power and communication systems, alternating current (AC) is the standard form of electric power used.
- **Power generation and transmission:**
  - Transmitting AC power at high voltages and lower currents is ideal as it minimizes transmission line losses, which are proportional to the square of the current (Power loss: $P_{loss} = I^2 R_{conductor}$)
  - High voltage transmission is a more cost-effective solution for power distribution.
  - At the load end, AC/DC rectifier circuits can convert AC to the required DC form.
- **Communication systems and information theory:**
  - Communication systems utilize various data modulation techniques, including amplitude, frequency, and phase modulation.
  - Advanced communication methods employ higher-order modulations for data transmission.

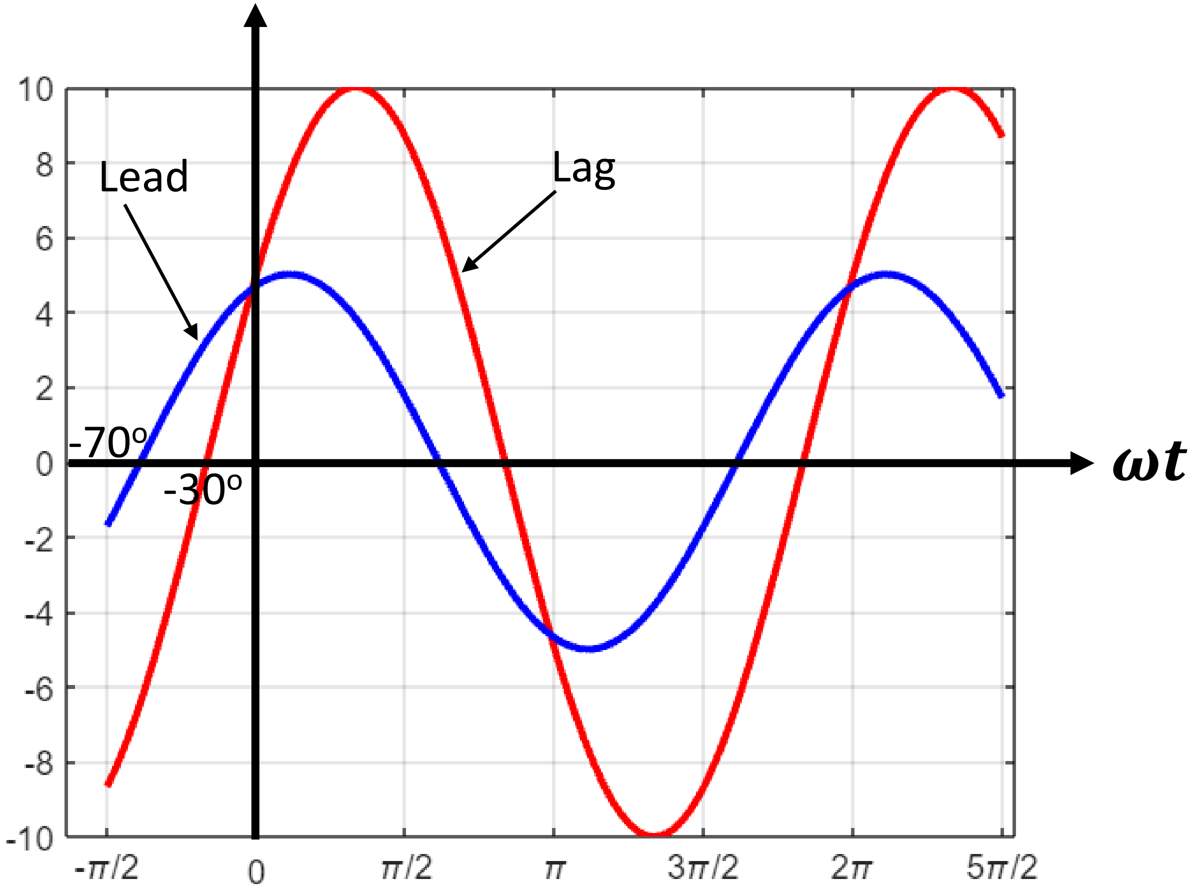
Time Domain (TD) Analysis

- Passive elements: R, L and C.
- Assume sinusoidal signals (periodic)
  - Integration and differentiation of sinusoidal signals are sinusoidal signals.
- $v(t) = V_m \sin(\omega t + \phi_v)\,, \quad i(t) = I_m \sin(\omega t + \phi_i)$
  - $v(t)$ *and* $i(t)$ are the instantaneous value of voltage/current
  - V_m *and* I_m are the max/peak values (amplitudes) of voltage/current
  - ω is the angular frequency (rad/sec) and is equal to $\omega = 2\pi f = \frac{2\pi}{T}$
  - f is the frequency in Hz
  - $T = \frac{1}{f}$ is the period in secs
  - ϕ is the initial phase angle
- Sinusoidal signals are defined by 3 parameters:
  - $V_m/I_m,\ \omega$ *and* ϕ.



