Effect of Spin-Torsion Interaction on Raychaudhuri Equation

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Abstract

Raychaudhuri equation is generalized in the parameterized absolute parallelism geometry. This version of absolute parallelism is more general than the conventional one. The generalization takes into account the suggested interaction between the quantum spin of the moving elementary particle and the torsion of the background gravitational field. The generalized Raychaudhuri equation obtained contains some extra terms, depending on the torsion of space-time, that would have some effects on the singularity theorems of general relativity. Under a certain condition, this equation could be reduced to the original Raychaudhuri equation without any need for a vanishing torsion.

KEY WORDS: Singularity - Absolute Parallelism Geometry- Path Equations, Anti-gravity, Torsion

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1 Introduction

It is well known that the Raychaudhuri equation plays an essential role in the study of space-time singularities [1]. Singularity theorems, established using this equation, show that the existence of singularities, in the solutions of general relativity (GR), is inevitable. Several attempts have been done to generalize or modify the Raychaudhuri equation in the hope that GR or other geometric field theories will be free from such singularities (cf.[2]). However, Senovilla [3] obtained some solutions of the field equations of GR which are not singular, by relaxing some of the assumptions of the singularity theorems of GR. Thus it is necessary to examine the roots of the singularity theorems, i.e. the Rayshaudhuri equation. This may give rise to other factors that affect the existence of singularities without any need to relax the assumptions mentioned above.

It is widely accepted that a collapsed object, before approaching a singular state, passes through a state in which matter degenerates into its elementary constituents. GR assumes that motion of elementary particles in a gravitational field is along geodesics of the metric, regardless of the spin of these particles. On the other hand, Raychaudhuri equation depends on the validity of the geodesic motion of these particles. If there is an interaction between the spin of the moving elementary particle and the background gravitational field, then there will be deviation from the geodesic motion and consequently the Raychaudhuri equation is to be modified to take this interaction into account.

One of the authors [4] suggested a version of *"Absolute Parallelism"* (AP)-Spaces in which curvature and torsion are simultaneously non-vanishing objects. This type of structure is known as the *"Parameterized Absolute Parallelism"* (PAP) space. Among other things, the non-symmetric connection of the AP-geometry is generalized, and consequently a new path equation is derived. This equation is suggested to represent trajectories of spinning particle in gravitational fields. The equation contains a term, that is suggested to represent a type of interaction between the quantum spin of the moving particle and the background gravitational field. This interaction has been called *"spin-gravity interaction" or "spin-torsion interaction"*. The equation, in its linearized form, has been used to interpret the discrepancy appeared in the results of the COW-experiment [5], and to discuss the time delay of massless particles coming from SN1987A [6].

The aim of the present work is to generalize the Raychaudhuri equation using the new general affine connection and the new path equation, in place of the geodesic of the metric. In section 2 a brief account on the recently established structure of the AP-space is given. Generalization of the Raychaudhuri equation with necessary definitions are given in section 3. Discussion of the results obtained and some remarks are given in section 4.

2 The Parameterized AP-geometry

The structure of the conventional AP-space is defined completely, in 4-dimensions, by a tetrad vector $\underset{i}{\lambda}^{\mu}$ ($i(=1,2,3,4)$ stands for the vector number and $\mu(=1,2,3,4)$ stands for the coordinate component) which is subject to the AP-condition,

$$\underset{i}{\lambda}^{\overset{\alpha}{+}}{}_{|\ \beta}=0 \quad . \tag{1}$$

The covariant components of $\underset{i}{\lambda}^{\alpha}$ is defined such that:

$$\underset{i}{\lambda}^{\alpha}\underset{i}{\lambda}_{\mu}=\delta^{\alpha}_{\mu}. \tag{2}$$

Summation convention is carried out over repeated indices whatever their position. The condition (1) defines a non-symmetric connection $\Gamma^{\alpha}_{.\mu\nu}(\overset{def.}{=}\underset{i}{\lambda}^{\alpha}\underset{i}{\lambda}_{\mu,\nu})$. A second order tensor, which can play the role of the metric tensor,is defined by,

$$g_{\alpha\beta}\overset{def.}{=}\underset{i}{\lambda}_{\alpha}\underset{i}{\lambda}_{\beta} \quad . \tag{3}$$

The torsion of this space is defined by the tensor,

$$\Lambda^{\alpha}_{.\mu\nu} \quad \overset{def.}{=} \quad \Gamma^{\alpha}_{.\mu\nu}-\Gamma^{\alpha}_{.\nu\mu} \quad . \tag{4}$$

