

## ON STABILITY OF TRIANGULAR POINTS OF THE RESTRICTED RELATIVISTIC ELLIPTIC THREE-BODY PROBLEM WITH TRIAXIAL AND OBLATE PRIMARIES

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**SUMMARY:** This paper investigates the location and linear stability of triangular points under combined effects of perturbations: triaxiality of a massive primary, oblateness of a less massive one, and relativistic corrections. The primaries in this system are assumed to move in elliptical orbits around their common barycenter. It is found that the locations of the triangular points are affected by the involved perturbations. The stability of orbits near these points is also examined. We observed that these points are stable for the mass ratio,  $\mu$ , range  $0 < \mu < \mu_c$ , where  $\mu_c$  is the critical mass ratio, and unstable for the range  $\mu_c \leq \mu \leq 0.5$ .

**Key words.** Celestial mechanics

### 1. INTRODUCTION

The importance of the Lagrangian points as possible locations for large space stations, which can be utilized in interplanetary navigation, much increases as time advances. This requires an accurate analysis of locations and linear stability of these points. The linear stability of triangular points was examined in several studies [see Musielak and Quarles (2014) for a review]. Bhatnagar and Hallan (1998) studied the linear stability of relativistic triangular points. They found that these points are unstable for the range of mass ratio  $0 \leq \mu \leq 0.5$ , despite the fact that the non-relativistic triangular points are stable for  $\mu < \mu_0 = 0.03852$ , where  $\mu_0$  is the Routh critical mass ratio. The same problem was revisited by Douskos and Perdios (2002) and Ahmed et al. (2006) whose results showed that the relativistic triangular points are linearly stable in the range of mass ratios less than a critical value  $\mu_c$ , i.e.  $0 \leq \mu < \mu_c$ . This critical value

was estimated by Douskos and Perdios (2002) to be  $\mu_c = \mu_0 - 17\sqrt{69}/486c^2$  while Ahmed et al. (2006) calculated it to be  $\mu_c = 0.03840$ .

Palit et al. (2009) analyzed the stability of circular orbits in the Schwarzschild-de Sitter spacetime. Yamada and Asada (2010) computed the relativistic corrections to the Sun-Jupiter libration points. Also, Yamada and Asada (2011) continued their work and investigated collinear solutions to the general relativistic three-body problem. They proved the uniqueness of the configuration for given system parameters (the masses and the end-to-end length). Ichita et al. (2011) investigated the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. For three finite masses, they found that the equilateral triangular configuration satisfies the post-Newtonian equation of motion in general relativity if and only if all three masses are equal. The post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem were re-examined by Yamada

and Asada (2012). They found that, for three finite masses, a triangular configuration satisfies the post-Newtonian equation of motion in general relativity if and only if it has the relativistic corrections to each side length.

Katour *et al.* (2014) studied the restricted three-body problem within the framework of the post-Newtonian approximation as well as the continuous radiation of the primaries plus the effect of oblateness of these two primaries. The authors computed perturbed locations for the triangular points under the considered force model. Yamada *et al.* (2015) examined the post-Newtonian effects on the stability of the triangular solution in relativistic three-body problem for general masses. They obtained, for three finite masses, a condition for stability of the triangular solution at the first post-Newtonian order.

The bodies in the restricted three-body problems are generally considered to be spherical in shape but, in fact, several heavenly bodies are either oblate spheroids or triaxial rigid bodies. The Earth, Jupiter and Saturn as well as some stars such as Archid and Archeron are sufficiently triaxial rigid bodies or oblate spheroids and they are significant in the study of celestial bodies and stellar systems. The lack of sphericity of heavenly bodies causes large perturbations (Narayan *et al.* 2015). Several studies examined a circular as well as elliptic restricted three-body problem with or without the triaxial and oblateness perturbations, and/or radiating sources. [e.g. (Singh and Ishwar 1999, Mital *et al.* 2009, Ishwar and Kushvah 2006, AbdulRaheem and Singh 2006, AbdulRaheem and Singh 2008, Vishnu Namboori *et al.* 2008, Kumar and Ishwar 2009)]. Abd El-Salam (2015) discussed the elliptic restricted three-body problem with oblate and triaxial primaries using the pulsating coordinates. The author found that the stability regions depend on (i) the eccentricity of the orbits, (ii) the oblateness coefficient, and (iii) the triaxial parameters of the primaries. Another observation is that, when  $e = 0.25$ , the stability region is destroyed for the Earth-Moon like system. Moreover, Elshaboury *et al.* (2016) considered the restricted three-body problem when the primaries are triaxial rigid bodies, and studied its basic dynamical features. They found that the triangular points are conditionally stable. Moreover, Yamada and Asada (2016) investigated the gravitational radiation reaction to Lagrange's equilateral triangular solution of the three-body problem in an analytic method. They found that the triangular configuration is adiabatically shrinking and is kept in equilibrium by increasing the orbital frequency due to the radiation reaction if the mass ratios satisfy the Newtonian stability condition.

