ELC1010

Electric Circuits (I) – Circuit Theory

Part (II) – AC Circuits
Lecture (2)

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Outline

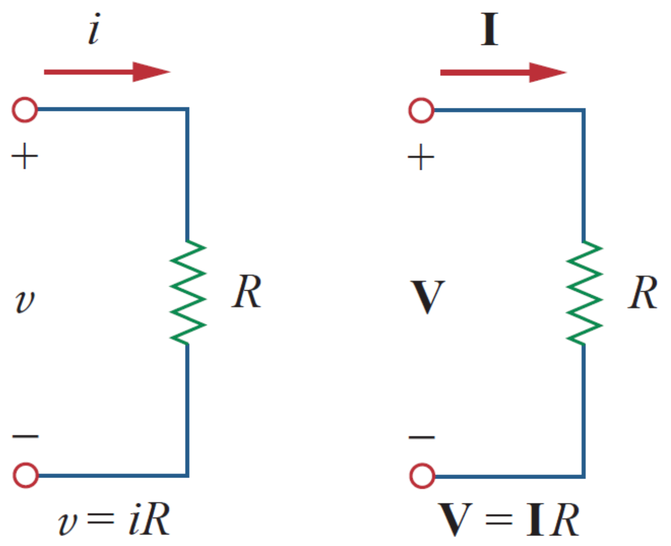
- V/I Phasor Relationship for R,L and C components.
- Method of solutions in phasor domain
- KCL, KVL in Phasor domain
- Parallel/series combinations
- Voltage/current divider
- Star/delta conversion
- Examples

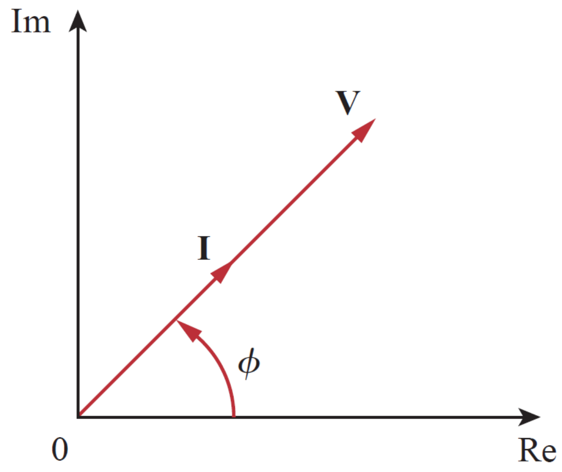
Circuit elements and phasors

- We need to find out how R, L, and C affect the magnitude and phase relationships between **V** and **I.**
- Once we do this, maybe we can rewrite ohm's law in the complex domain and do everything we did for DC exactly the same way
  - This is valid ONLY for steady-state sinusoid stimulus
- But when we rewrite ohm's law, maybe we will find a new definition for the constant of proportionality
  - This defines imepdances and reactances

Phasor Relationship for Resistor

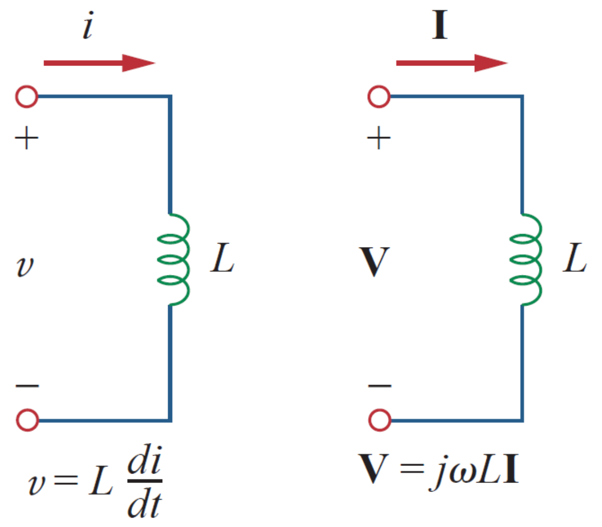
- Resistor current:
  - $i = I_m \cos(\omega t + \Phi) = Re\{I_m e^{j(\omega t + \phi)}\}$
  - $\boldsymbol{I} = I_m \angle \Phi$
- Resistor voltage:
  - $v = iR = RI_m \cos(\omega t + \Phi)$
  - $v = R \times Re\{I_m e^{j(\omega t + \phi)}\}$
  - $\boldsymbol{V} = RI_m \angle \Phi$
- Phasor relation:
  - $\boldsymbol{V} = R\boldsymbol{I}$
  - $\boldsymbol{V}$&$\boldsymbol{I}$ are in phase

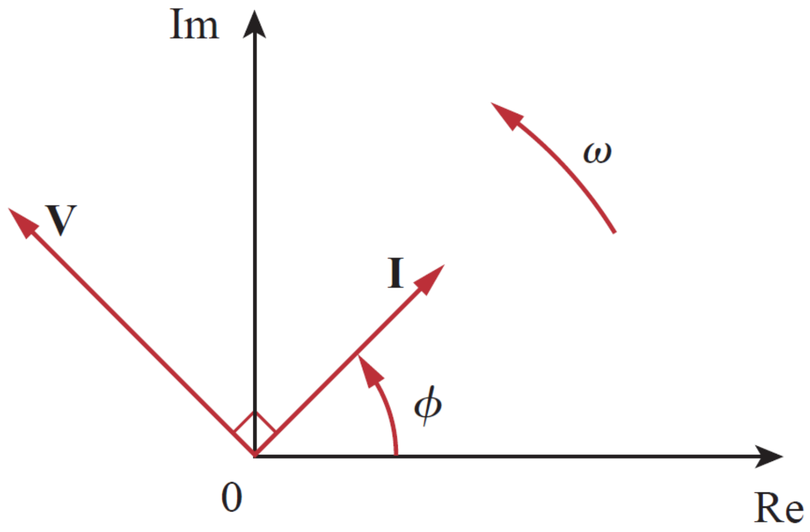




Phasor Relationship for Inductor

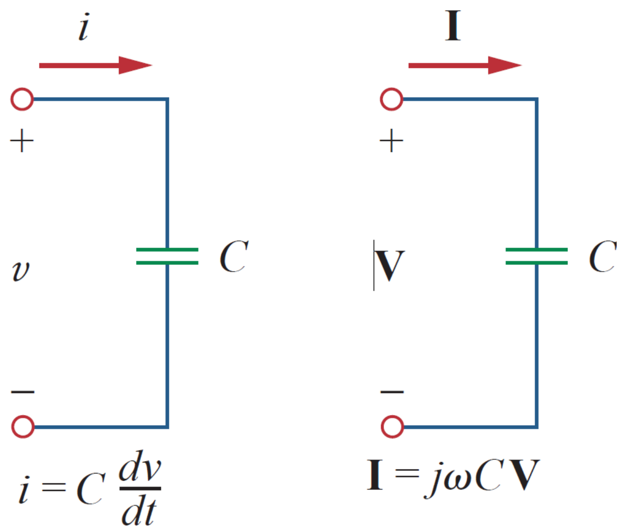
- Inductor current:
  - $i = I_m \cos(\omega t + \Phi) = Re\{I_m e^{j(\omega t+\phi)}\}$
  - $\boldsymbol{I} = I_m \angle \Phi$
- Inductor voltage:
  - $v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \Phi)$
  - $v = \omega L I_m \cos(\omega t + \Phi + 90^o)$
  - $v = Re\{\omega L I_m e^{j(\omega t+\phi+90^o)}\}$
  - $\boldsymbol{V} = \omega L I_m \angle(\Phi + 90^o) = j\omega L I_m \angle \Phi$
- Phasor relation:
  - $\boldsymbol{V} = j\omega L \times \boldsymbol{I}$
  - $\boldsymbol{I}$ lags $\boldsymbol{V}$ by 90°