Certain considerations [4] lead to the following general linear connection,

$$\nabla^{\mu}_{.\alpha\beta} \quad \overset{def.}{=} \quad a\left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\}+b\Gamma^{\mu}_{\alpha\beta} \tag{5}$$

where a, b are parameters and $\left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\}$ is the Christoffel symbol defined using (3). The tensor derivative using (5) is defined by,

$$A_{\mu||\nu} \quad \overset{def.}{=} \quad A_{.\mu,\nu}-A_{\alpha}\nabla^{\alpha}_{.\mu\nu} \quad , \tag{6}$$

$$A^{\mu}_{\ ||\nu} \quad \overset{def.}{=} \quad A^{\mu}_{,\nu}+A^{\alpha}\nabla^{\mu}_{.\alpha\nu} \quad ,$$

where A_{μ} is an arbitrary vector. Metricity is achieved upon taking the condition,

$$a+b \quad = \quad 1, \tag{7}$$

$$\text{i.e.} \qquad g_{\mu\nu||\sigma} \quad = \quad 0. \tag{8}$$

The non-commutation of the general tensor derivatives is expressed by [4]

$$A_{\mu||\nu\sigma} - A_{\mu||\sigma\nu} \quad = \quad A_{\alpha}B^{\alpha}_{.\mu\nu\sigma} - bA_{\mu||\varepsilon}\Lambda^{\varepsilon}_{.\nu\sigma} \tag{9}$$

where,

$$B^{\alpha}_{.\mu\nu\sigma} \quad \stackrel{def.}{=} \quad \nabla^{\alpha}_{.\mu\sigma,\nu} - \nabla^{\alpha}_{.\mu\nu,\sigma} + \nabla^{\alpha}_{.\varepsilon\nu}\nabla^{\varepsilon}_{.\mu\sigma} - \nabla^{\alpha}_{.\varepsilon\sigma}\nabla^{\varepsilon}_{.\mu\nu} \quad . \tag{10}$$

Using the general affine connection (5) and the metricity condition (8),the following path equation is obtained [4],

$$\frac{dZ^{\mu}}{d\tau} + \left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\} Z^{\alpha}Z^{\beta} \quad = \quad -b\Lambda_{(\alpha\beta)}^{\ \ \ \ \mu} Z^{\alpha}Z^{\beta} \qquad , \tag{11}$$

where parantheses are used for symmetrization, $Z^{\mu}(\stackrel{def.}{=} \frac{dx^{\mu}}{d\tau})$ is the tangent to the path and τ is the parameter varying along the path. The parameter (b) is suggested to take the form [4],

$$b \quad = \quad \frac{n}{2}\alpha\gamma \qquad , \tag{12}$$

where n is a natural number, α is the fine structure constant and γ is a dimensionless parameter to be fixed by experiment or observation. The term on the R.H.S. of (11) is suggested [4] to represent a type of interaction between the intrinsic quantum spin, of the moving particle and the torsion of the background gravitational field. The natural number n of (12) takes the values $0, 1, 2, \ldots$ for particles with spin $0, \frac{1}{2}, 1, \ldots$ respectively.

Equation (11) is suggested to represent the motion of spinning elementary particle in a gravitational field. It is worth of mention that for spinless particles and slowly rotating macroscopic bodies $n = 0$, and thus equation (11) will reduce to the ordinary geodesic equation.

An important note is that in the case $a = 1, b = 0$, the affine connexion (5) reduces to Christoffel symbol and the geometry becomes Riemannian, while in the case $a = 0, b = 1$, the connexion (5) reduces to $\Gamma^{\alpha}_{.\mu\nu}$ and the geometry becomes the conventional AP-geometry. The geometry, depending on the connection (5) and the condition (7), given briefly in this section is called the *"Parameterized Absolute Parallelism"* (PAP) geometry. For more details, the reader is referred to reference [7].

3 Generalization of Raychaudhuri scheme

In this section, we are going to generalize the original Raychaudhuri scheme [8], in order to study the effect of *spin-torsion interaction* on the structure of this equation.