This work is a continuation of previous studies that attempt to accurately identify the location of Lagrangian points and investigate their stability under perturbation forces. The goal of this paper is to study locations and the linear stability of the triangular points under effects due to the triaxiality of the more massive primary, the oblateness of the less massive one, as well as the inclusion of relativistic corrections.

## 2. EQUATIONS OF MOTION

Let  $m_1$  and  $m_2$  denote the masses of the bigger and smaller primaries, respectively, and let  $m$  be the mass of an infinitesimal body. We take the units so that the sum of the masses and the distance between the two primaries is unity. Also, the unit of time is chosen to make the gravitational constant unity. Let the mass ratio  $\mu$  be defined as  $\mu = m_2/(m_1 + m_2)$  and therefore  $1 - \mu = m_1/(m_1 + m_2)$  with  $m_1 > m_2$ . The coordinates of  $m_1$  and  $m_2$  are  $(-\mu, 0)$  and  $(1 - \mu, 0)$ , respectively. Let  $\sigma_i$  ( $i = 1, 2$ ) and  $A_2$  be the triaxiality and oblateness coefficients of the bigger and smaller primaries, respectively, where  $\sigma_i, A_2 \ll 1$ . The x-axis is taken along the line joining the primaries in which  $r_1$  and  $r_2$  are the distances of  $m$  from  $m_1$  and  $m_2$ , respectively. A sketch to illustrate the system is given in Fig. 1.

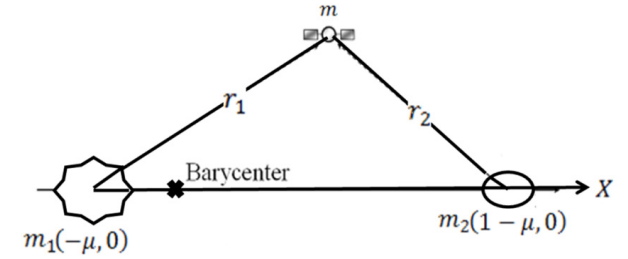


Fig. 1. A sketch of the restricted three-body system.

The equations of motion of an infinitesimal body in the elliptic restricted problem with a triaxial and oblate primaries in a dimensionless, barycentric, and non-uniformly rotating system following McCusky (1963) are given by:

$$\frac{d^2x}{df^2} - 2f \frac{dy}{df} = (1 + e \cos f)^{-1} \frac{\partial U_e}{\partial x},$$

$$\frac{d^2y}{df^2} + 2f \frac{dx}{df} = (1 + e \cos f)^{-1} \frac{\partial U_e}{\partial y},$$

where  $f$  is the true anomaly of  $m_1$ ,  $e$  is the eccentricity of the orbit, and  $U_e$  is a potential-like function of the system which is given by:

$$\begin{aligned} U_e = & \frac{1}{2} (x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{n^2} \left[ \frac{(1-\mu)}{2r_1^3} (2\sigma_1 - \sigma_2) \right. \\ & - \frac{3(1-\mu)}{2r_1^5} (\sigma_1 - \sigma_2) y^2 + \frac{\mu A_2}{2r_2^3} \\ & + \frac{1}{c^2} \left\{ \frac{(x^2 + y^2)}{2} [\mu(1-\mu) - 3] + \frac{1}{8} [(\dot{x} + y)^2 \right. \\ & + (\dot{y} - x)^2]^2 + \frac{3}{2} \left( \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \right) [(\dot{x} + y)^2 \\ & + (\dot{y} - x)^2] - \frac{1}{2} \left( \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \right)^2 - \frac{\mu(1-\mu)}{2} \left[ \frac{1}{r_1} \right. \\ & + \left( \frac{1}{r_1} - \frac{1}{r_2} \right) (1 - 3\mu - 7x - 8\dot{y}) + y^2 \left( \frac{\mu}{r_1^3} \right. \\ & \left. \left. \left. + \frac{(1-\mu)}{r_2^3} \right) \right] \right\} \left. \right]. \end{aligned} \quad (1)$$