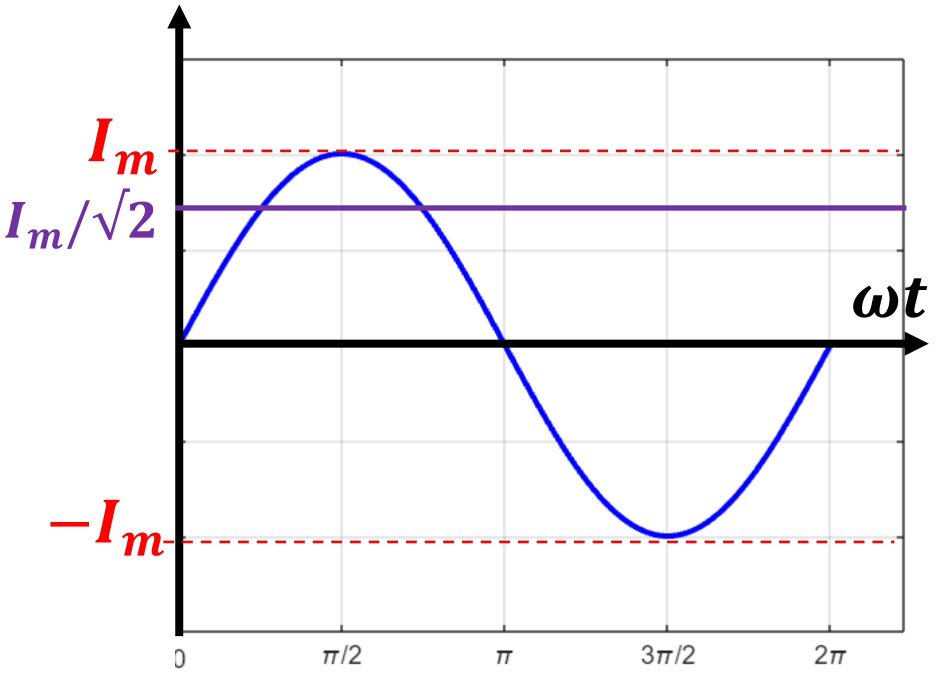
Lead/Lag Phase Relationship in Time Domain

- Phase relationship between $v\ and\ i$:
  - Lead
  - Lag
- *Example:*
  - $v(t) = 10\sin(\omega t + 30^o)\ V$
  - $i(t) = 5\sin(\omega t + 70^o)\ V$
  - i leads v by 40^o
  - v lags i by 40^o



Average and Effective (RMS) Values

- $i(t) = I_m \sin(\omega t)$



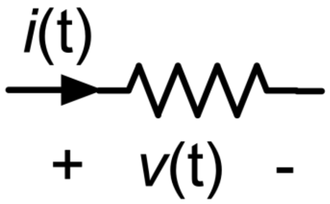
- $I_{avg} = \frac{1}{T}\int_0^T i(t)\ dt = \frac{1}{2\pi}\int_0^{2\pi} i(t)\ d(\omega t)$
- $I_{avg} = \frac{1}{2\pi}\int_0^{2\pi} I_m \sin(\omega t)\, d(\omega t) = \frac{I_m}{2\pi}(-\cos(\omega t))\Big|_0^{2\pi} = 0$

- $I_{rms} = I_{eff} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2(t)\ d(\omega t)}$
- $I_{rms} = \sqrt{\frac{I_m^2}{2\pi}\int_0^{2\pi} \sin^2(\omega t)\, d(\omega t)} = \sqrt{\frac{I_m^2}{4\pi}\int_0^{2\pi}(1-\cos^2(\omega t))\, d(\omega t)}$

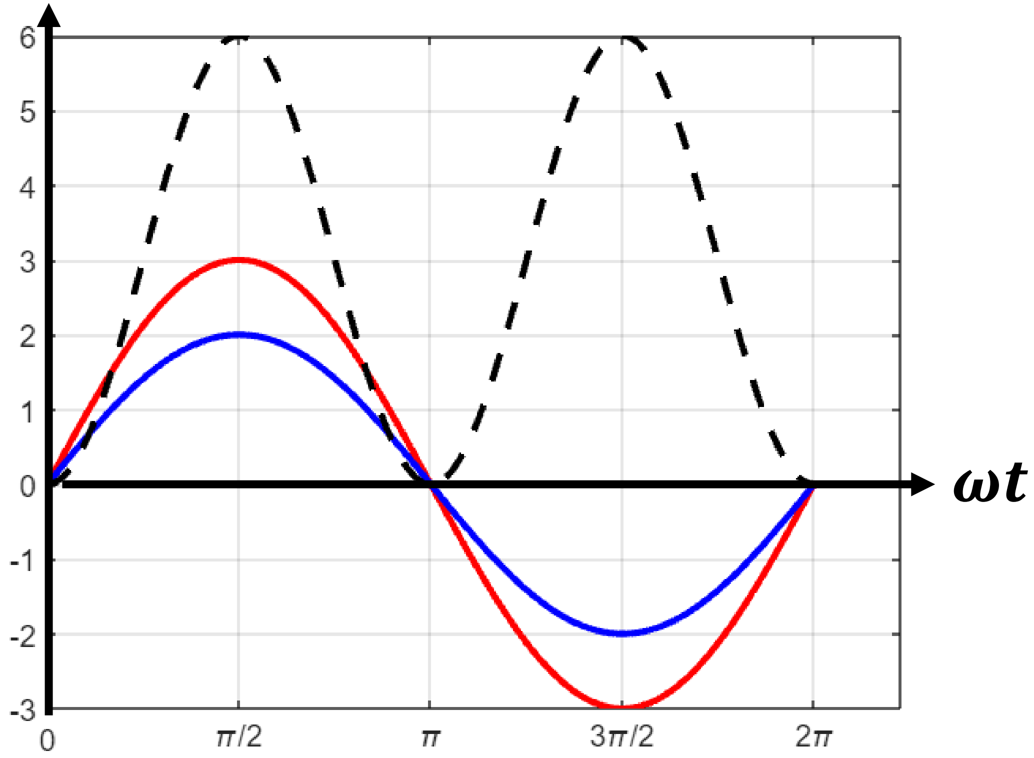
$$= \sqrt{\frac{I_m^2}{4\pi}(2\pi - 0)} = \frac{I_m}{\sqrt{2}}$$