According to the PAP-geometry, reviewed briefly in section 2, the velocity components of a spinning particle, assumed to move along (11), in a gravitational field is given by,

$$Z^{\mu} \stackrel{def.}{=} \frac{dx^{\mu}}{d\tau} \quad . \tag{13}$$

Using this vector,we can define the following tensors:

$$Z_{\mu\nu} \stackrel{def.}{=} Z_{\mu||\nu} \quad , \tag{14}$$

$$\Theta \stackrel{def.}{=} Z^{\mu}{}_{||\mu} \quad , \tag{15}$$

$$\Omega_{\mu\nu} \stackrel{def.}{=} Z_{[\mu||\nu]} \quad , \tag{16}$$

$$\Sigma_{\mu\nu} \stackrel{def.}{=} Z_{(\mu||\nu)} - \frac{1}{3}\frac{\Theta}{Z^2}P_{\mu\nu} \quad , \tag{17}$$

where brackets are used for anti-symmetrization and,

$$P_{\mu\nu} \stackrel{def.}{=} Z^2 g_{\mu\nu} - Z_{\mu}Z_{\nu} \quad , \tag{18}$$

and

$$Z^2 \stackrel{def.}{=} Z^{\mu}Z_{\mu} \quad . \tag{19}$$

The tensors $\Omega_{\mu\nu}, \Sigma_{\mu\nu}$ are generalization of the vorticity and shear tensors, in Raychaudhuri treatment respectively, while the scalar Θ is a generalization of the expansion.The tensor $P_{\mu\nu}$ can be used as a generalization of the projection tensor, as it projects any tensor on the hypersurface perpendicular to the vector Z^{μ} as will appear in due course.

It can be easily shown that,

$$P_{\mu}^{.\nu} P_{.\nu}^{\mu} = 3Z^4 \quad , \tag{20}$$

$$P_{.\mu}^{\mu} = 3Z^2 \quad . \tag{21}$$

The non-symmetric tensor $Z_{\mu\nu}$ can now be written in the form:

$$Z_{\mu\nu} = \Omega_{\mu\nu} + \Sigma_{\mu\nu} + \frac{1}{3}\frac{\Theta}{Z^2}P_{\mu\nu} \quad . \tag{22}$$

It has been shown [5], using the 1st integral of the path equation (11), that Z is constant, thus

$$\left(Z^{\beta}Z_{\beta}\right)_{||\alpha} = 0 \quad .$$

As a consequence of (8),the last expression can be written as,

$$Z^{\beta}Z_{\beta||\alpha} = 0 \quad , \qquad (i)$$

while the new path equation (11) can be written in the equivalent form,

$$Z^{\beta}Z^{\alpha}_{.||\beta} = 0 \quad . \qquad (ii)$$

Now using *(i)* , *(ii)*, the following results can be easily obtained:

$$\Sigma_{\alpha\beta}Z^{\beta} = 0 \quad , \tag{23}$$

$$\Omega_{\alpha\beta}Z^{\beta} = 0 \quad , \tag{24}$$

from which it is clear that $\Sigma_{\alpha\beta}, \Omega_{\alpha\beta}$ are defined only on the hypersurface perpendicular to the Z^{μ} vector, i.e. they are purely spatial tensors. Hence we can verify the following results:

$$P^{\alpha\beta}\Sigma_{\alpha\beta} = 0 \quad , \tag{25}$$

$$P^{\alpha\beta}\Omega_{\alpha\beta} = 0 \quad . \tag{26}$$

Let us now define the following quantities,

$$\Omega^{2} \overset{def.}{=} \frac{1}{2}\Omega_{\alpha\beta}\Omega^{\alpha\beta} \quad , \tag{27}$$

$$\Sigma^{2} \overset{def.}{=} \frac{1}{2}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta} \quad . \tag{28}$$

Using (19), (24), (25), (26) and (27), and noting that $\Omega_{\alpha\beta}$ is antisymmetric, we can write

$$Z^{\alpha}_{.\beta}Z^{\beta}_{.\alpha} = -2\Omega^{2} + 2\Sigma^{2} + \frac{1}{3}\Theta^{2} \tag{29}$$

From the new path equation *(ii)* we can write

$$\left(Z^{\alpha}Z^{\beta}_{.\ ||\alpha}\right)_{||\gamma} = 0 \quad ,$$

i.e.

$$Z^{\alpha}_{.\ ||\gamma} Z^{\beta}_{.\ ||\alpha} + Z^{\alpha} Z^{\beta}_{.\ ||\alpha\gamma} = 0 \quad . \qquad (iii)$$

Using (9), we can write,

$$Z^{\beta}_{.\ ||\alpha\gamma} = Z^{\sigma} B^{\cdot\beta}_{\sigma.\alpha\gamma} + Z^{\beta}_{.\ ||\gamma\alpha} - b Z^{\beta}_{.\ ||\sigma} \Lambda^{\sigma}_{.\alpha\gamma} \quad . \qquad (iv)$$