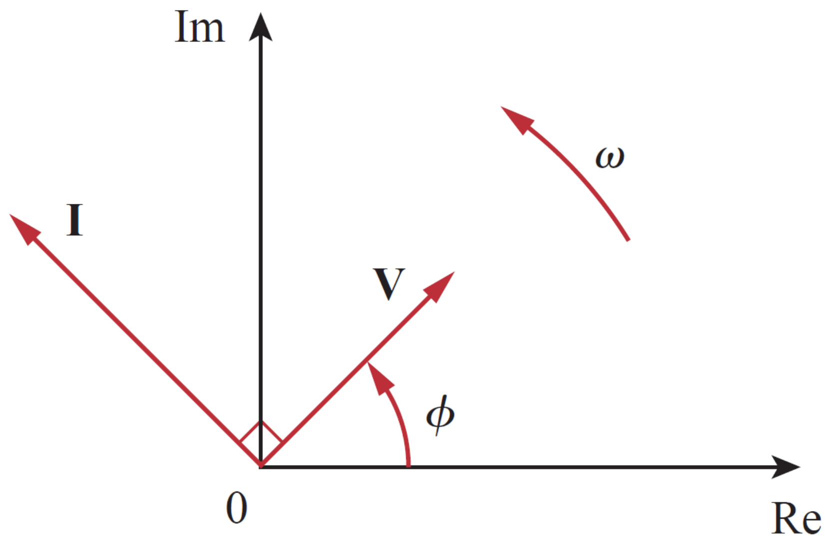




Phasor Relationship for Capacitor

- Capacitor voltage:
  - $v = V_m \cos(\omega t + \Phi) = Re\{V_m e^{j(\omega t+\phi)}\}$
  - $\boldsymbol{V} = V_m \angle \Phi$
- Capacitor current:
  - $i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \Phi)$
  - $i = \omega C V_m \cos(\omega t + \Phi + 90^o)$
  - $i = Re\{\omega C V_m e^{j(\omega t+\phi+90^o)}\}$
  - $\boldsymbol{I} = \omega C V_m \angle(\Phi + 90^o) = j\omega C V_m \angle \Phi$
- Phasor relation:
  - $\boldsymbol{I} = j\omega C \times \boldsymbol{V} \rightarrow \boldsymbol{V} = \frac{1}{j\omega C}\boldsymbol{I}$
  - $\boldsymbol{I}$ leads $\boldsymbol{V}$ by 90°





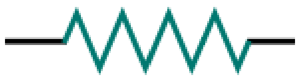
Summary of voltage-current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Impedance and Admittance

- The impedance Z of a circuit is the ratio of the phasor voltage $\boldsymbol{V}$ to the phasor current $\boldsymbol{I}$, measured in ohms (Ω)
- Ohm's law is now $\boldsymbol{V} = \boldsymbol{I}Z$, always and for everyone
- Just calculate the right Z

R



$$V = RI$$

$$Z = R$$

L

$$V = j\omega L I$$

$$Z = j\omega L$$

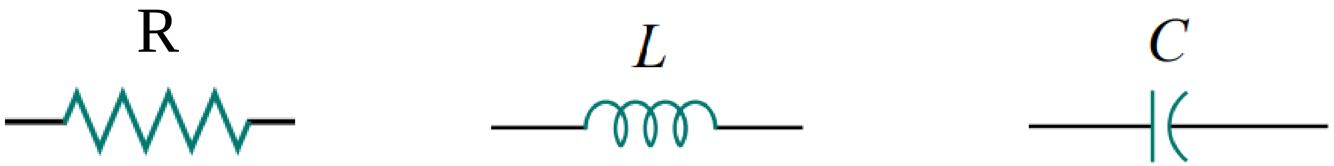
C

$$V = \frac{I}{j\omega C}$$

$$Z = \frac{1}{j\omega C}$$

Impedance and Admittance

- The admittance Y of a circuit is the ratio of the phasor current $\boldsymbol{I}$ to the phasor voltage $\boldsymbol{V}$, measured in siemens (S or Ω^{-1})



$$I = V/R \qquad I = \frac{V}{j\omega L} \qquad I = j\omega CV$$

$$Y = \frac{1}{R} \qquad Y = \frac{1}{j\omega L} \qquad Y = j\omega C$$

Impedance and Admittance

- For a circuit with multiple elements, the equivalent Z can be calculated using parallel and series combinations. It ends up being a complex number.
- Impedance $\boldsymbol{Z} = \boldsymbol{R} + \boldsymbol{jX}$
  - $\boldsymbol{R}$ is the equivalent, while X is the reactance
  - $\boldsymbol{Z}$ is inductive if $\boldsymbol{X}$ is +ve
  - $\boldsymbol{Z}$ is capacitive if $\boldsymbol{X}$ is –ve
  - $\boldsymbol{Z}, \boldsymbol{R},$ and $\boldsymbol{X}$ have the units of Ω
- Admittance $\boldsymbol{Y} = \boldsymbol{G} + \boldsymbol{jB}$
  - $\boldsymbol{G}$ is the conductance, while B is the susceptance
  - $\boldsymbol{Y}$ is inductive if $\boldsymbol{B}$ is -ve
  - $\boldsymbol{Y}$ is capacitive if $\boldsymbol{B}$ is +ve
  - $\boldsymbol{Y}, \boldsymbol{G},$ and $\boldsymbol{B}$ have the units of Ω^{-1}

