Compressibility of Fluids

Bulk Modulus:

The compressibility of the fluid expresses how the volume and the density of a given mass of the fluid be change under the effect of an applied pressure. The ***bulk modulus of elasticity*** is defined as

$$E_v = -\frac{dP}{d\mathbb{V}/\mathbb{V}}$$

where dP is the differential change in pressure,
$d\mathbb{V}$ is a differential change in volume,
and $\mathbb{V}$ is the initial volume.
The negative sign indicates that the increase in pressure causes a decrease in volume.

$$\because m = \rho V$$

$$\therefore\ E_v = \frac{dP}{d\rho/\rho}$$

E_v has the units of Pa=N/m^2 (in SI) or psi= lb/in^2 (in BG).

Large values for the bulk modulus indicate that the fluid is relatively ***incompressible***. Whereas liquids are usually considered to be incompressible, gases are generally considered compressible.

Chapter 3
Elementary Fluid Dynamics
Bernoulli Equation

The inviscid fluid flow is governed by pressure and gravity forces. In such case, Newton's second law reads

Net pressure force on a particle + net gravity force on particle
= (*particle mass*) (*particle acceleration*)

A *streamline*: A line is everywhere tangent to the velocity field. If the flow is steady, the streamlines are fixed lines in space. For unsteady flows the streamlines may change with time.

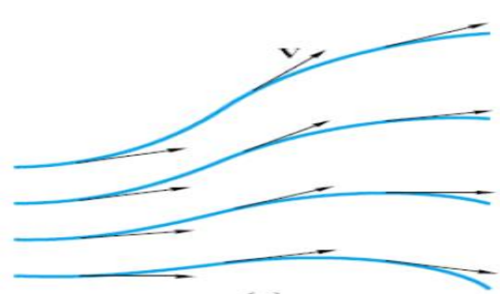

Fig.1
Streamlines

As the fluid particle moves, both gravity and pressure forces do work on the particle. The work-energy principle states that "*The work done on a particle by all forces acting on the particle is equal to the change of the kinetic energy of the particle.*". Along the streamline of a given flow, we can express work-energy principle as

$$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = contant$$

This is called ***Bernoulli equation***.

Here,

- dP is the variation in pressure at a given point.
- ρ is the density of the fluid.
- V is the fluid velocity.
- g is the acceleration due to gravity.
- z is the elevation.

The elevation term (gz) is related to the potential energy of the particle. Bernoulli equation is the most used and the most abused equation in fluid mechanics.

For a steady, inviscid, incompressible flow, along the streamline

$$P + \frac{\mathbf{1}}{2}\rho V^2 + \gamma z = constant\ .$$

$\gamma = \rho g$ is the specific weight of the fluid (N/m^3 or lb/ft^3).

This indicates that between two points (1) and (2) **along a given streamline** of inviscid steady incompressible flow,

$$\boldsymbol{P_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2}.$$

Across the stream line, the sum of pressure, elevation, and velocity effects is constant.

$$\boldsymbol{\int \frac{dP}{\rho} + \int \frac{V^2}{\mathcal{R}} d\hat{n} + gz = contant}.$$

Here, $\mathcal{R}$ is the local radius of curvature of the streamlines.

$\hat{n}$ defines the normal direction.

For steady, inviscid, incompressible flow, across the stream line

$$\boldsymbol{P + \rho \int \frac{V^2}{\mathcal{R}} d\hat{n} + \gamma z = contant}\ .$$

Example 1:

Consider the inviscid, incompressible, steady flow shown in Fig. 2. From section *A* to *B* the streamlines are straight, while from *C* to *D* they follow circular paths. Describe the pressure variation between points (1) and (2), and points (3) and (4).