This potential-like function of the restricted relativistic elliptic three-body problem (RRE3BP) is the sum of three components:  $U_e = U_c + U_{obl.\&tri.} + U_r$ , which are:  $U_c$  the classical potential,  $U_{obl.\&tri.}$  the potential due to triaxiality and oblateness of the bigger and smaller primaries, and  $U_r$  the potential due to the relativistic corrections (Katour et al. 2014, Jagadish and Jessica 2010). The solution of equations  $\frac{\partial U_e}{\partial x} = \frac{\partial U_e}{\partial y} = 0$  with  $y \neq 0$  gives the locations of the triangular equilibrium points. When the primaries are neither oblate spheroid nor triaxial, i.e when  $\sigma_i = A_2 = 0$  ( $i = 1, 2$ ), and  $\frac{1}{c^2} \ll 1$ , the solution of these equations are  $r_1 = r_2 = 1$ , which is the known equilateral solution of the classical restricted three-body problem (Szebehely 1967). Hence, taking into consideration the perturbations due to the participating bodies, we can assume that:

$$r_i = 1 + \delta_i, \quad \delta_i \ll 1, (i = 1, 2), \quad (2)$$

where  $\delta_i$  are small quantities. The locations of the triangular points  $x_{L_{4,5}}$ ,  $y_{L_{4,5}}$  for the studied system as given by Zahra et al. (2016) are:

$$\begin{aligned} x_{L_{4,5}} &= \frac{1}{2} - \mu + \frac{1}{n^2} \frac{D_2 E_1 - D_1 E_2}{E_1 F_2 - E_2 F_1} - \frac{1}{n^2} \frac{D_2 F_1 - D_1 F_2}{E_2 F_1 - E_1 F_2} \\ y_{L_{4,5}} &= \pm \frac{\sqrt{3}}{2} \left[ 1 - \frac{2}{3n^2} \left( \frac{D_2 F_1 - D_1 F_2}{E_2 F_1 - E_1 F_2} + \frac{D_2 E_1 - D_1 E_2}{E_1 F_2 - E_2 F_1} \right) \right], \end{aligned} \quad (3)$$

where the coefficients  $D_1$ ,  $E_1$ ,  $F_1$ ,  $D_2$ ,  $E_2$ , and  $F_2$  are functions of  $\sigma_i$ ,  $A_2$ ,  $\mu$  and  $1/c^2$ . Calculations of these coefficients are discussed in details in Zahra et al. (2016).

### 3. THE STABILITY OF THE TRIANGULAR POINTS

To examine the stability of orbits in the vicinity of triangular libration points we linearized the equations of motion around an equilibrium point. Let  $x = x_0 + \Delta x$ ,  $y = y_0 + \Delta y$ , where  $\Delta x$  and  $\Delta y$  are small displacements in  $(x_0, y_0)$  so that the linearized equations can be written as:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{1}{1 + \text{ecos } f} [x U_{xx}^{L_{4,5}} + y U_{xy}^{L_{4,5}}], \\ \ddot{y} + 2\dot{x} &= \frac{1}{1 + \text{ecos } f} [x U_{xy}^{L_{4,5}} + y U_{yy}^{L_{4,5}}]. \end{aligned} \quad (4)$$

Thus, the characteristic equation corresponding to the linearized equations is:

$$\lambda^4 - (U_{xx}^{L_{4,5}} + U_{yy}^{L_{4,5}} - 4) \lambda^2 + U_{xx}^{L_{4,5}} U_{yy}^{L_{4,5}} - (U_{xy}^{L_{4,5}})^2 = 0. \quad (5)$$

To carry out the stability analysis, let us suppose that  $\lambda = i\phi$ , so that Eq. (5) can be re-written in the form:

$$\phi^4 - \mathcal{U}_{L_{4,5}} \phi^2 + \mathcal{V}_{L_{4,5}} = 0, \quad (6)$$

where:

$$\mathcal{U}_{L_{4,5}} = 4 - (U_{xx}^{L_{4,5}} + U_{yy}^{L_{4,5}}),$$

$$\mathcal{V}_{L_{4,5}} = U_{xx}^{L_{4,5}} U_{yy}^{L_{4,5}} - (U_{xy}^{L_{4,5}})^2.$$