Notes about impedance

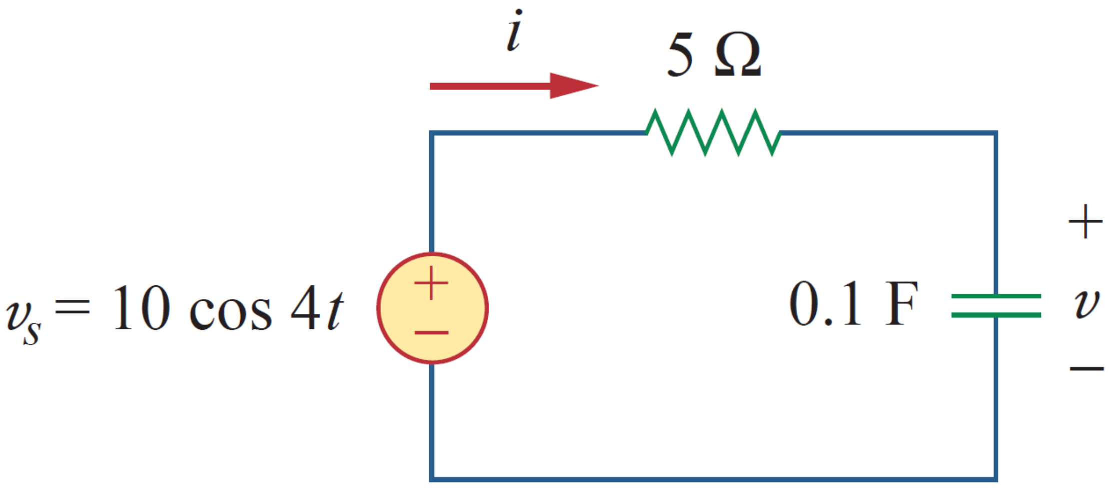
- Resistive impedance is real
  - Resistance is the only thing that dissipates power
- Capacitive and inductive impedance is imaginary
  - They store power
- What part of a complex Z dissipates power and what part stores?
- A purely resistive impedance will not change phase

Impedance and frequency

- Resistive impedance $Z_R = R$ is not a function of frequency
- Capacitance impedance is $Z_C = 1/j\omega C$
- So is inductive impedance $Z_L = j\omega L$
- At DC $\omega = 0$
  - $Z_C = \frac{1}{j\omega C} = \infty$ → Capacitors are O.C.
  - $Z_L = j\omega L = 0$ → Inductors are S.C.
  - $Z_R = R$ → Only resistors remain
- At very high frequencies $\omega \rightarrow \infty$
  - $Z_C = \frac{1}{j\omega C} = 0$ → Capacitors are S.C.
  - $Z_L = j\omega L = \infty$ → Inductors are O.C.
  - $Z_R = R$ → Only resistors remain
- This is the basis of filters

Example 1.0

- For the circuit shown find $v(t)$ and $i(t)$
- $V_S = 10\angle 0°,\ \omega = 4 rad/sec$
- $Z = R + \frac{1}{j\omega C}$
- $Z = 5 - j2.5\ \Omega$
- $I = \frac{V_S}{Z} = \frac{10}{5-j2.5}$
- $V = \frac{10}{5-j2.5} \times (-j2.5) = 10 \times \frac{-j2.5}{5-j2.5}$ (potential divider)



Note:

- For any complex number z $= x + jy$,
- The conjugate is $z^* = x - jy$
- $z \times z^* = (x + jy)(x - jy) = x^2 - jxy + jxy - j^2y^2$
- $Hence, z \times z^* = x^2 + y^2 = |z|^2$

Example 1.0

- $I = \frac{10}{5-j2.5} \frac{5+j2.5}{5+j2.5}$
- $I = \frac{50+j25}{31.25} = 1.6 + j0.8$
- $I = 1.79\angle 26.57° \; A$
- $i(t) = 1.79 \cos(4t + 26.57°) \; A$

- $V = IZ_C = \frac{I}{j\omega C} = \frac{1.79\angle 26.57°}{j0.4} = \frac{1.79\angle 26.57°}{0.4\angle 90}$
- $V = 4.47\angle - 63.43°$
- $v(t) = 4.47 \cos(4t - 63.43°)$

Summary of AC Circuits

- Voltage and current and written as magnitude and phase
- $v(t) = V_m \cos(\omega t + \phi) = Re\left(V_m e^{j\omega t + j\phi}\right)$
  - $V = V_m \angle \phi$
- $i(t) = I_m \cos(\omega t + \theta) = Re\left(I_m e^{j\omega t + j\theta}\right)$
  - $I = I_m \angle \theta$
- Impedance is defined as $Z = \frac{V}{I}$
  - For resistors $Z = R$
  - For inductors $Z = j\omega L$
  - For capacitors $Z = \frac{1}{j\omega C}$

Kirchhoff's Laws in the Phasor Domain

- KVL and KCL are valid in phasor domain

- **KVL**
  - Let $v_1, v_2 \ldots, v_n$ be the voltages around a closed loop.
  - Then:

$$\boldsymbol{v_1 + v_2 + \cdots + v_n = 0}$$

  - In the sinusoidal steady state, each voltage may be written in cosine form

$$\boldsymbol{V_{m1}\cos(\omega t + \Phi_1) + V_{m2}\, cos(\omega t + \Phi_2) + \cdots + V_{mn}\, cos(\omega t + \Phi_n) = 0}$$

Kirchhoff's Laws in the Phasor Domain

- Then

$$\boldsymbol{Re\{V_{m1}e^{j\Phi_1}e^{j\omega t} + V_{m2}e^{j\Phi_2}e^{j\omega t} + \cdots + V_{mn}e^{j\Phi_n}e^{j\omega t}\} = 0}$$

$$\boldsymbol{Re\{V_{m1}e^{j\Phi_1}\} + Re\{V_{m2}e^{j\Phi_2}\} + \cdots + Re\{V_{mn}e^{j\Phi_n}\} = 0}$$

