

A Global Approach to Absolute Parallelism Geometry*

Nabil L. Youssef ^{1,2} and Waleed A. Elsayed ^{1,2}

¹Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt

²Center for Theoretical Physics (CTP),
British University in Egypt (BUE)

Emails: nlyoussef@sci.cu.edu.eg, nlyoussef2003@yahoo.fr
waleedelsayed@sci.cu.edu.eg, waleed.a.elsayed@gmail.com

Abstract. In this paper we provide a *global* investigation of the geometry of parallelizable manifolds (or absolute parallelism geometry) frequently used for application. We discuss the different linear connections and curvature tensors from a global point of view. We give an existence and uniqueness theorem for a remarkable linear connection, called the canonical connection. Different curvature tensors are expressed in a compact form in terms of the torsion tensor of the canonical connection only. Using the Bianchi identities, some interesting identities are derived. An important special fourth order tensor, which we refer to as Wanas tensor, is globally defined and investigated. Finally a “double-view” for the fundamental geometric objects of an absolute parallelism space is established: The expressions of these geometric objects are computed in the parallelization basis and are compared with the corresponding local expressions in the natural basis. Physical aspects of some geometric objects considered are pointed out.

Keywords: Absolute parallelism geometry, Parallelization vector field, Parallelization basis, Canonical connection, Dual connection, Bianchi identities, Wanas tensor.

MSC 2010: 53C05, 53A40, 51P05.

PACS 2010: 02.04.Hw, 45.10.Na, 04.20.-q, 04.50.-h.

*arXiv: 1209.1379 [gr-qc]

0. Introduction

At the beginning of the 20th century, the importance of geometry in physical applications has been illuminated by Albert Einstein. He has advocated a new philosophy known as "The Geometerization Philosophy". This philosophy can be summarized in the following statement: "To understand nature, one has to start with geometry and end with physics" [13]. In 1915, Einstein used this philosophy to understand the essence of Gravity, starting with a 4-dimensional Riemannian geometry, ending with a successful theory for gravity; the General theory of Relativity (GR) [3]. After the success of the theory, by testing its predictions and applications, many authors have directed their attention to the use of geometry to solve physical problems.

Einstein in his continuous attempts to understand more physical interactions, has searched for a wider geometry to unify gravity and electromagnetism. The problem with Riemannian geometry, however, is that it has only ten degrees of freedom (the components of the metric tensor in four dimensions) which are just sufficient to describe gravity. Thus to construct a successful geometric theory that would encompass both gravity and electromagnetism, one needs to enlarge the number of degrees of freedom. This can be done in two different ways: either by increasing the dimension of the underlying space (à la-Kaluza-Klein) or by replacing the Riemannian structure by another geometric structure having more degrees of freedom (without increasing the dimension of the underlying space). In tackling the problem of unification, Einstein has chosen the second alternative. This led him to consider Absolute Parallelism geometry (AP-geometry) [2] which has sixteen degrees of freedom (the number of components of the vector fields forming the parallelization); six extra degrees of freedom are gained. Many developments of AP-geometry have been achieved (e.g., [10], [12], [14]). Theories constructed in this geometry (e.g., [6], [7], [10]) together with applications (e.g., [8], [9], [11]) show the advantages of using AP-geometry in physics.

In this paper, we establish a global approach to AP-geometry. The global formulation of the different geometric aspects of AP-geometry has many advantages. Some advantages of the global formalism are:

- It could give more insight into the infra-structure of physical theories constructed in the context of AP-geometry. Moreover, it may offer the opportunity to unify field theories in a more economic scheme.
- It helps better understand the meaning and the essence of the geometric objects and formulae without being trapped into the complexity of indices. As a consequence, it reduces the probability of mistake
- It connects AP-geometry with the modern language of the differential geometry.
- In local coordinates some important expressions, such as the Lie bracket $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]$, disappear. Consequently, the contribution, geometrical or physical, of all Lie brackets are completely hidden. Such expressions do not vanish in global formalism. This may produce new geometric or physical information.