Element Response: Resistors

- Let $i(t) = I_m \sin(\omega t)$



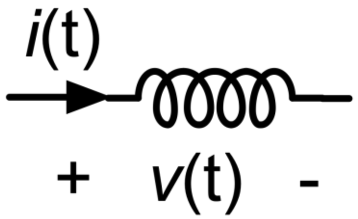
- Resistor (R):
  - $v(t) = R \times i(t)$
  - $v(t) = RI_m \sin(\omega t) = V_m \sin(\omega t)$
  - $V_m = RI_m \ \rightarrow V_{rms} = \frac{RI_m}{\sqrt{2}} = RI_{rms}$
  - $v(t)\ and\ i(t)$ are in phase

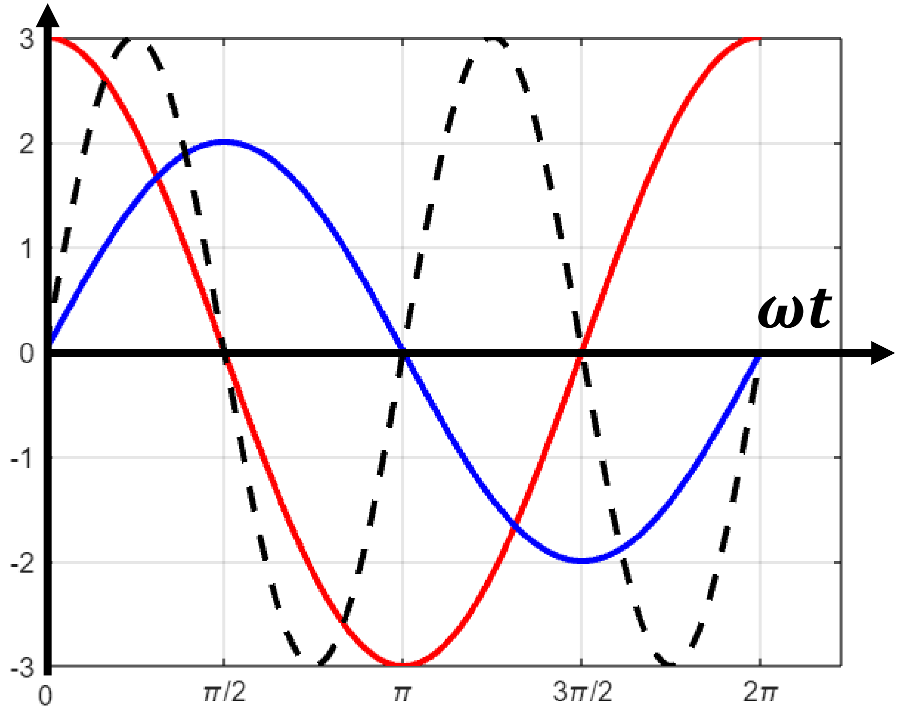


- Instantaneous power:
  - $p(t) = i(t) \times v(t) = I_m V_m \sin^2(\omega t)$
- Average power:
  - $P_{avg} = \frac{1}{2\pi} \int_0^{2\pi} I_m V_m \sin^2(\omega t)\, d(\omega t) = \frac{V_m I_m}{2} = V_{rms} I_{rms}$
  - $P_{avg} = \frac{V_m I_m}{2} = \frac{1}{2} I_m^2 R = \frac{V_m^2}{2R}$
  - $P_{avg} = V_{rms} I_{rms} = I_{rms}^2 R = \frac{V_{rms}^2}{R}$