- Let

$$\boldsymbol{V}_k = V_{mk}e^{j\Phi_k}$$

- Then

$$\boldsymbol{V}_1 + \boldsymbol{V}_2 + \cdots + \boldsymbol{V}_n = 0$$

- So, Kirchhoff's voltage law holds for phasors

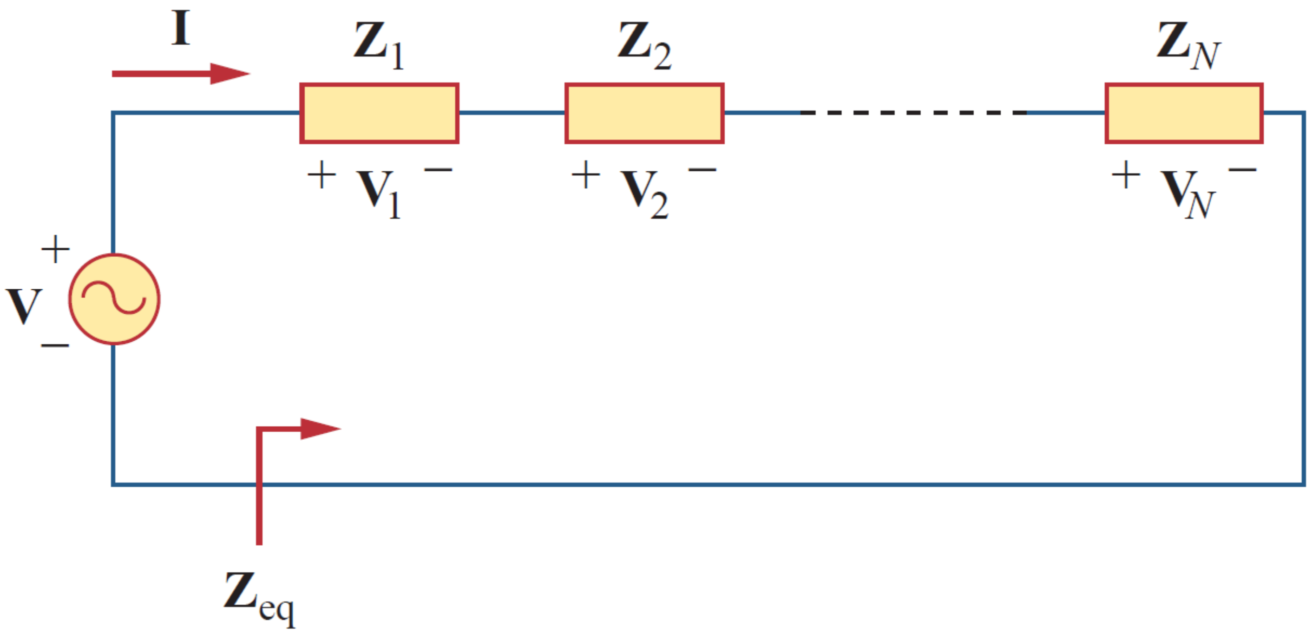
Kirchhoff’s Laws in the Phasor Domain

- Similarly, Kirchhoff’s current law holds for phasors at all nodes

$$\boldsymbol{I}_1 + \boldsymbol{I}_2 + \cdots + \boldsymbol{I}_n = 0$$

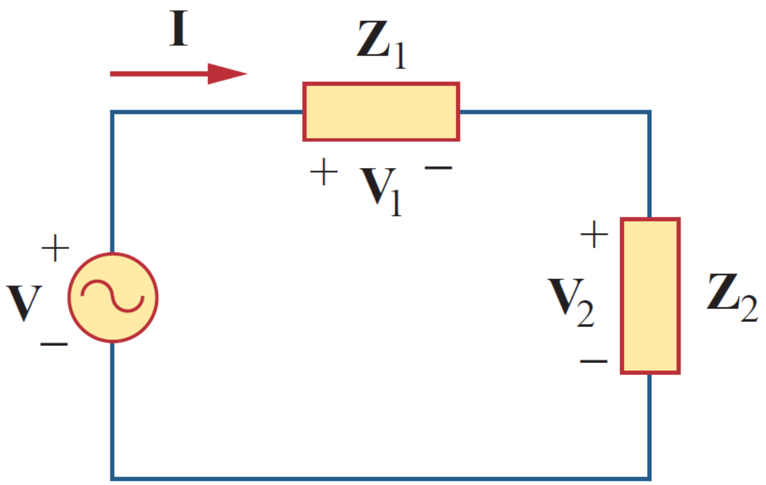
Impedance Combination

- For N series connected impedances, the same current I flows through the impedances
- $V = V_1 + V_2 + \cdots + V_N = I(Z_1 + Z_2 + \cdots + Z_N)$
- $Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + \cdots + Z_N$



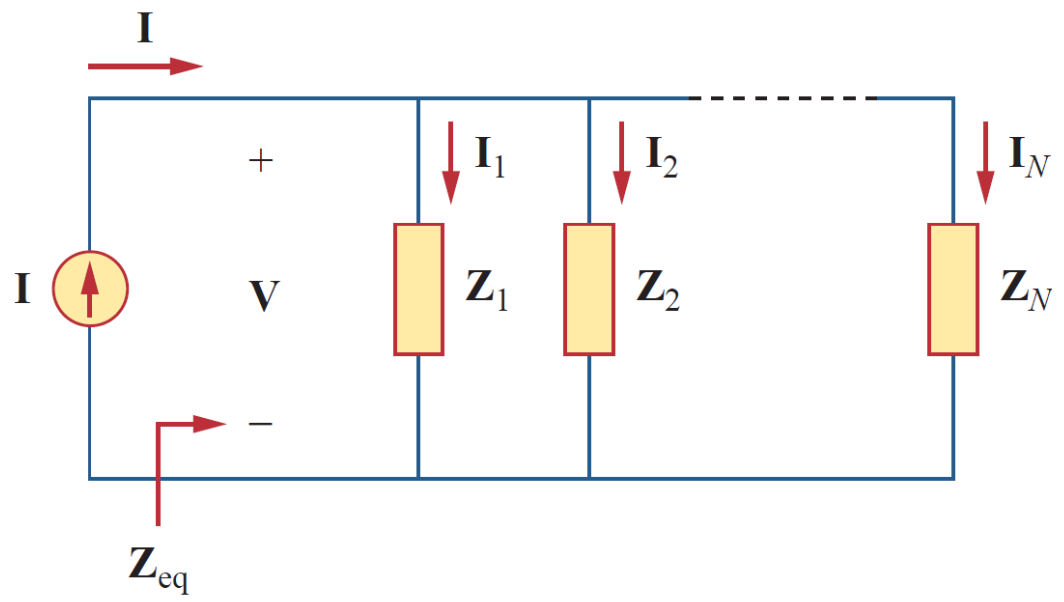
Voltage Divider

- For two impedances connected in series
  - $I = \frac{V}{Z_1+Z_2}$
  - $V_1 = IZ_1, V_2 = IZ_2$
  - $V_1 = \frac{Z_1}{Z_1+Z_2}V, V_2 = \frac{Z_2}{Z_1+Z_2}V$



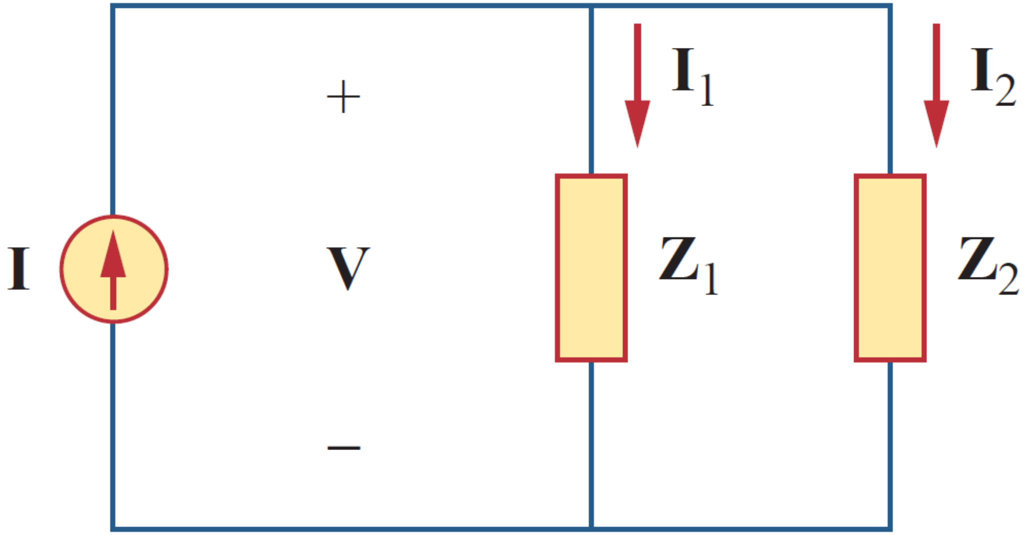
Admittance Combination

- For N Parallel connected impedance, voltage is the same across all impedances
- Applying KCL
  - $I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right)$
  - $\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}$
  - $Y_{eq} = Y_1 + Y_2 + \cdots + Y_N$