- The local formalism represents roughly a **micro** viewpoint or a micro approach whereas the global formalism represents a **macro** viewpoint. The two viewpoints are not alternatives but rather complementary and are indispensable both for geometry and physics.
- As global results hold on the entire manifold (not only on coordinate neighborhoods), they also hold locally. The converse is not true; a result may hold locally but not globally. Moreover, one can easily shift from global to local; it suffices to view the global result in a coordinate neighborhood.

These are the main motivations of the present work, where all results obtained are formulated in a prospective modern coordinate free form.

The paper is organized in the following manner. In section 1, we define globally the basic elements of the AP-geometry and prove an existence and uniqueness theorem for a remarkable linear connection which we call the canonical connection (a (flat) connection for which the parallelization vector fields are parallel). We also study some properties of this connection. In section 2, we define three other natural connections (the dual, symmetric and Riemannian connections) and investigate their properties together with the tensor fields associated to them. In section 3, we express the curvature tensors of the above mentioned three connections in a simple and compact form in terms of the torsion tensor of the canonical connection only. We then use the Bianchi identities to derive some further interesting identities. In section 4, we give a global treatment of the W-tensor and investigate some of its properties. In section 5, we present a double-view for the fundamental geometric objects of AP-geometry: On one hand, we consider the local expressions of these geometric objects in the natural basis and, on the other hand, we compute their expressions in the parallelization basis, and then compare between the two sets of expressions.

It should finally be noted that this work is based mainly on [14].

Throughout the present paper we use the following notation:

- M : an n -dimensional smooth real manifold,
- $\mathfrak{F}(M)$: the \mathbb{R} -algebra of C^∞ functions on M ,
- $\mathfrak{X}(M)$: the $\mathfrak{F}(M)$ -module of vector fields on M ,
- $T_x M$: the tangent space to M at $x \in M$,
- $T_x^* M$: the cotangent space to M at $x \in M$.

We make the assumption that all geometric objects we consider are of class C^∞ .

1. Canonical connection

In this section, we give the definition of an AP-space and prove an existence and uniqueness theorem for a remarkable linear connection, which we call the canonical connection. Also, we prove some properties concerning this connection.

Definition 1.1. [1] *A parallelizable manifold is a pair (M, X) , where M is an n -dimensional smooth manifold and X ($i = 1, \dots, n$) are n independent vector fields defined globally on M . The vector*

fields X_1, \dots, X_n are said to form a parallelization on M .

Such a space is also known in the literature as an *Absolute Parallelism space* or a *Teleparallel space*. For simplicity, we will rather use the expressions “*AP-space* and *AP-geometry*”.

Since X_i are n independent vector fields on M , $\{X_i(x) : i = 1, \dots, n\}$ is a basis of $T_x M$ for every $x \in M$. Any vector field $Y \in \mathfrak{X}(M)$ can be written globally as $Y = Y^i X_i$, where $Y^i \in \mathfrak{F}(M)$. Here we use the notation Y^i to denote the components of Y with respect to X_i . Einstein summation convention will be applied on Latin indices whatever their position is (even if the two repeated indices are downword).

Definition 1.2. *The n differential 1-forms $\Omega_i : \mathfrak{X}(M) \longrightarrow \mathfrak{F}(M)$ defined by*

$$\Omega_i(X_j) = \delta_{ij} \quad (1.1)$$

are called the parallelization forms.

Clearly, if $Y = Y^i X_i$, then

$$\Omega_i(Y) = Y^i, \quad \Omega_i(Y) X_i = Y. \quad (1.2)$$

It follows directly from (1.1) that $\{\Omega_x = \Omega_i|_{T_x M} : i = 1, \dots, n\}$ is the dual basis of the parallelization basis $\{X_i(x) : i = 1, \dots, n\}$ for every $x \in M$. We call $\{\Omega_x : i = 1, \dots, n\}$ the dual parallelization basis of $T_x^* M$. The parallelization forms Ω_i are independent in the $\mathfrak{F}(M)$ -module $\mathfrak{X}^*(M)$.