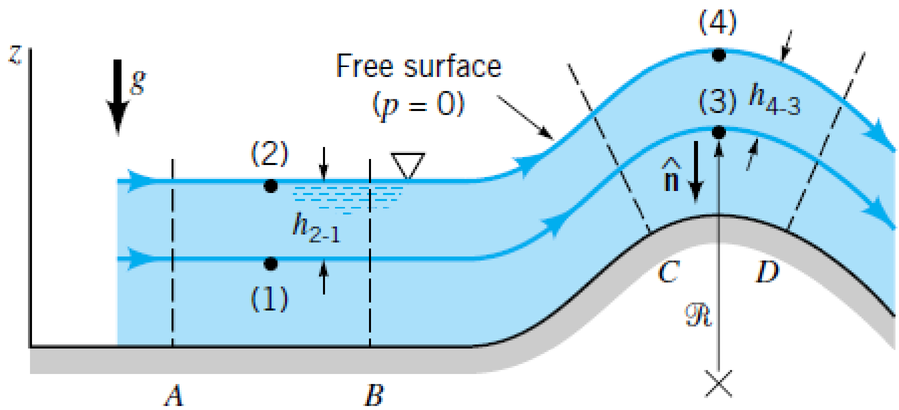

Fig. 2

Solution:

With the above assumptions and the fact that $\mathcal{R} = \infty$ for the portion from *A* to *B*, Eq. 3.14 becomes

$$p + \gamma z = \text{constant}$$

The constant can be determined by evaluating the known variables at the two locations using $p_2 = 0$ (gage), $z_1 = 0$, and $z_2 = h_{2-1}$ to give

$$p_1 = p_2 + \gamma(z_2 - z_1) = p_2 + \gamma h_{2-1} \qquad \textbf{(Ans)}$$

Note that since the radius of curvature of the streamline is infinite, the pressure variation in the vertical direction is the same as if the fluid were stationary.

However, if we apply Eq. 3.14 between points (3) and (4) we obtain (using $dn = -dz$)

$$p_4 + \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}}(-dz) + \gamma z_4 = p_3 + \gamma z_3$$

With $p_4 = 0$ and $z_4 - z_3 = h_{4-3}$ this becomes

$$p_3 = \gamma h_{4-3} - \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}} dz \qquad \textbf{(Ans)}$$

To evaluate the integral, we must know the variation of V and $\mathcal{R}$ with z. Even without this detailed information we note that the integral has a positive value. Thus, the pressure at (3) is less than the hydrostatic value, γh_{4-3}, by an amount equal to $\rho \int_{z_3}^{z_4} (V^2/\mathcal{R})\, dz$. This lower pressure, caused by the curved streamline, is necessary to accelerate the fluid around the curved path.

Confined Flows

Consider a fluid flowing through a fixed volume such as a tank that has one inlet (1) and one outlet (2) as shown in Fig. 3.

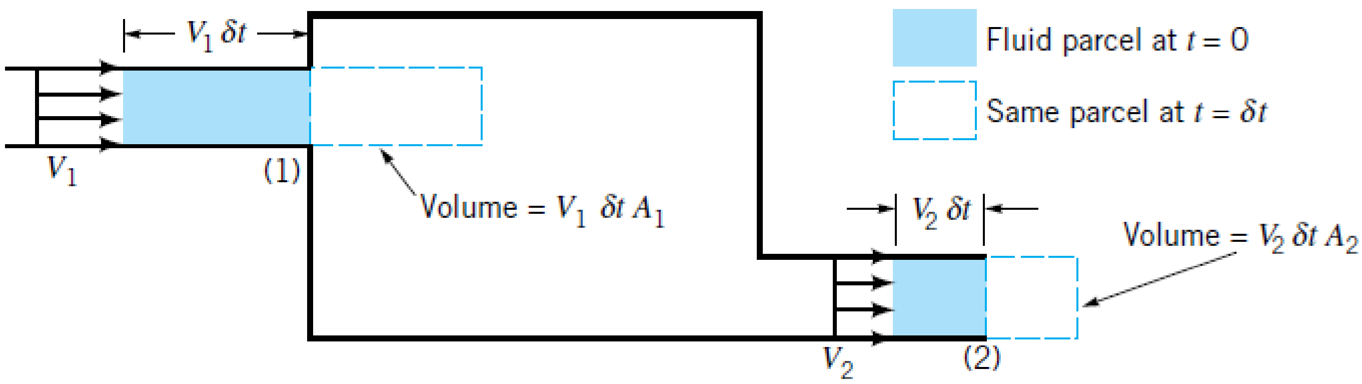

Fig.3: Steady flow into and out of tank.