The roots of Eq. (6) determine stability of the triangular Lagrangian points. The required partial derivatives of the potential  $U_e$  with respect to  $x$  and  $y$ , after setting the locations of  $L_{4,5}$  at  $r_1 = 1 + \delta_1$  and  $r_2 = 1 + \delta_2$  and retaining terms up to the first order in the small quantities  $\delta_1$  and  $\delta_2$ , can be written as follows:

$$\begin{aligned} U_{xx}^{L_{4,5}} &= 1 - [(1-\mu)(1-3\delta_1) + \mu(1-3\delta_2)] + 3[(1-\mu)(x \\ &+ \mu)^2(1-5\delta_1) + \mu(x+\mu-1)^2(1-5\delta_2)] + \frac{1}{n^2} [-3\mu \times \\ &A_2(1-5\delta_1) + y^2(1-\mu)(\sigma_1 - \sigma_2)[7.5(1-7\delta_1) \\ &- 52.5(x+\mu)^2(1-9\delta_1)] - 1.5(1-\mu)(2\sigma_1 - \sigma_2) \times \\ &(1-5\delta_1) + 7.5A_2\mu(x+\mu-1)^2(1-7\delta_2) + 7.5(1-\mu) \times \\ &(x+\mu)^2(2\sigma_1 - \sigma_2)(1-7\delta_1) + \frac{1}{c^2} \{(\mu - \mu^2 - 3) + 0.5x^2 \\ &+ 0.5(x^2 + y^2) + [(1-\mu)(1-\delta_1) + \mu(1-\delta_2)] [(1-\mu) \times \\ &(1-3\delta_1) + \mu(1-3\delta_2)] - 3[(1-\mu)(x+\mu)^2(1-5\delta_1) \\ &+ \mu(x+\mu-1)^2(1-5\delta_2)] - [(1-\mu)(x+\mu)(1-3\delta_1) \\ &+ \mu(x+\mu-1)(1-3\delta_2)]^2 - 6x[(1-\mu)(x+\mu)(1-3\delta_1) \\ &+ \mu(x+\mu-1)(1-3\delta_2)] - 1.5(x^2 + y^2) [(1-\mu)(x+\mu) \times \\ &(1-3\delta_1) + \mu(x+\mu-1)(1-3\delta_2)] - 3[(1-\mu)(x+\mu)^2 \times \\ &(1-5\delta_1) + \mu(x+\mu-1)^2(1-5\delta_2)] + 3[(1-\mu) \times \\ &(1-\delta_1) + \mu(1-\delta_2)] - 0.5\mu(1-\mu)[3(x+\mu)^2(1-5\delta_1) \\ &- (1-5\delta_1) + 14[(x+\mu)(1-3\delta_1) - (x+\mu-1)(1 \\ &- 3\delta_2)] + (-1 + 3\mu + 7x)[[3(x+\mu)^2(1-5\delta_1) - 3 \times \\ &(x+\mu-1)^2(1-5\delta_1)] - (1-3\delta_1) + (1-3\delta_2)] + y^2 \times \\ &[15(1-\mu)(-1+x+\mu)^2(1-7\delta_2) - 3(1-\mu)(1-5\delta_2) \\ &+ 15\mu(x+\mu)^2(1-7\delta_1) - 3\mu(1-7\delta_2)]\}], \end{aligned} \quad (7)$$