Lemma 1.3. *Let D be a linear connection on M . The D -covariant derivative of Ω_i vanishes if and only if the D -covariant derivative of X_i vanishes.*

Proof. For every $Y, Z \in \mathfrak{X}(M)$, we have, by (1.2) and (1.1),

$$(D_Y \Omega_i)(Z) = (D_Y \Omega_i)(\Omega_j(Z) X_j) = -\Omega_j(Z) \Omega_i(D_Y X_j).$$

Consequently, by (1.2),

$$((D_Y \Omega_i)(Z)) X_i = -\Omega_j(Z) D_Y X_j,$$

from which the result follows. □

Theorem 1.4. *On an AP-space (M, X_i) , there exists a unique linear connection ∇ for which the parallelization vector fields X_i are parallel.*

Proof. First we prove the uniqueness. Assume that ∇ is a linear connection satisfying the condition $\nabla_i X_i = 0$. For all $Y, Z \in \mathfrak{X}(M)$, we have

$$\nabla_Y Z = \nabla_Y(\Omega_i(Z) X_i) = \Omega_i(Z) \nabla_Y X_i + (Y \cdot \Omega_i(Z)) X_i = (Y \cdot \Omega_i(Z)) X_i.$$

Hence, the connection ∇ is uniquely determined by the relation

$$\nabla_Y Z = (Y \cdot \Omega_i(Z)) X_i. \quad (1.3)$$

To prove the existence, let $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ be defined by (1.3). We show that ∇ is a linear connection on M . In fact, let $Y, Y_1, Y_2, Z, Z_1, Z_2 \in \mathfrak{X}(M)$, $f \in \mathfrak{F}(M)$. It is clear that $\nabla_{Y_1+Y_2}Z = \nabla_{Y_1}Z + \nabla_{Y_2}Z$ and $\nabla_Y(Z_1 + Z_2) = \nabla_Y Z_1 + \nabla_Y Z_2$. Moreover,

$$\begin{aligned}\nabla_{fY}Z &= ((fY) \cdot \Omega(Z))_i X = f(Y \cdot \Omega(Z))_i X = f \nabla_Y Z, \\ \nabla_Y(fZ) &= (Y \cdot \Omega(fZ))_i X = \left(Y \cdot (f \Omega(Z)) \right)_i X \\ &= f(Y \cdot \Omega(Z))_i X + (Y \cdot f) \Omega(Z)_i X \\ &= f \nabla_Y Z + (Y \cdot f)Z, \text{ by (1.2) and (1.3).}\end{aligned}$$

It remains to show that ∇ satisfies the condition $\nabla_i X = 0$:

$$\nabla_Y X = (Y \cdot \Omega(X))_i X = (Y \cdot \delta_{ij})_i X = 0.$$

This completes the proof. \square

As a consequence of Lemma 1.3, we also have $\nabla_i \Omega = 0$. Hence

$$\nabla_i X = 0, \quad \nabla_i \Omega = 0. \quad (1.4)$$

This property is known (locally) in the literature as the AP-condition.

Definition 1.5. *Let (M, X) be an AP-space. The unique linear connection ∇ on M defined by (1.3) will be called the canonical connection of (M, X) .*

The canonical connection is of crucial importance because almost all geometric objects in the AP-space will be built up of it, as will be seen throughout the paper.

Now we give an intrinsic formula of the torsion tensor T of ∇ .