Element Response: Inductors

- Let $i(t) = I_m \sin(\omega t)$

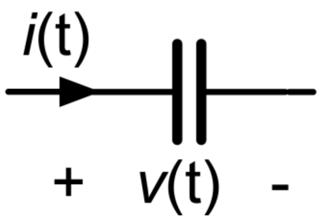




- Inductors (L):
  - $v(t) = L \times \frac{di(t)}{dt}$ (Faraday's law)
  - $v(t) = \omega L I_m \cos(\omega t) = V_m \cos(\omega t)$
  - Reactance: $X_L = \omega L$ (Ω)
  - $V_m = X_L I_m \rightarrow V_{rms} = \frac{X_L I_m}{\sqrt{2}} = X_L I_{rms}$
  - $v(t)$ leads $i(t)$ by 90^o or $i(t)$ lags $v(t)$ by 90^o
- Instantaneous power:
  - $p(t) = i(t) \times v(t) = I_m V_m \sin(\omega t)\cos(\omega t) = \frac{V_m I_m}{2}\sin(2\omega t) = V_{rms} I_{rms} \sin(2\omega t)$
- Average power:
  - $P_{avg} = \frac{1}{2\pi}\int_0^{2\pi} I_m V_m \sin(\omega t)\cos(\omega t)\, d(\omega t) = \frac{V_m I_m}{4\pi}\int_0^{2\pi} \sin(2\omega t)\, d(\omega t) = 0$
  - Power is stored/retuned to the source → net average power = 0

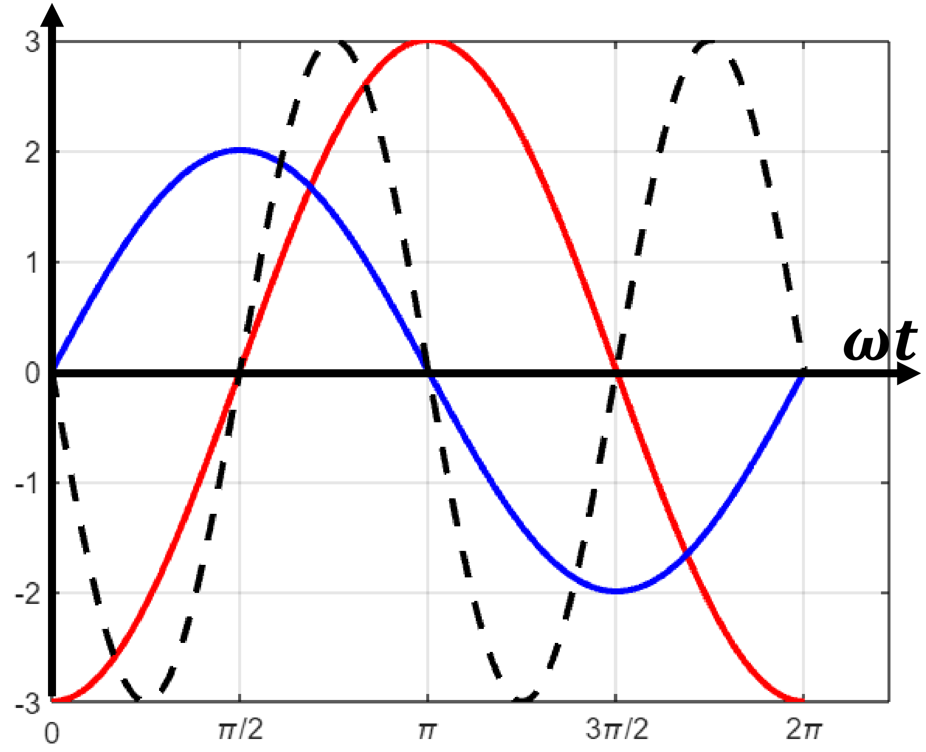
Element Response: Capacitor

- Let $i(t) = I_m \sin(\omega t)$
- Capacitor (C):
  - $i(t) = C \times \frac{dv(t)}{dt}$
  - $v(t) = \frac{1}{C}\int i(t)\,dt$
  - $v(t) = \frac{-1}{\omega C} I_m \cos(\omega t) = V_m \cos(\omega t)$
  - Reactance: $X_C = \frac{-1}{\omega C}\,(\Omega)$
  - $V_m = X_C I_m \rightarrow V_{rms} = \frac{X_C I_m}{\sqrt{2}} = X_C I_{rms}$
  - $i(t)$ leads $v(t)$ by 90^o or $v(t)$ lags $i(t)$ by 90^o