If the flow is steady,

∴ There is no additional accumulation of fluid within the tank volume.

The continuity equation (conservation of mass) states that "*mass cannot be created or destroyed*".

∴ The rate of fluid flow into the tank (at 1) = The rate of flow out of the tank (at 2)

$$\boldsymbol{The\ mass\ flowrate} = \frac{\boldsymbol{dm}}{\boldsymbol{dt}} = \dot{\boldsymbol{m}} \quad (\boldsymbol{slugs/s} \equiv \boldsymbol{kg/s})$$

$$\dot{\boldsymbol{m}} = \frac{\boldsymbol{d(\rho\mathbb{V})}}{\boldsymbol{dt}} = \boldsymbol{\rho}\frac{\boldsymbol{d\mathbb{V}}}{\boldsymbol{dt}} = \boldsymbol{\rho Q}$$

For an incompressible fluid, $\boldsymbol{\rho}$ is constant.

Q is the ***volume flowrate*** ≡ ***the volume per unit time(*** $ft^3/s \equiv m^3/s$).

The volume of the fluid crossing the perpendicular outlet area (A) in a time (δt) with a velocity (V) is $\boldsymbol{VA\delta t}$.

$$\therefore Q(the\ volume\ per\ unit\ time) = \frac{\boldsymbol{d\mathbb{V}}}{\boldsymbol{dt}} = \frac{\boldsymbol{VA\delta t}}{\boldsymbol{\delta t}} = \boldsymbol{VA}$$

and,

$$\dot{\boldsymbol{m}} = \boldsymbol{\rho Q} = \boldsymbol{\rho VA}$$

For a steady flow, $\dot{\boldsymbol{m}}\,(\boldsymbol{1}) = \dot{\boldsymbol{m}}\,(\boldsymbol{2})$

$$\therefore \boldsymbol{\rho_1 V_1 A_1 = \rho_2 V_2 A_2}$$

For an incompressible fluid ($\rho = constant$), we have

$$\boldsymbol{V_1 A_1 = V_2 A_2}$$

$$\therefore \boldsymbol{Q_1 = Q_2}$$

Example 2:

A stream of water of diameter d=0.1 m flows steadily from a tank of diameter D=1.0 m as shown in Fig. 4(a). Determine the flowrate (Q) needed from the inflow pipe if the water depth remains constant, h = 2.0 m.

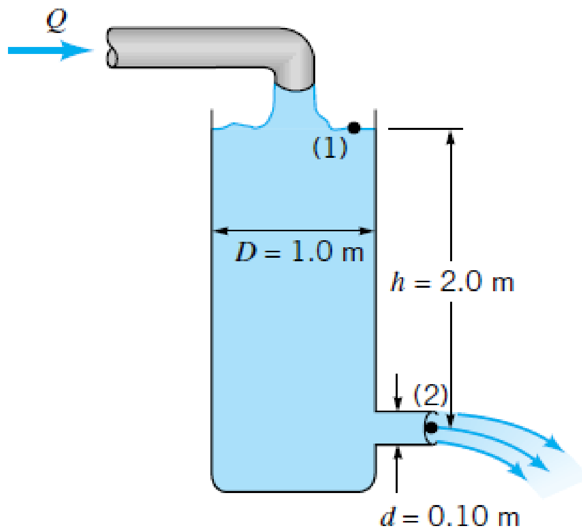

Fig. 4

For steady, inviscid, incompressible flow, the Bernoulli equation applied between points (1) and (2) is

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 \qquad (1)$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2 \qquad (2)$$

Although the water level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq. 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \qquad (3)$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}} = \sqrt{\frac{2(9.81\text{ m/s}^2)(2.0\text{ m})}{1-(0.1\text{m}/1\text{m})^4}} = 6.26\text{ m/s}$$

Thus,

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\text{ m})^2(6.26\text{ m/s}) = 0.0492\text{ m}^3\text{/s} \qquad \textbf{(Ans)}$$

Problems: (3.26, 3.30, Example 3.8)

References: "***Fundamentals of Fluid Mechanics***"; Munson, Young and Okishi, Ch. 3