$$\begin{aligned} U_{yy}^{L_{4,5}} &= 1 - [(1-\mu)(1-3\delta_1) - \mu(1-3\delta_2)] + y^2(3\mu \times \\ &(1-5\delta_2) + 3(1-\mu)(1-5\delta_1)) + \frac{1}{n^2} [-1.5(1 \\ &-\mu)(2\sigma_1 - \sigma_2)(1-5\delta_2) - 3(1-\mu)(\sigma_1 - \sigma_2) \times \\ &(1-5\delta_1) - 1.5\mu A_2(1-5\delta_2) + y^2[7.5\mu A_2(1 \\ &- 7\delta_2) + (1-\mu)(\sigma_1 - \sigma_2)[37.5(1-7\delta_1) \\ &- 52.5y^2(1-9\delta_1)] + 7.5(1-\mu)(2\sigma_1 - \sigma_2)](1 \\ &- 7\delta_1) + \frac{1}{c^2} \{[(1-\mu)(1-\delta_1) + \mu(1-\delta_2)] [3\mu y^2 \times \\ &(1-5\delta_2) + 3 \times (1-\mu)y^2(1-5\delta_2) + (1-\mu)(1 \\ &- 3\delta_1) + \mu(1-3\delta_2)] - (\mu - \mu^2 - 3) + y^2 + 0.5(x^2 \\ &+ y^2) [3(3\mu y^2(1-5\delta_2) + 3(1-\mu)y^2(1-5\delta_2) \\ &- (1-\mu)(1-3\delta_1) - \mu(1-3\delta_2))] + 1] - y^2 [6((1 \end{aligned}$$

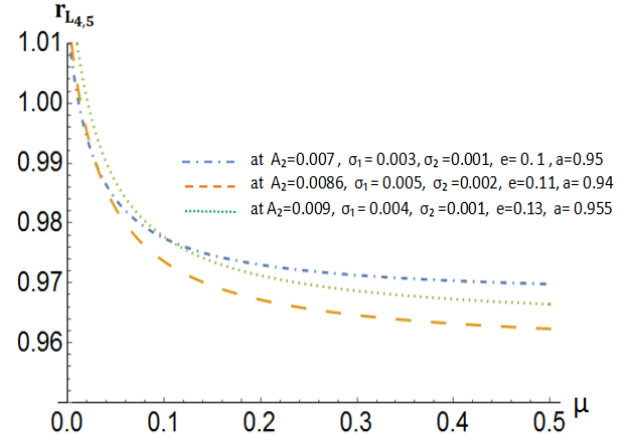
$$\begin{aligned}
& -\mu(1-3\delta_1) + \mu(1-3\delta_2)) + (- (1-\mu)(1-3\delta_1) \\
& -\mu(1-3\delta_2))^2] + 3[(1-\mu)(1-\delta_1) + \mu(1-\delta_2)] \\
& -0.5\mu(1-\mu)[3y^2(1-5\delta_2) - (1-3\delta_2) + y^2[15 \times \\
& (1-\mu)y^2(1-7\delta_2) - 3(1-\mu)(1-5\delta_2) + 15\mu y^2(1 \\
& -7\delta_1) - 3\mu(1-5\delta_1) + (-1+3\mu+7x)(-3y^2(1 \\
& -5\delta_2) + 3y^2(1-5\delta_1) - (1-3\delta_1) + (1-3\delta_2)) \\
& + 2((1-\mu)(1-3\delta_2) + \mu(1-3\delta_1)) - 12y^2((1-\mu) \times \\
& (1-5\delta_2) + \mu(1-5\delta_1))] \}, \quad (8)
\end{aligned}$$

and:

$$\begin{aligned}
U_{xy}^{L_{4,5}} = & (1-\mu)(x+\mu)(1-5\delta_1)y \left[ 3 + \frac{1}{n^2} [-52.5(\sigma_1 \right. \\
& -\sigma_2)(1-4\delta_2)y^2 + 15(\sigma_1 - \sigma_2)(1-2\delta_1) \\
& + 7.5(2\sigma_1 - \sigma_2)(1-2\delta_1)] + 3\mu(x+\mu-1)y \times \\
& (1-5\delta_2)(1+2.5A_2(1-2\delta_1)) - \frac{1}{c^2} \{ [(1-\mu)(x \\
& +\mu)(1-3\delta_2) + \mu(x+\mu-1)(1-3\delta_1)] [(1-\mu) \times \\
& (1-3\delta_1) + \mu(1-3\delta_2)] y - 4.5y(x^2 + y^2) [(1 \\
& -\mu)(x+\mu)(1-5\delta_1) + \mu(x+\mu-1)(1-5\delta_2)] \\
& + 3xy[(1-\mu)(1-3\delta_1) + \mu(1-3\delta_2)] - xy \\
& + 3y[(1-\mu)(x+\mu)(1-3\delta_1) + \mu(x+\mu-1)(1 \\
& -3\delta_2) + [(1-\mu)(x+\mu)(1-5\delta_1) + (1-5\delta_2) \times \\
& \mu(x+\mu-1)] [(1-\mu)(1-\delta_1) + \mu(1-\delta_2)] \\
& + 0.5\mu(1-\mu)[3(x+\mu)y(1-5\delta_1) + 15y^3[(1 \\
& -\mu)(x+\mu-1)(1-7\delta_2) + \mu(x+\mu)(1-7\delta_1)] \\
& - 3y(-1+3\mu+7x)[(x+\mu)(1-5\delta_1) - (x+\mu \\
& -1)(1-5\delta_2)] - 7y[(1-3\delta_1) + (1-3\delta_2)] \\
& - 6y[(1-\mu)(x+\mu-1)(1-5\delta_2) + \mu(x+\mu) \times \\
& (1-5\delta_1)] \} \}. \quad (9)
\end{aligned}$$