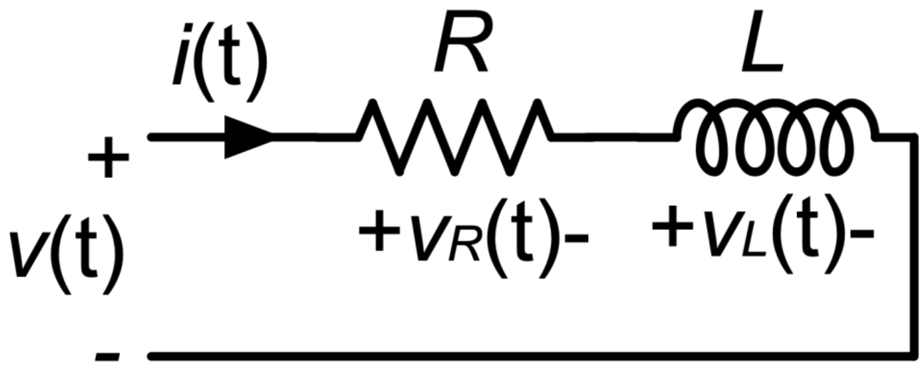
$q = Cv$

$\frac{dq}{dt} = i = C\frac{dv}{dt}$

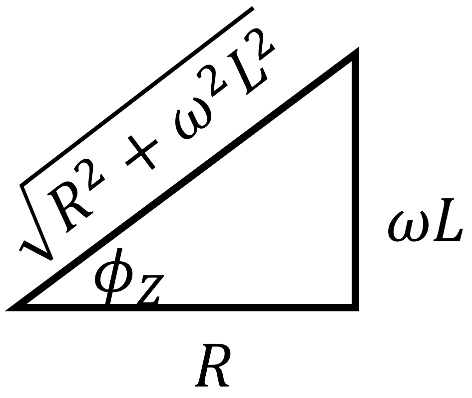


- Instantaneous power:
  - $p(t) = i(t) \times v(t) = I_m V_m \sin(\omega t)\cos(\omega t) = \frac{V_m I_m}{2}\sin(2\omega t) = V_{rms} I_{rms} \sin(2\omega t)$
- Average power:
  - $P_{avg} = \frac{1}{2\pi}\int_0^{2\pi} I_m V_m \sin(\omega t)\cos(\omega t)\,d(\omega t) = \frac{V_m I_m}{4\pi}\int_0^{2\pi} \sin(2\omega t)\,d(\omega t) = 0$
  - Power is stored/retuned to the source → net average power = 0

R,L Series Combination



- $i(t) = I_m \sin(\omega t)$
- $v(t) = v_R(t) + v_L(t) = Ri(t) + L\frac{di(t)}{dt}$
- $v(t) = RI_m \sin(\omega t) + \omega L I_m \cos(\omega t)$
- $v(t) = I_m[R\sin(\omega t) + \omega L \cos(\omega t)]$
- $v(t) = I_m\sqrt{R^2 + \omega^2 L^2}\sin(\omega t + \phi_z) = V_m \sin(\omega t + \phi_z)$
- $\phi_z = \tan^{-1}\left(\frac{\omega L}{R}\right)$
- $V_m = I_m\sqrt{R^2 + \omega^2 L^2}$
- Impedance: $|Z| = \frac{V_m}{I_m} = \frac{V_{rms}}{I_{rms}} = \sqrt{R^2 + \omega^2 L^2}$ (Ω)
- $v(t)$ leads $i(t)$ by ϕ_z or $i(t)$ lags $v(t)$ by ϕ_z
- Indictive circuit



Consider the function

$$f(x) = a\sin(x) + b\cos(x)$$

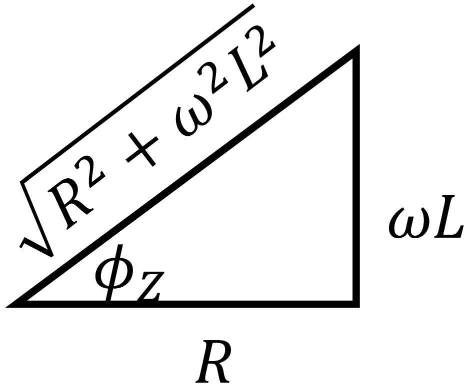
We shall show that this is a sinusoidal wave

$$f(x) = A\sin(x + \phi)$$

and find that the amplitude is $A = \sqrt{a^2 + b^2}$ and the phase $\phi = \arctan\frac{b}{a}$

R,L Series Combination

- $\phi_z = \tan^{-1}\left(\frac{\omega L}{R}\right)$
  - $0 \le \phi_z \le \frac{\pi}{2}$, ϕ_z is +ve
  - $\phi_z = 0$ → Resistor only
  - $\phi_z = \pi/2$ → Inductor only