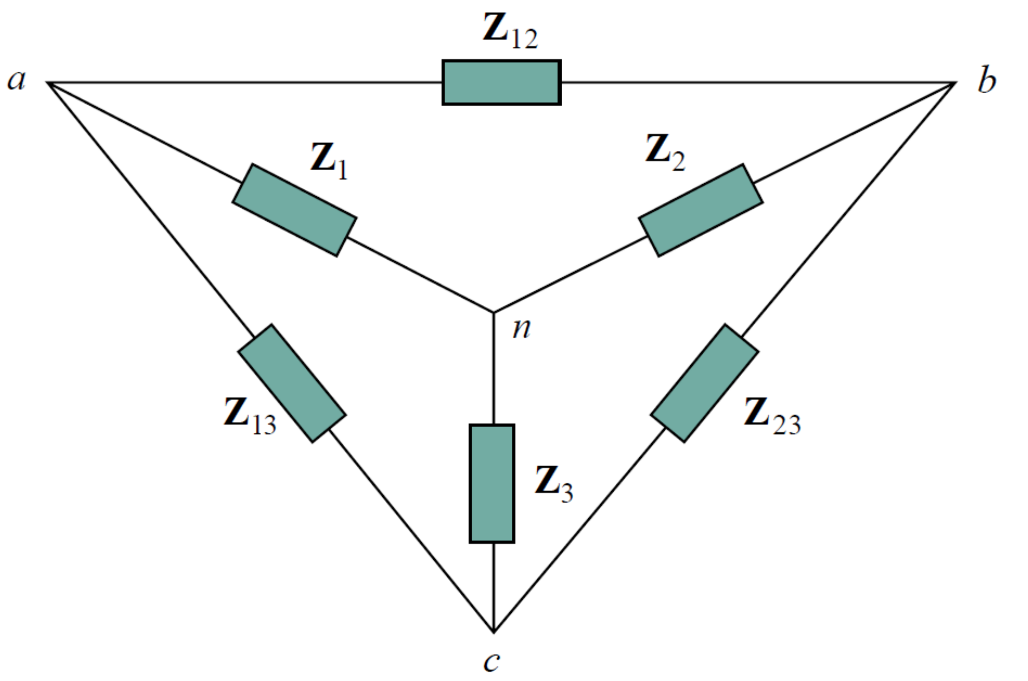
Current Divider

- $Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{Y_1+Y_2} = \frac{1}{1/Z_1+1/Z_2} = \frac{Z_1 Z_2}{Z_1+Z_2}$
- $V = IZ_{eq} = I\frac{Z_1 Z_2}{Z_1+Z_2} = I_1 Z_1 = I_2 Z_2$
- $I_1 = \frac{Z_2}{Z_1+Z_2} I, I_2 = \frac{Z_1}{Z_1+Z_2} I$



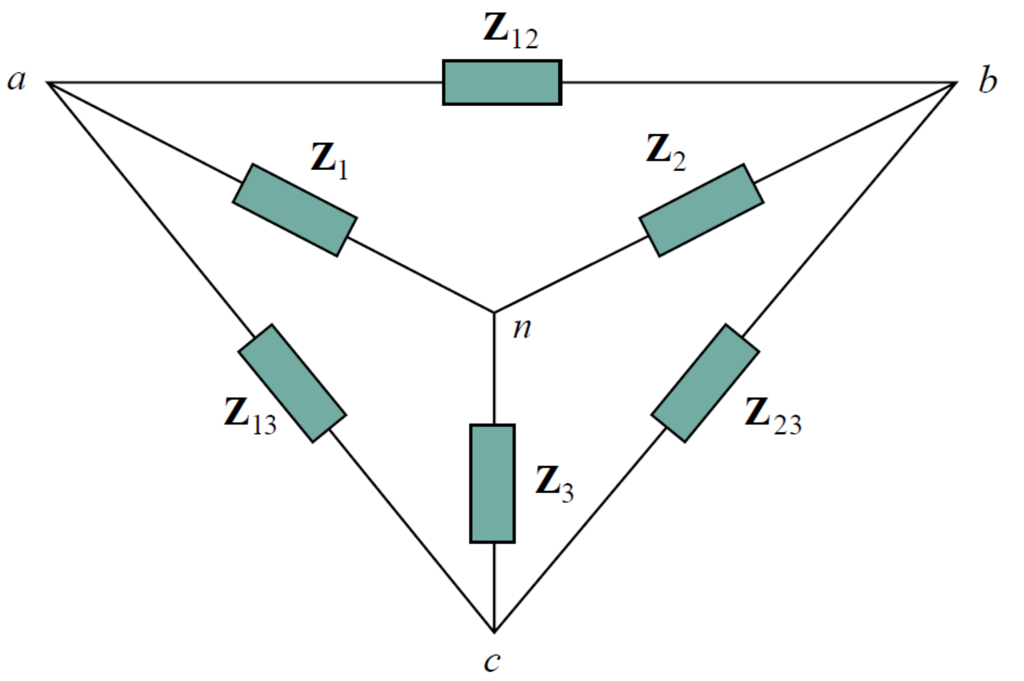
Star-Delta Conversion

- $Z_{12} = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}$
- $Z_{23} = Z_2 + Z_3 + \frac{Z_2 Z_3}{Z_1}$
- $Z_{13} = Z_1 + Z_3 + \frac{Z_1 Z_3}{Z_2}$