#### 4. NUMERICAL REPRESENTATIONS AND DISCUSSION

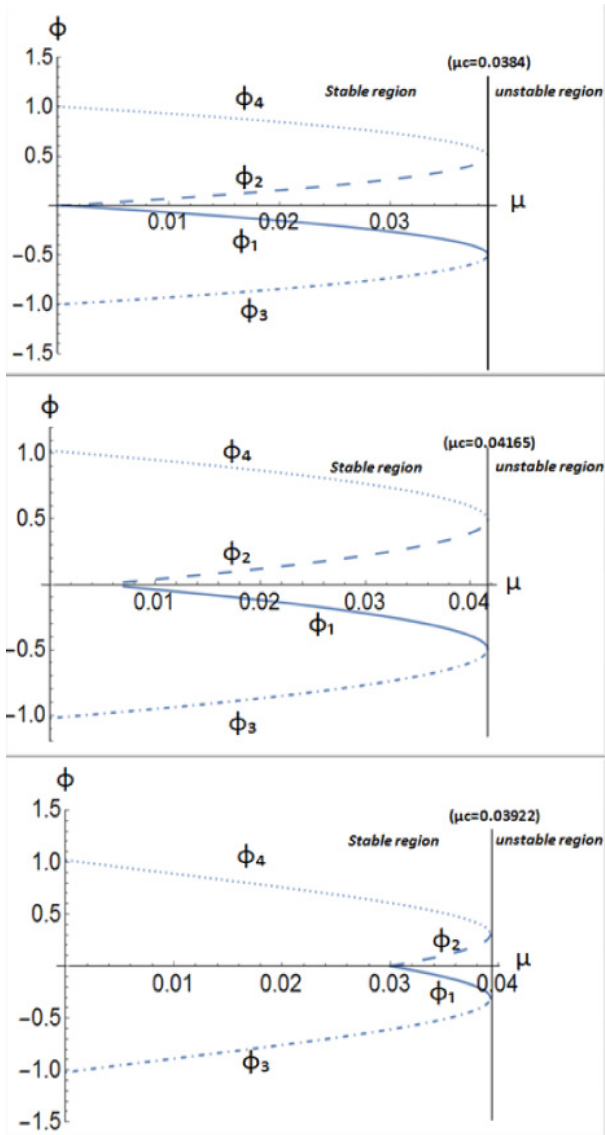
In this work, we investigated the linear stability of triangular points for the RRE3BP in which one of the two primaries is triaxial  $m_1$  and the other is an oblate spheroid  $m_2$ . The variation in position of Lagrangian points  $(x_{L_{4,5}}, y_{L_{4,5}})$  has been calculated numerically for several dynamical systems and the shift from the classical case has also been obtained, see Table 1. In Fig. 2, we illustrate variation in magnitude of the radius vector of the two triangular points,  $r_{L_{4,5}}$ , throughout the whole range of the mass ratio  $\mu$  taking into account combined effects of perturbed forces. From our numerical results, it is clear that the considered force model shifts the location of the triangular points from the classical case.



**Fig. 2.** Variations in location of the triangular points versus the mass ratio  $\mu$ , for different dynamical systems with relativistic corrections (see the plot key).

**Table 1:** Comparison of positions of triangular points  $(x_{L_{4,5}}, y_{L_{4,5}})$  in classical  $(x_{L_{4,5}}, y_{L_{4,5}})_{cl}$  and perturbed  $(x_{L_{4,5}}, y_{L_{4,5}})_{per}$  cases for several dynamical systems in which the oblateness of the second primary is set to  $A_2 = 0.0018$ .