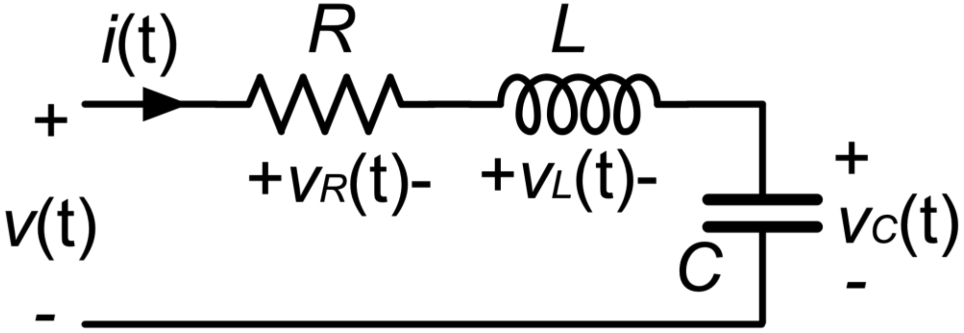


- $p(t) = i(t) \times v(t) = I_m V_m \sin(\omega t)\sin(\omega t + \phi_z)$
- $p(t) = \frac{I_m V_m}{2}\left[\cos(\phi_z) - \cos(2\omega t + \phi_z)\right]$
- $P_{avg} = \frac{1}{2\pi}\int_0^{2\pi} I_m V_m \sin(\omega t)\sin(\omega t + \phi_z)\, d(\omega t)$
- $P_{avg} = \frac{I_m V_m}{4\pi}\int_0^{2\pi}\left[\cos(\phi_z) - \cos(2\omega t + \phi_z)\right] d(\omega t)$
- $P_{avg} = \frac{I_m V_m}{2}\cos(\phi_z) = V_{rms} I_{rms} \cos(\phi_z)$

R,L Series Combination

- Power Factor: $pf = \cos(\phi_z)$
  - $0 \leq pf \leq 1$
  - $pf = 1 \rightarrow \phi_z = 0 \rightarrow$ Resistor only
  - $pf = 0 \rightarrow \phi_z = \pi/2 \rightarrow$ Inductor only
- When power factor =1 → unity power factor
- Assume we have $v(t) = V_m \sin(\omega t)$
- Then, $i(t) = \frac{V_m}{|Z|} \sin(\omega t - \phi_z) = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \sin(\omega t - \tan^{-1}\frac{\omega L}{R})$

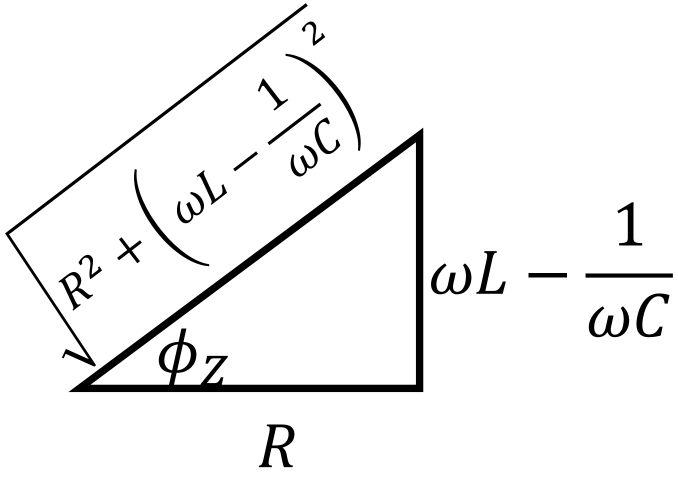
R,L,C Series Combination



- General Case: R,L, and C are in series
- $i(t) = I_m \sin(\omega t)$
- $v(t) = v_R(t) + v_L(t) + v_C(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$
- $v(t) = RI_m \sin(\omega t) + \omega L I_m \cos(\omega t) - \frac{1}{\omega C} I_m \cos(\omega t)$
- $v(t) = I_m \left[R\sin(\omega t) + \left(\omega L - \frac{1}{\omega C}\right)\cos(\omega t)\right] = V_m \sin(\omega t + \phi_z)$
- $\phi_z = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$
- $V_m = I_m\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$
- Impedance: $|Z| = \frac{V_m}{I_m} = \frac{V_{rms}}{I_{rms}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\ (\Omega)$

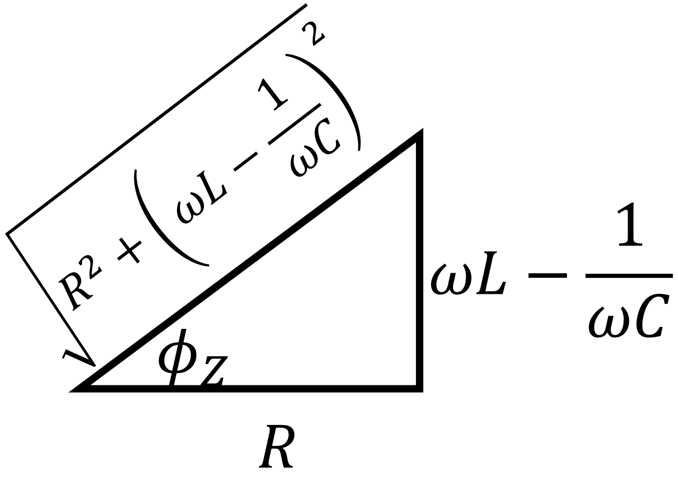
R,L,C Series Combination

- $\phi_z = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$
- $-\frac{\pi}{2} \leq \phi_z \leq \frac{\pi}{2}$
- If $\omega L > \frac{1}{\omega C}$
  - ϕ_z is positive → Inductive Circuit
  - $v(t)$ leads $i(t)$ by ϕ_z
- If $\omega L < \frac{1}{\omega C}$
  - ϕ_z is negative → Capacitive Circuit
  - $i(t)$ leads $v(t)$ by ϕ_z
- If $\omega L = \frac{1}{\omega C}$
  - ϕ_z is 0 → Resistive Circuit (Series Resonance Condition) → $|Z| = R$
  - $i(t)$ and $v(t)$ are in phase