Proposition 1.6. *The torsion tensor T of the canonical connection is given by*

$$T(Y, Z) = \Omega_i(Y) \Omega_j(Z) [X_j, X_i]. \quad (1.5)$$

Proof. The torsion tensor T of ∇ is defined, for all $Y, Z \in \mathfrak{X}(M)$, by

$$T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z].$$

Using the AP-condition (1.4), we get

$$\begin{aligned}T(Y, Z) &\stackrel{(1.2)}{=} T(\Omega_i(Y) X_i, \Omega_j(Z) X_j) = \Omega_i(Y) \Omega_j(Z) T(X_i, X_j) \\ &= \Omega_i(Y) \Omega_j(Z) (\nabla_{X_j} X_i - \nabla_{X_i} X_j - [X_i, X_j]) = \Omega_i(Y) \Omega_j(Z) [X_j, X_i].\end{aligned} \quad \square$$

Theorem 1.7. *Let (M, X) be an AP-space. The canonical connection of (M, X) is flat.*

Proof. The result follows from the definition of the curvature tensor R of ∇ :

$$R(Y, Z)V = \nabla_Y \nabla_Z V - \nabla_Z \nabla_Y V - \nabla_{[Y, Z]} V$$

and the AP-condition (1.4). \square

Remark 1.8. [14] It is for this reason that many authors think that the AP-space is a flat space. This is by no means true. In fact, it is meaningless to speak of curvature without reference to a connection. All we can say is that the AP-space is flat with respect to its canonical connection. However, there are other three natural connections on an AP-space which are nonflat, as will be shown later.

2. Other linear connections on an AP-space

In this section, we define a metric on an AP-space and investigate the properties of three other natural connections on the space. Moreover, we define the contortion tensor and give its relation to the torsion tensor (1.5).

Theorem 2.1. *Let (M, X_i) be an AP-space and Ω_i the parallelization forms on M . Then*

$$g := \Omega_i \otimes \Omega_i \quad (2.1)$$

defines a metric tensor on M .

Proof. Clearly g is a symmetric tensor of type $(0, 2)$ on M . For all $Y \in \mathfrak{X}(M)$, we have

$$g(Y, Y) = (\Omega_i \otimes \Omega_i)(Y, Y) = \sum_{i=1}^n (\Omega_i(Y))^2 \geq 0.$$

Moreover,

$$g(Y, Y) = 0 \implies \sum_{i=1}^n (\Omega_i(Y))^2 = 0 \implies \Omega_i(Y) = 0 \quad \forall i \implies \Omega_i(Y) X_i = 0 \stackrel{(1.2)}{\implies} Y = 0.$$

Hence, g is a metric tensor on M . □

Remark 2.2. It is clear that:

$$(a) \quad g(X_i, X_j) = \delta_{ij}. \quad (2.2)$$

$$(b) \quad g(X_i, Y) = \Omega_i(Y). \quad (2.3)$$

Property **(a)** shows that the parallelization vector fields X_i are g -orthonormal and **(b)** provides the duality between X_i and Ω_i via g .

Lemma 2.3. *Let (M, X_i) be an AP-space. A linear connection D on M is a metric connection if and only if*

$$\Omega_i(D_V X_j) + \Omega_j(D_V X_i) = 0.$$

Proof. By simple calculation, using (2.2) and (2.3), one can show that

$$(D_V g)(X_i, X_j) = -\Omega_i(D_V X_j) - \Omega_j(D_V X_i),$$

from which the result follows. □

The last lemma together with the AP-condition (1.4) give rise to the next result.

Proposition 2.4. *The canonical connection is a metric connection.*

Proposition 2.5. *On an AP-space there are three other (built-in) linear connections:*

(a) *The dual connection $\tilde{\nabla}$ given by*

$$\tilde{\nabla}_Y Z := \nabla_Z Y + [Y, Z]. \quad (2.4)$$

(b) *The symmetric connection $\hat{\nabla}$ given by*

$$\hat{\nabla}_Y Z := \frac{1}{2}(\nabla_Y Z + \nabla_Z Y + [Y, Z]). \quad (2.5)$$

(c) *The Riemannian connection $\overset{\circ}{\nabla}$ is given by [5]*

$$\begin{aligned} 2g(\overset{\circ}{\nabla}_Y Z, V) &= Y \cdot g(Z, V) + Z \cdot g(V, Y) - V \cdot g(Y, Z) \\ &\quad - g(Y, [Z, V]) + g(Z, [V, Y]) + g(V, [Y, Z]). \end{aligned} \quad (2.6)$$

The proof is straightforward and we omit it.