Syst- em	Mass ratio $\mu$	Perturbation factors				$(x_{L_{4,5}})_{cl}$	$(x_{L_{4,5}})_{per}$	$\Delta x_{L_{4,5}}$	$(\pm y_{L_{4,5}})_{cl}$	$(\pm y_{L_{4,5}})_{per}$	$\pm \Delta y_{L_{4,5}}$
		$\sigma_1$	$\sigma_2$	$a$	$e$						
1	0.001	0.003	0.001	0.9	0.01	0.499	0.471180652573	0.02781	0.8660	0.849990	0.01601
2	0.003	0.004	0.004	0.91	0.04	0.497	0.456152542006	0.04547	0.8660	0.842510	0.02349
3	0.008	0.005	0.003	0.92	0.08	0.492	0.465959051389	0.02604	0.8660	0.851189	0.014811
4	0.01	0.006	0.002	0.93	0.1	0.490	0.441303840394	0.04869	0.8660	0.838122	0.027878
5	0.02	0.007	0.003	0.94	0.2	0.480	0.443797947233	0.036203	0.8660	0.845459	0.020541
6	0.028	0.008	0.006	0.95	0.3	0.472	0.458411841108	0.013589	0.8660	0.858587	0.007413
7	0.03	0.02	0.01	0.98	0.4	0.470	0.416030856127	0.053969	0.8660	0.83516	0.03084
8	0.04	0.04	0.025	1.08	0.5	0.460	0.403007814005	0.056993	0.8660	0.82878	0.03722



**Fig. 3.** The stability regions (from Eq. (6)) for a range of mass ratios, for several dynamical systems: panel (a) is the classical case where  $A_2 = \sigma_1 = \sigma_2 = 0$ , panels (b and c) are the systems:  $A_2 = 0.00018$ ,  $\sigma_1 = \sigma_2 = 0$ , and  $A_2 = 0.00018$ ,  $\sigma_1 = 0.06$ ,  $\sigma_2 = 0.008$ , respectively. The critical mass ratio,  $\mu_c$ , is also shown for each system. Different line styles indicate the four roots  $\phi_i$ .

The stability of the triangular points was also investigated by solving Eq. (6) for several dynamical systems including the classical case of spherical objects. The critical mass ratio,  $\mu_c$ , after which the system becomes unstable, was also computed. Calculations of the critical mass ratio for several dynamical systems are listed in Table 2. Some of these systems are represented in the set of Fig. 3 for a mass ratio  $\mu \in (0, 0.05)$ . In Fig. 3, we noticed that

the stability regions are shifted in each case due to the inclusion of considered perturbations. In all the studied cases, the stability regions are found to be completely destroyed at critical values in the range  $0.03282 \leq \mu_c \leq 0.04623$ .

**Table 2.** The calculated values of the critical mass ratio  $\mu_c$  for different dynamical systems.

System	Parameters of the primaries			Critical mass ratio
	$m_1$		$m_2$	
	$\sigma_1$	$\sigma_2$	$A_2$	$\mu_c$
1	0	0	0	0.0384
2	0	0	0.00018	0.04165
3	0.006	0.001	0.00018	0.04530
4	0.005	0	0.0002	0.04549
5	0.05	0.002	0.0003	0.04270
6	0	0.004	0.001	0.03912
7	0.06	0.008	0.0018	0.03922
8	0.05	0.01	0.0019	0.04623
9	0.07	0.02	0.002	0.03282
10	0.08	0.04	0.008	0.04391

## 5. CONCLUSION

We studied the linear stability of triangular equilibrium points  $L_{4,5}$  of an elliptic restricted three-body problem, in the post-Newtonian framework. The two primaries were chosen so that the more massive primary is triaxial and the less massive one is an oblate spheroid. We included the relativistic corrections as an additional perturbing effect. For the locations of the triangular points, computed by Zahra et al. (2016), we found that these locations are affected by the perturbations considered in this work. The stability regions were also tested and the results showed that these regions are affected by the perturbing factors included in this work. We found that, under combined effects of these perturbed forces, the triangular points are stable for  $0 < \mu < \mu_c$ , and unstable for  $\mu_c \leq \mu \leq 0.5$ .