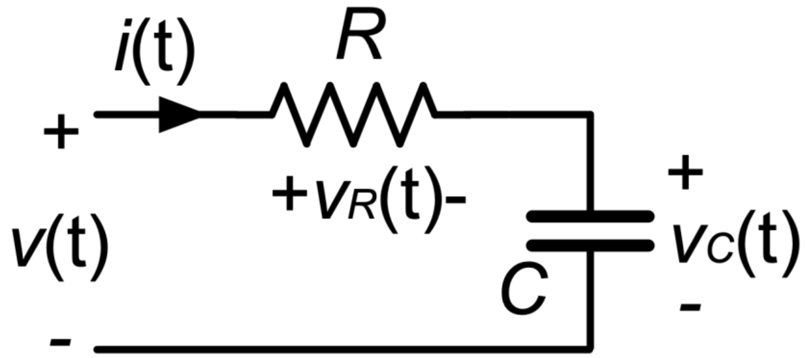
R,L,C Series Combination

- $p(t) = i(t) \times v(t) = I_m V_m \sin(\omega t) \sin(\omega t + \phi_z)$
- $P_{avg} = \frac{I_m V_m}{4\pi} \int_0^{2\pi} [\cos(\phi_z) - \cos(2\omega t + \phi_z)]\, d(\omega t)$
- $P_{avg} = \frac{I_m V_m}{2} \cos(\phi_z) = V_{rms} I_{rms} \cos(\phi_z)$
- Given that: $|Z| = \frac{V_{rms}}{I_{rms}}$
  - $P_{avg} = I_{rms}^2 |Z| \cos(\phi_z) = I_{rms}^2 R$
  - Note that $P_{avg} \neq \frac{V_{rms}^2}{R}$ in this case as $v(t)$ is across R,L, and C altogether
- Power Factor: $pf = \cos(\phi_z)$
  - $0 \leq pf \leq 1$
  - $pf = 1$ → $\phi_z = 0$ → Resistor only
  - $pf = 0$ → $\phi_z = \pm\pi/2$ → Inductor or Capacitor only → Need to know lead/lag?

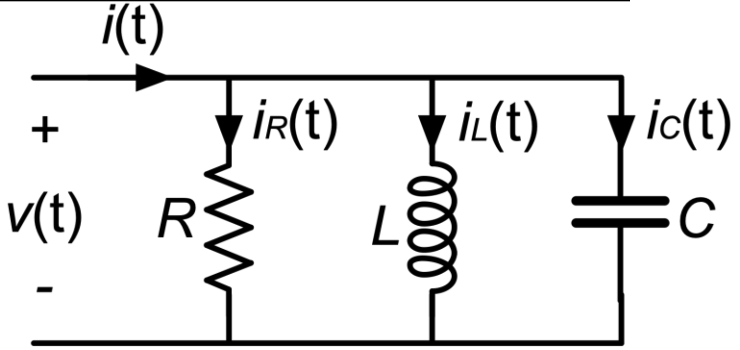


R,C Series Combination

- Special case from the previous discussion ($L = 0$)
- $|Z| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$
- $\phi_z = \tan^{-1}\left(\frac{-\frac{1}{\omega C}}{R}\right) = -\tan^{-1}\left(\frac{1}{\omega RC}\right)$
  - ϕ_z is negative → Capacitive Circuit
  - $i(t)$ leads $v(t)$ by ϕ_z
- $P_{avg} = V_{rms} I_{rms} \cos(\phi_z)$



R,L,C Parallel Combination



- $v(t) = V_m \sin(\omega t)$
- $i(t) = i_R(t) + i_L(t) + i_C(t) = \frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt + C\frac{dv(t)}{dt}$
- $i(t) = \frac{V_m}{R}\sin(\omega t) - \frac{1}{\omega L}V_m\cos(\omega t) + \omega C V_m\cos(\omega t)$
- $i(t) = V_m\left[\frac{1}{R}\sin(\omega t) + \left(\omega C - \frac{1}{\omega L}\right)\cos(\omega t)\right] = I_m\sin(\omega t + \phi_Y)$
- $\phi_Y = \tan^{-1}\left(\frac{\omega C - \frac{1}{\omega L}}{\frac{1}{R}}\right)$
- $I_m = V_m\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$
- Admittance: $|Y| = \frac{I_m}{V_m} = \frac{I_{rms}}{V_{rms}} = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}\ (1/\Omega)$