Remark 2.6. One can easily show that:

(a) $\tilde{\nabla}_Y Z = \nabla_Y Z - T(Y, Z).$

(b) $\hat{\nabla}_Y Z = \nabla_Y Z - \frac{1}{2}T(Y, Z) = \frac{1}{2}(\nabla_Y Z + \tilde{\nabla}_Y Z).$

(c) $\hat{\nabla}$ and $\overset{\circ}{\nabla}$ are torsionless whereas ∇ and $\tilde{\nabla}$ have the same torsion up to a sign.

Here T is the torsion tensor of the canonical connection ∇ . Since there are no other torsion tensors in the space, we can say that T is the torsion of the space.

In Riemannian geometry the Riemannian connection has no explicit expression. However, in AP-geometry we can have an *explicit expression* for the Riemannian connection $\overset{\circ}{\nabla}$ as shown in the following.

Theorem 2.7. *Let (M, X_i) be an AP-space. Then the Riemannian connection $\overset{\circ}{\nabla}$ can be written in the form:*

$$\overset{\circ}{\nabla}_Y Z = \hat{\nabla}_Y Z - \frac{1}{2}(\mathcal{L}_{X_i} g)(Y, Z) X_i, \quad (2.7)$$

where \mathcal{L}_Y is the Lie derivative with respect to $Y \in \mathfrak{X}(M)$.

Proof. By replacing V in (2.6) by X and using (2.3), we get

$$2\overset{\circ}{\Omega}(\overset{\circ}{\nabla}_Y Z) = Y \cdot \overset{\circ}{\Omega}(Z) + Z \cdot \overset{\circ}{\Omega}(Y) - \overset{\circ}{X} \cdot g(Y, Z) + g(Y, [\overset{\circ}{X}, Z]) + g(Z, [\overset{\circ}{X}, Y]) + \overset{\circ}{\Omega}([Y, Z]).$$

Taking into account (1.2) and (1.3), the above equation reads

$$\begin{aligned} 2\overset{\circ}{\nabla}_Y Z &= \nabla_Y Z + \nabla_Z Y - \left(\overset{\circ}{X} \cdot g(Y, Z) \right) \overset{\circ}{X} + g(Y, [\overset{\circ}{X}, Z]) \overset{\circ}{X} + g(Z, [\overset{\circ}{X}, Y]) \overset{\circ}{X} + [Y, Z] \\ &= 2\widehat{\nabla}_Y Z - \left(\overset{\circ}{X} \cdot g(Y, Z) - g(Y, [\overset{\circ}{X}, Z]) - g(Z, [\overset{\circ}{X}, Y]) \right) \overset{\circ}{X}, \text{ by (2.5)} \\ &= 2\widehat{\nabla}_Y Z - (\mathcal{L}_{\overset{\circ}{X}} g)(Y, Z) \overset{\circ}{X}. \end{aligned} \quad \square$$

Corollary 2.8. *In an AP-space, the Riemannian connection and the symmetric connection coincide if, and only if, the parallelization vector fields are Killing vector fields:*

$$\overset{\circ}{\nabla} = \widehat{\nabla} \iff \mathcal{L}_{\overset{\circ}{X}} g = 0 \quad \forall i.$$