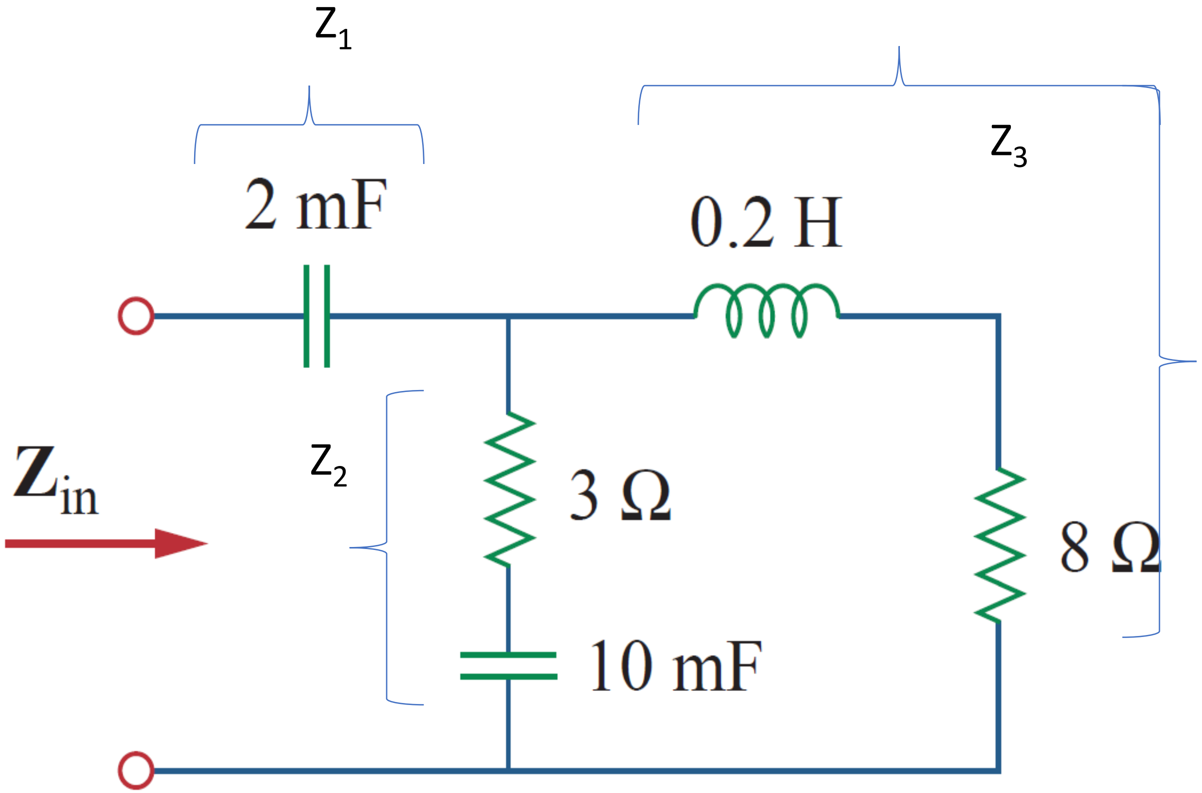
Delta-Star Conversion

- $Z_1 = \frac{Z_{12}Z_{13}}{Z_{12}+Z_{13}+Z_{23}}$
- $Z_2 = \frac{Z_{12}Z_{23}}{Z_{12}+Z_{13}+Z_{23}}$
- $Z_3 = \frac{Z_{13}Z_{23}}{Z_{12}+Z_{13}+Z_{23}}$



Example 1.0

- Find the input impedance of the circuit shown. Assume that the circuit operates at ω = 50 rad/s



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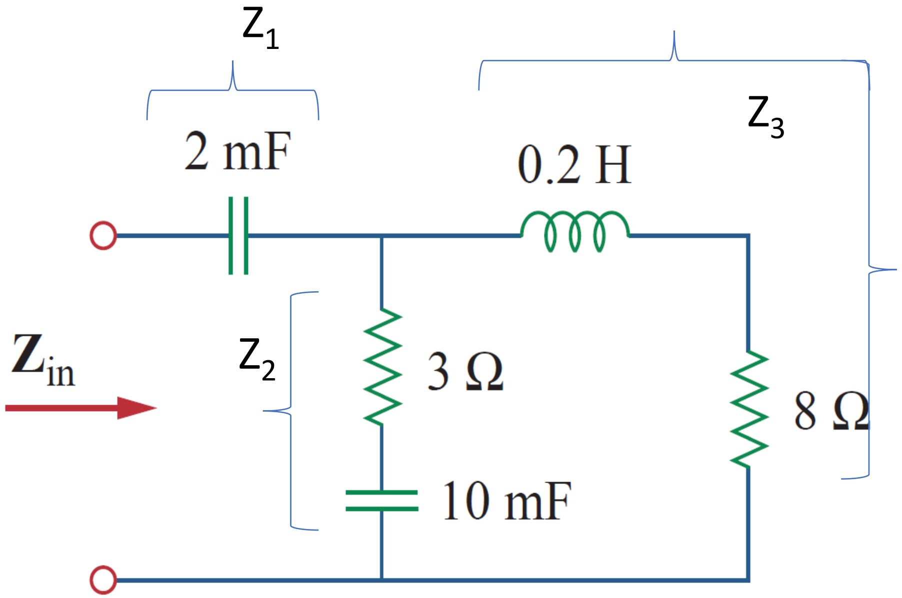
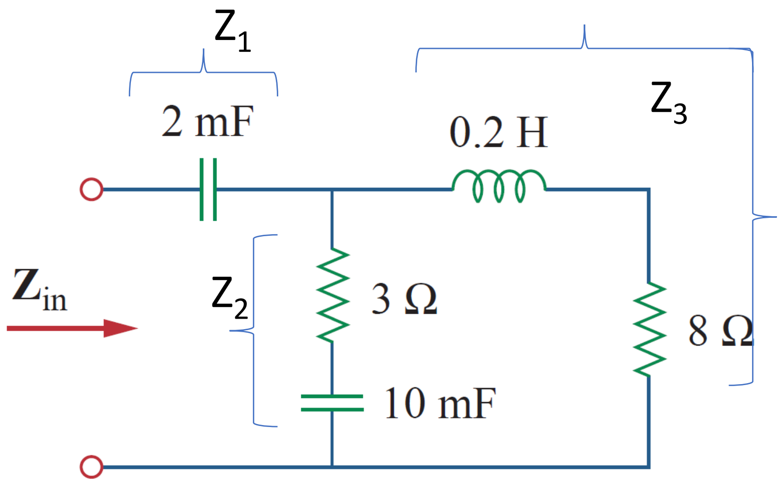


- $Z_1 = Impedance\ of\ 2mF\ Capacitor$
  - $Z_1 = \frac{1}{j\omega C} = -j10$
- $Z_2 = R + \frac{1}{j\omega C} = 3 - j2$
- $Z_3 = Impedance\ of\ the\ 0.2H\ inductor\ in\ series\ with\ the\ 8\Omega\ resistor$
  - $Z_3 = 8 + j10$