R,L,C Parallel Combination

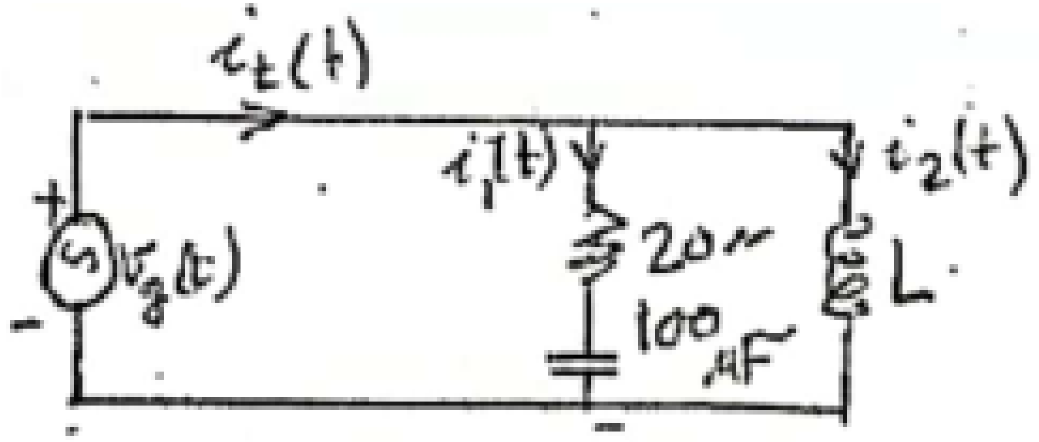
- $\phi_Y = \tan^{-1}\left(\frac{\omega C - \frac{1}{\omega L}}{\frac{1}{R}}\right)$
- $-\frac{\pi}{2} \leq \phi_Y \leq \frac{\pi}{2}$
- If $\omega C > \frac{1}{\omega L}$
  - ϕ_Y is positive → Capacitive Circuit
  - $i(t)$ leads $v(t)$ by ϕ_Y
- If $\omega C < \frac{1}{\omega L}$
  - ϕ_Y is negative → Inductive Circuit
  - $v(t)$ leads $i(t)$ by ϕ_Y
- If $\omega C = \frac{1}{\omega L}$
  - ϕ_Y is 0 → Resistive Circuit (Parallel Resonance Condition) → $|Y| = \frac{1}{R}$
  - $i(t)$ and $v(t)$ are in phase

R,L,C Parallel Combination

- $p(t) = i(t) \times v(t) = I_m V_m \sin(\omega t)\sin(\omega t + \phi_Y)$
- $P_{avg} = \frac{I_m V_m}{4\pi}\int_0^{2\pi}[\cos(\phi_Y) - \cos(2\omega t + \phi_Y)]\,d(\omega t)$
- $P_{avg} = \frac{I_m V_m}{2}\cos(\phi_Y) = V_{rms} I_{rms}\cos(\phi_Y)$
- Given that: $|Y| = \frac{I_{rms}}{V_{rms}}$
  - $P_{avg} = V_{rms}^2 |Y| \cos(\phi_Y) = V_{rms}^2\left(\frac{1}{R}\right) = \frac{V_{rms}^2}{R}$
  - Note that $P_{avg} \neq I_{rms}^2 R$ in this case as $i(t)$ is the total R,L, and C current
- Power Factor: $pf = \cos(\phi_Y)$
  - $0 \leq pf \leq 1$
  - $pf = 1$ → $\phi_Y = 0$ → Resistor only
  - $pf = 0$ → $\phi_Y = \pm\pi/2$ → Capacitor or Inductor only → Need to know lead/lag?

Example

- If $V_g = 100\sin(500t)\ V, i(t) = 2.5\sin(500t)\ \ A$.
- Find $i_1(t), i_2(t), L\ and\ P_{avg}$.
- Sketch the waveforms of all quantities on the same graph.
- Solution:
- $\omega = 500\ rad/sec$
- $\frac{-1}{\omega C} = \frac{-10^6}{500\times100} = -20\Omega$
- R and C in series:
- $|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = 20\sqrt{2}\Omega$
- $\phi_z = -\tan\left(\frac{1}{\omega CR}\right) = -45^o$



Example

- $i_1(t) = \frac{V_m}{|Z|}\sin(500t - \phi_Z) = \frac{100}{20\sqrt{2}}\sin(500t + 45^o)\,A$
- $i_2(t) = i(t) - i_1(t) = 2.5\sin(500t) - \frac{5}{\sqrt{2}}\sin(500t + 45^o)\ \ A$
- $i_2(t) = -2.5\cos(500t) = 2.5\sin(\omega t - 90^o)\ \ A$
- $I_{m2} = 2.5 = \frac{V_m}{\omega L}$
- $L = \frac{V_m}{\omega I_{m2}} = 0.08H$
- $P_{avg} = \frac{1}{2}V_m I_m \cos(\phi_Z) = 0.5 \times 100 \times 2.5 = 125\,W$
- $P_{avg} = \frac{1}{2}I_{m1}^2 R = 0.5 \times \left(\frac{5}{\sqrt{2}}\right)^2 \times 20 = 125\,W$