Definition 2.9. *The contortion tensor C is defined by the formula:*

$$C(Y, Z) = \nabla_Y Z - \overset{\circ}{\nabla}_Y Z. \quad (2.8)$$

The contortion tensor may also be written in the form:

$$C(Y, Z) = (\overset{\circ}{\nabla}_Y \overset{\circ}{\Omega})(Z) \overset{\circ}{X}. \quad (2.9)$$

In fact, using (1.2) and (1.3), we have for all $Y, Z \in \mathfrak{X}(M)$,

$$C(Y, Z) = \nabla_Y Z - \overset{\circ}{\nabla}_Y Z = (Y \cdot \overset{\circ}{\Omega}(Z)) \overset{\circ}{X} - \overset{\circ}{\Omega}(\overset{\circ}{\nabla}_Y Z) \overset{\circ}{X} = (\overset{\circ}{\nabla}_Y \overset{\circ}{\Omega})(Z) \overset{\circ}{X}.$$

The identities (2.8) and (2.9) show that the geometry of an AP-space can be built up from the Riemannian connection instead of the canonical connection:

$$\nabla_Y Z = \overset{\circ}{\nabla}_Y Z + (\overset{\circ}{\nabla}_Y \overset{\circ}{\Omega})(Z) \overset{\circ}{X}.$$

The next proposition establishes the mutual relations between the torsion and contortion tensors.

Proposition 2.10. *The following identities hold:*

(a) $T(Y, Z) = C(Y, Z) - C(Z, Y).$

(b) $C(Y, Z) = \frac{1}{2} \left(T(Y, Z) + T(\overset{\circ}{X}, Y, Z) \overset{\circ}{X} + T(\overset{\circ}{X}, Z, Y) \overset{\circ}{X} \right).$

From which,

(a)' $T(Y, Z, V) = C(Y, Z, V) - C(Z, Y, V).$

(b)' $C(Y, Z, V) = \frac{1}{2} \left(T(Y, Z, V) + T(V, Y, Z) + T(V, Z, Y) \right),$

where $C(Y, Z, V) = g(C(Y, Z), V)$ and $T(Y, Z, V) = g(T(Y, Z), V)$.

Consequently, the torsion tensor vanishes if and only if the contortion tensor vanishes.

Proof. Let $Y, Z, V \in \mathfrak{X}(M)$. Then,

(a) The first identity gives the torsion tensor in terms of the contortion tensor.

$$\begin{aligned} T(Y, Z) &= \nabla_Y Z - \nabla_Z Y - [Y, Z] \\ &= (\nabla_Y Z - \nabla_Z Y) - (\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Z Y), \text{ since } \overset{\circ}{\nabla} \text{ is torsionless} \\ &= (\nabla_Y Z - \overset{\circ}{\nabla}_Y Z) - (\nabla_Z Y - \overset{\circ}{\nabla}_Z Y) = C(Y, Z) - C(Z, Y). \end{aligned}$$

(b) The second identity gives the contortion tensor in terms of the torsion tensor. In the following proof we make use of (2.7), Remark 2.6, (1.2), Remark 2.2 and (1.5).

$$\begin{aligned} 2C(Y, Z) &= 2\nabla_Y Z - 2\overset{\circ}{\nabla}_Y Z = 2\nabla_Y Z - 2\widehat{\nabla}_Y Z + (\mathcal{L}_X g)(Y, Z) X_i \\ &= 2\nabla_Y Z - 2\nabla_Y Z + T(Y, Z) + \Omega_j(Y) \Omega_k(Z) (\mathcal{L}_X g)(X_j, X_k) X_i \\ &= T(Y, Z) + \Omega_j(Y) \Omega_k(Z) \left(X_i \cdot g(X_j, X_k) - g([X_i, X_j], X_k) - g([X_i, X_k], X_j) \right) X_i \\ &= T(Y, Z) - \left(g(T(Y, X_i), Z) + g(T(Z, X_i), Y) \right) X_i \\ &= T(Y, Z) + \left(T(X_i, Y, Z) + T(X_i, Z, Y) \right) X_i. \quad \square \end{aligned}$$

Remark 2.11. $T(Y, Z, V)$ is skew-symmetric in the first two arguments whereas $C(Y, Z, V)$ is skew-symmetric in the last two arguments.