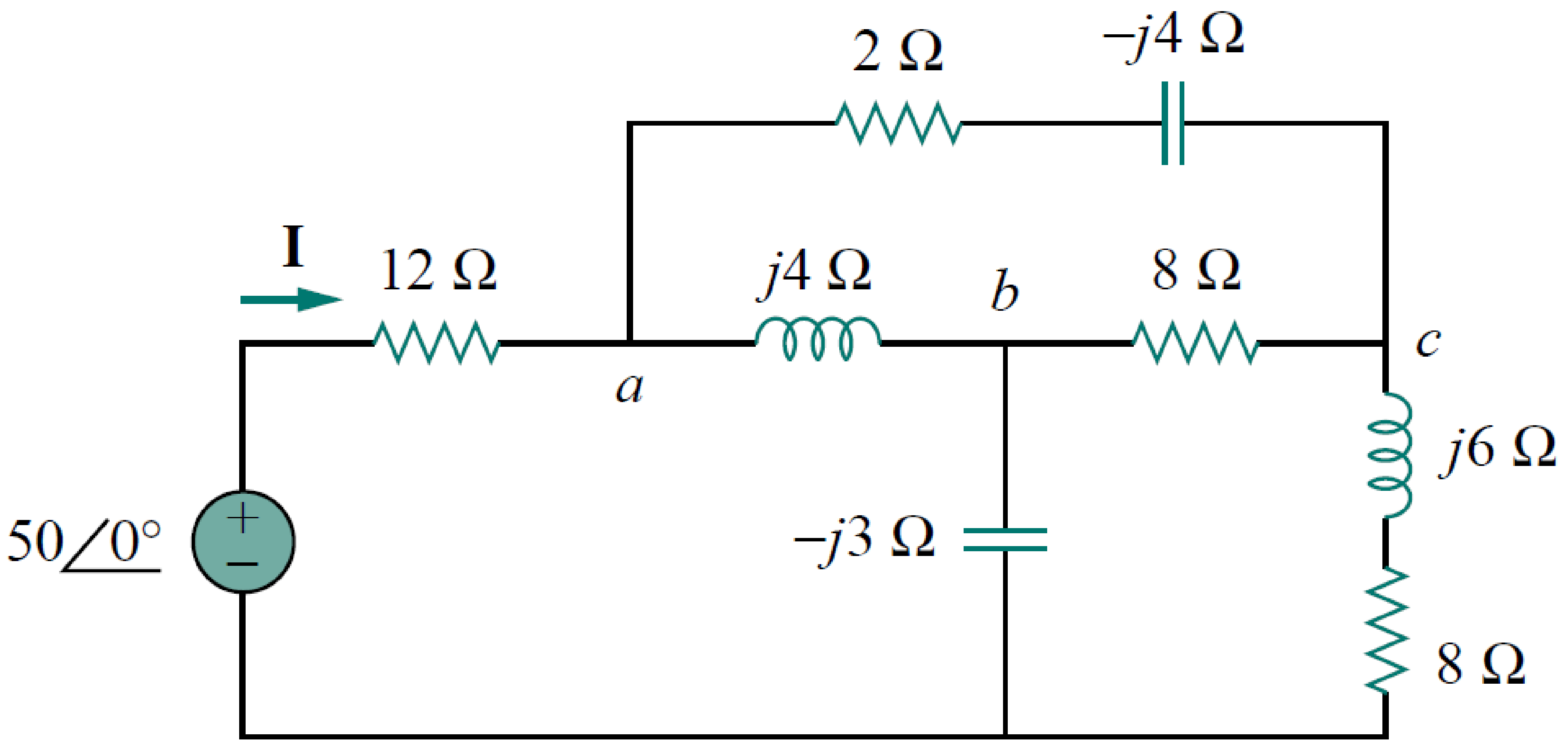
Example 1.0



- $Z_1 = \frac{1}{j\omega C} = -j10, Z_2 = R + \frac{1}{j\omega C} = 3 - j2, Z_3 = 8 + j10$
- The input impedance is $Z_{in} = Z_1 + Z_2 || Z_3$
  - $Z_{in} = -j10 + \frac{(3-j2)(8+j10)}{11+j8} = -j10 + \frac{44+j14}{11+j8}$
  - $Z_{in} = -j10 + \frac{(44+j14)(11-j8)}{11^2+8^2} = -j10 + 3.22 - j1.07$
  - $Z_{in} = 3.22 - j11.07$

Example 2.0

- Find the current I for the circuit shown



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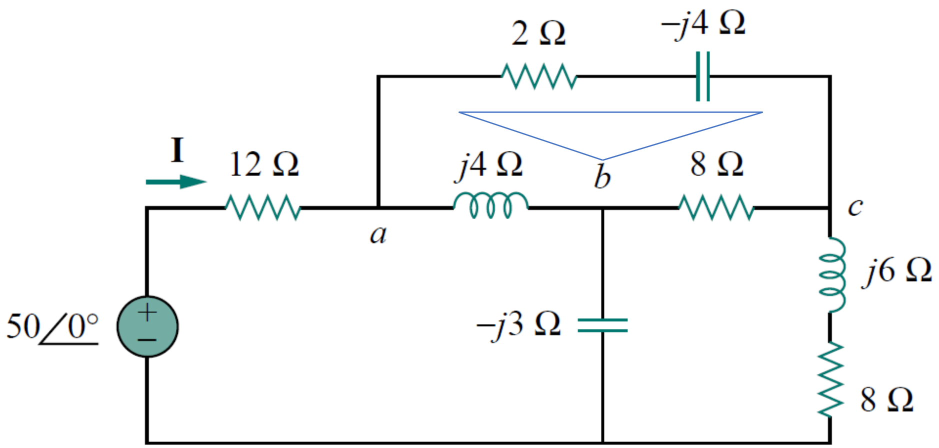
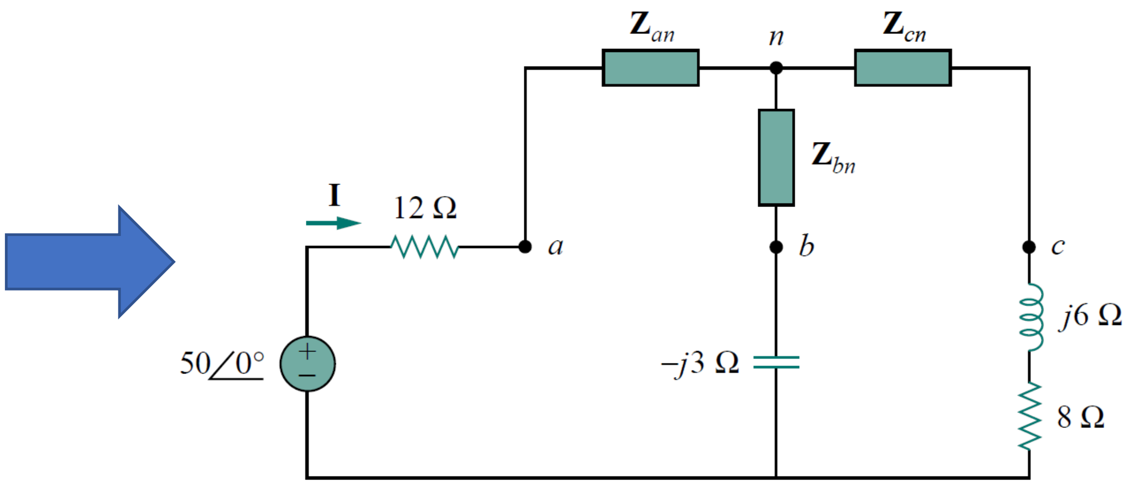
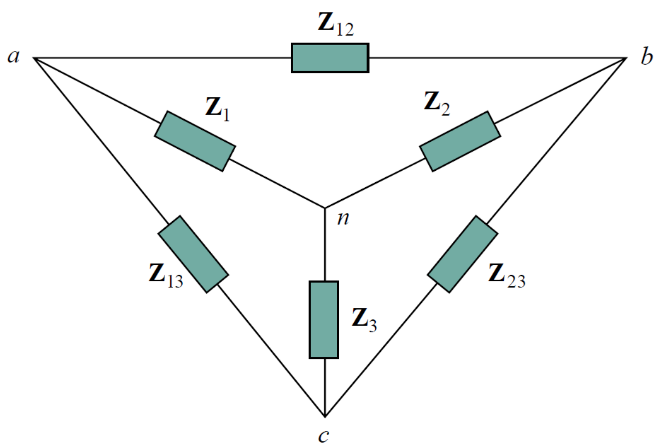