Substituting from *(iv)* into *(iii)* we get,

$$Z^{\alpha}_{.\ ||\gamma} Z^{\beta}_{.\ ||\alpha} + Z^{\alpha}(Z^{\sigma} B^{\cdot\beta}_{\sigma.\alpha\gamma} + Z^{\beta}_{.\ ||\gamma\alpha} - b Z^{\beta}_{.\ ||\sigma} \Lambda^{\sigma}_{.\alpha\gamma}) = 0 \quad .$$

Contracting the last equation by setting $\beta = \gamma$, we get after using (15), (27) and (28),

$$\frac{d\Theta}{d\tau} = 2\Omega^2 - 2\Sigma^2 - \frac{1}{3}\Theta^2 - Z^{\alpha} Z^{\sigma} B_{\sigma\alpha} + b Z^{\alpha} Z^{\beta}_{.\ ||\sigma} \Lambda^{\sigma}_{.\alpha\beta} \quad . \qquad (30)$$

where,

$$B_{\sigma\alpha} \overset{def.}{=} B^{\cdot\beta}_{\sigma.\alpha\beta} \quad . \qquad (31)$$

Substituting from (16) and (17) into (30),we get

$$\frac{d\Theta}{d\tau} = 2\Omega^2 - 2\Sigma^2 - \frac{1}{3}\Theta^2 - Z^{\alpha} Z^{\sigma} B_{\sigma\alpha} + b Z^{\alpha}(\Omega^{\beta}_{.}\sigma + \Sigma^{\beta}_{.}\sigma)\Lambda^{\sigma}_{.\alpha\beta} + b\frac{\Theta}{3} Z^{\alpha} C_{\alpha} \quad .(32)$$

$$\text{where} \qquad C_{\alpha} = \Lambda^{\beta}_{.\alpha\beta} \qquad \text{and} \qquad Z_{\sigma} Z^{\alpha} Z^{\beta} \Lambda^{\sigma}_{.\alpha\beta} = 0 \quad .$$

Equation (32) is a generalized form of Raychaudhuri equation in which the suggested interaction, between the quantum spin of the moving elementary particle and the torsion of the background gravitational field, is taken into account.

4 Discussion

In the PAP-geometry, (having a general non-symmetric connection, and simultaneously non-vanishing torsion and curvature), Raychaudhuri equation is generalized. The generalization takes into account the suggested interaction between the quantum spin of the moving particle and the background space-time torsion produced by a gravitational field.

If we compare the generalized equation (32) with the original Raychaudhuri equation, developed in Riemannian geometry [8], we find some extra terms. Some of those terms appeared as a consequence of the generalized definitions (13)-(19), i.e. due to the use of the general affine connection (5) with the condition (7). These terms are hidden in all the terms of (32) except the last two terms in which (b)

appears explicitly. In general all these terms depend on the new parameter b. The vanishing of this parameter will reduce (32) to the original Raychaudhuri equation. The vanishing of b indicates that the interaction between quantum spin and torsion is switched off. In other words, the vanishing of the parameter b indicates that the moving particle is spinless.

Equation (32) can be compared with a corresponding one established in spaces with torsion [2]. For example, in the Riemann-Cartan space in which Einstein-Cartan theory is constructed, the torsion is coupled with the intrinsic spin density of the material particles, which vanishes in the absence of spin [9]. In this case we encounter a problem, that is, such theories are not viable in conventional AP-spaces [10], since the vanishing of the torsion will reduce the space-time to a flat one. In the present treatment, torsion is not connected to the spin density, but to the gravitational field. For gravity theories constructed in the AP-space (cf.[11]), (even in the case of GR written in this space) the space time torsion has non-vanishing components whether or not the spin is present.

The present treatment overcomes the above mentioned problem. The R.H.S of equation (11) represents, as stated before, a type of interaction between the quantum spin and the torsion of the background gravitational field. There is no need for the torsion to vanish in order to switch off this interaction. The interaction will vanish automatically for spinless particles ($n = 0$), in virtue of (12). More investigation is needed to explore the effect of the extra term on the singularity theorems.

Finally, we would like to point out that, the interaction of matter with space-time curvature gives rise to an **attractive** force (gravity), while its interaction with space-time torsion gives rise to a **repulsive** one (anti-gravity)[4]. This behavior has been used to interpret the accelerating expansion of the Universe [12].

Acknowledgments

The authors would like to thank late Dr. M.Melek, and Dr. M.E.Kahil for many discussions.

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