## REFERENCES

- Abd El-Salam, F. A.: 2015, *Astrophys. Space Sci.*, **357**, 15.  
 AbdulRaheem, A. and Singh, J.: 2006, *Astron. J.*, **131**, 1880.  
 AbdulRaheem, A. and Singh, J.: 2006, *Astrophys. Space Sci.*, **317**, 393.  
 Ahmed, M., Abd El-Salam, K. F. A. and Abd El-Bar, S. E.: 2006, *American Journal of Applied Sciences*, **3**, 1993.

- Bhatnagar, K. B. and Hallan, P. P.: 1998, *Celest. Mech. Dyn. Astron.*, **69**, 271.
- Douskos, C. N. and Perdios, E. A.: 2002, *Celest. Mech. Dyn. Astron.*, **28**, 317.
- Elshaboury, S. M., Abouelmagd, E. I., Kalantonis, V. S. and Perdios, E. A.: 2016, *Astrophys. Space Sci.*, **361**, 315.
- Ichita, T., Yamada, K. and Asada H.: 2011, *Physical Review D*, **83**, 084026.
- Ishwar, B. and Kushvah, B. S.: 2006, *J. Dyn. Syst. Geom. Theories*, **4**, 79.
- Katour, D. A., Abd El-Salam, F. A. and Shaker, M. O.: 2014, *Astrophys. Space Sci*, **351**, 143.
- Kumar, S. and Ishwar, B.: 2009, *AIP Conf. Proc.*, **1146**, 456.
- McCusky, S. W.: 1963, Introduction to Celestial Mechanics, Addison-Wesley Pub. Co..
- Mital, A., Ahmad, I. and Bhatnagar, K. B.: 2009, *Astrophys. Space Sci*, **323**, 65.
- Musielak, Z. E. and Quarles, B.: 2014, *Rep. Prog. Phys.*, **77**, 065901.
- Narayan A., Pandey, K. K. and Shrivastava, S. K.: 2015, *International Journal of Advanced Astronomy*, **3**, 97.
- Palit, A., Panchenko, A., Migranov, N. G., Bhadra, A. and Nandi, K. K.: 2009, *International Journal of Theoretical Physics*, **48**, 1271.
- Singh, J. and Ishwar, B.: 1999, *Bull. Astr. Soc. India*, **27**, 415.
- Singh, J. and Jessica, M. B.: 2010, *Astrophys. Space Sci*, **332**, 319.
- Szebehely, V.: 1967, Theory of Orbits: The Restricted Problem of Three Bodies, Academic Press, New York and London.
- Vishnu Namboodiri, N. I., Reddy, D. S. and Sharma, R. K.: 2008, *Astrophys. Space Sci.*, **318**, 161.
- Yamada, K. and Asada, H.: 2010, *Physical Review D*, **82**, 104019.
- Yamada, K. and Asada, H.: 2011, *Physical Review D*, **83**, 024040.
- Yamada, K. and Asada, H.: 2012, *Physical Review D*, **86**, 124029.
- Yamada, K., Tsuchiya, T. and Asada, H.: 2015, *Physical Review D*, **91**, 124016.
- Yamada, K. and Asada, H.: 2016, *Physical Review D*, **93**, 084027.
- Zahra, K., Radwan, M. and Awad, M. E.: 2016, *Non-linear Analysis and Differential Equations*, **4**, 609.

## О СТАБИЛЬНОСТИ ТРИАНГУЛАРНЫХ ТАЧАКА У РЕЛАТИВИСТИЧКОМ ЕЛИПТИЧКОМ ОГРАНИЧЕНОМ ПРОБЛЕМУ ТРИ ТЕЛА СА ТРООСНИМ СПЉОШТЕНИМ ПРИМАРНИМ КОМПОНЕНТАМА

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Оригинални научни рад

Овај рад истражује положаје и линеарну стабилност триангуларних тачака услед комбинованог пертурбативног дејства: троосне масивније примарне компоненте, спљоштености секундарне компоненте и релативистичких корекција. Претпоставља се да се примарна компонента креће по елиптичкој орбити око заједничког центра

маса. Нађено је да на положај триангуларних тачака утичу наведене пертурбације и анализирана је стабилност орбита у околини ових тачака. Примећено је да су тачке стабилне за однос маса  $\mu$  у распону  $0 < \mu < \mu_c$ , где је  $\mu_c$  критичан однос маса, а нестабилне за  $\mu_c \leq \mu \leq 0.5$ .