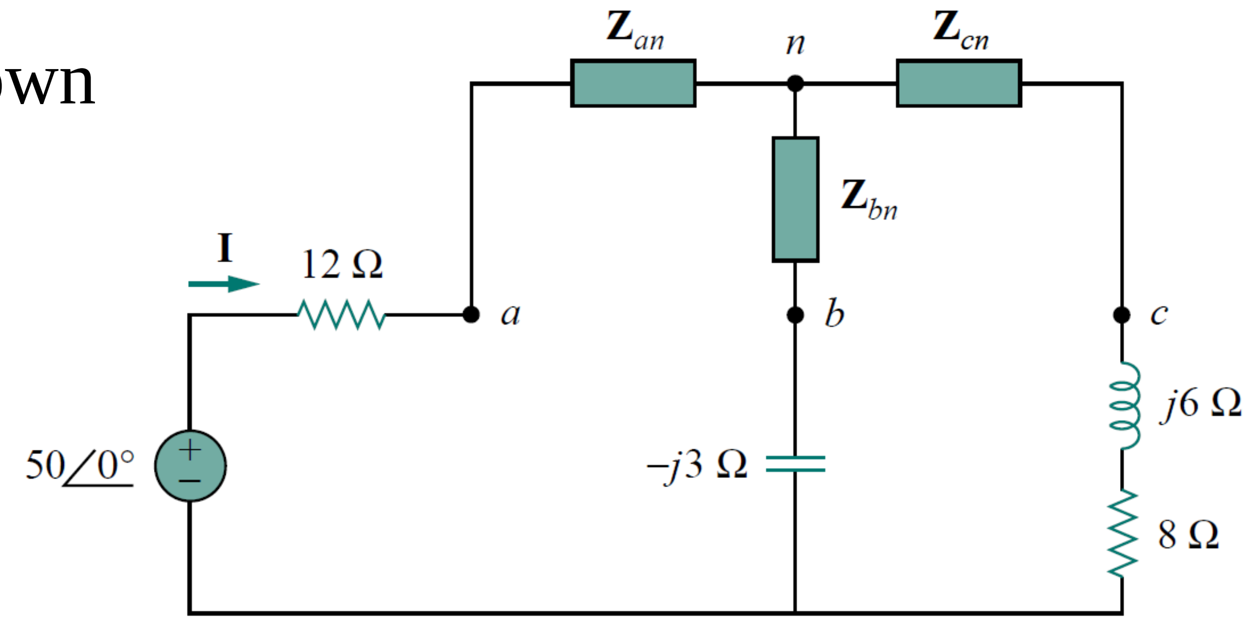
- $Z_{cn} = \frac{8(2-j4)}{10} = 1.6 - j3.2$
- $Z_{an} = \frac{j4(2-j4)}{10} = 1.6 + j0.8$
- $Z_{bn} = \frac{j4 \times 8}{10} = j3.2$



$$Z_1 = \frac{Z_{12}Z_{13}}{Z_{12}+Z_{13}+Z_{23}}$$
$$Z_2 = \frac{Z_{12}Z_{23}}{Z_{12}+Z_{13}+Z_{23}}$$
$$Z_3 = \frac{Z_{13}Z_{23}}{Z_{12}+Z_{13}+Z_{23}}$$

Example 2.0

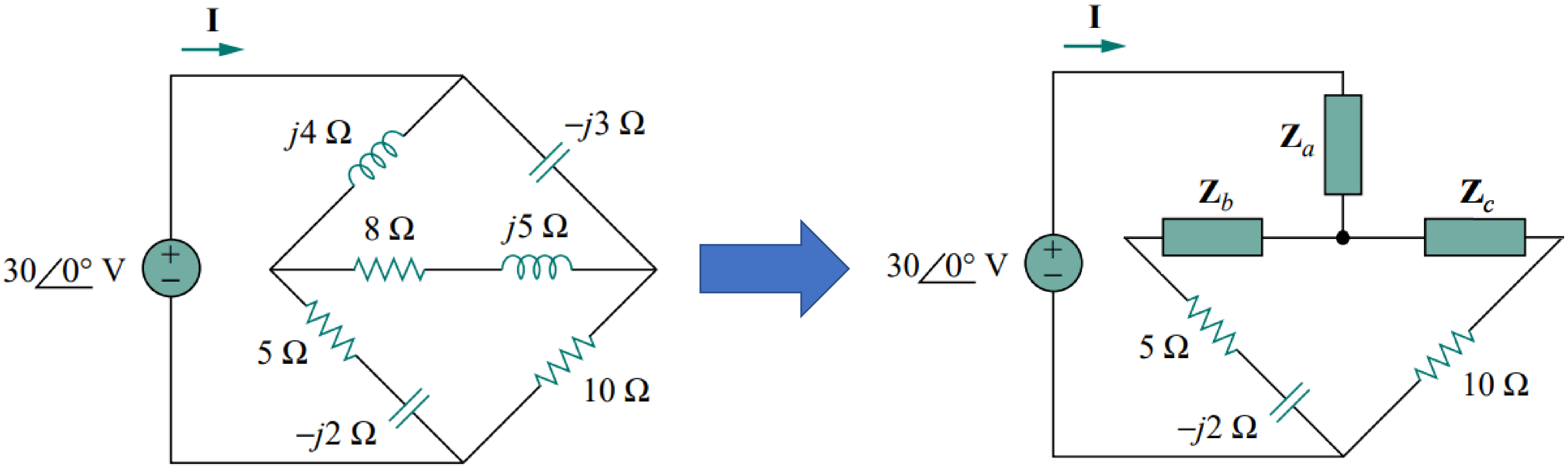
- Find the current I for the circuit shown



- $Z = 12 + Z_{an} + (Z_{bn} - j3)||(Z_{cn} + 8 + j6)$
- $Z = 12 + 1.6 + j0.8 + (j0.2)||(9.6 + j2.8)$
- $Z \approx 13.6 + j1 = 13.64\angle 4.2°$
- $I = \frac{V}{Z} = \frac{50\angle 0°}{13.64\angle 4.2°} = 3.67\angle -4.2°$

Example 3.0

- Find the current I for the circuit shown



- $I = 6.36\angle 3.8°$