Definition 2.12. Let (M, X_i) be an AP-space. The contracted torsion or the basic form B is defined, for every $Y \in \mathfrak{X}(M)$ by

$$B(Y) := Tr\{Z \mapsto T(Z, Y)\}.$$

This 1-form is known (locally) in the literature as the basic vector. In terms of the metric tensor (2.1), using (2.2), the basic form can be written as

$$B(Y) = g(T(X_i, Y), X_i) = T(X_i, Y, X_i). \quad (2.10)$$

Using Proposition 2.10(b)', $B(Y)$ can also be expressed in the form

$$B(Y) = C(X_i, Y, X_i).$$

Making use of (2.10) and (1.5), we have

$$B(Y) = \Omega_j(Y) \Omega_i([X_j, X_i]).$$

It should be noted that in the above three expressions and in similar expressions summation is carried out on repeated mesh indices, although they are situated in different argument positions.

Proposition 2.13. *Concerning the four connections of the AP-space, the difference tensors are given by:*

(a) $\nabla_Y Z - \tilde{\nabla}_Y Z = T(Y, Z).$

(b) $\nabla_Y Z - \hat{\nabla}_Y Z = \frac{1}{2}T(Y, Z).$

(c) $\nabla_Y Z - \overset{\circ}{\nabla}_Y Z = C(Y, Z).$

(d) $\tilde{\nabla}_Y Z - \hat{\nabla}_Y Z = -\frac{1}{2}T(Y, Z).$

(e) $\tilde{\nabla}_Y Z - \overset{\circ}{\nabla}_Y Z = C(Z, Y).$

(f) $\hat{\nabla}_Y Z - \overset{\circ}{\nabla}_Y Z = \frac{1}{2}(\mathcal{L}_{X_i}g)(Y, Z)X_i.$

Proof. Properties (a), (b), (d) follow from Remark 2.6, (c) is the definition of the contortion tensor, (e) follows from (2.8) and the fact that $\overset{\circ}{\nabla}$ is torsionless, and (f) follows from (2.7). \square

As a consequence of the above proposition, we have the following useful relations.

Corollary 2.14. *For every $Y, Z, V \in \mathfrak{X}(M)$, we have the following relations:*

(a) $(\nabla_V T)(Y, Z) - (\tilde{\nabla}_V T)(Y, Z) = \mathfrak{S}_{Y,Z,V} \left\{ T(V, T(Y, Z)) \right\}.$

(b) $(\nabla_V T)(Y, Z) - (\hat{\nabla}_V T)(Y, Z) = \frac{1}{2} \mathfrak{S}_{Y,Z,U} \left\{ T(V, T(Y, Z)) \right\}.$

(c) $(\nabla_V T)(Y, Z) - (\overset{\circ}{\nabla}_V T)(Y, Z) = -T(Y, C(V, Z)) + T(Z, C(V, Y)) + C(V, T(Y, Z)),$

where $\mathfrak{S}_{Y,Z,V}$ denotes the cyclic permutation of Y, Z, V and summation.

3. Curvature tensors and Bianchi identities

In an AP-space the curvature R of the canonical connection ∇ vanishes identically. This section is devoted to show that the other three curvature tensors \tilde{R}, \hat{R} and $\overset{\circ}{R}$, associated with $\tilde{\nabla}, \hat{\nabla}$ and $\overset{\circ}{\nabla}$ respectively, do not vanish. Also, we show that the vanishing of R enables us to express these three curvature tensors in terms of the torsion tensor only.

Theorem 3.1. *The three curvature tensors \tilde{R}, \hat{R} and $\overset{\circ}{R}$ of the connections $\tilde{\nabla}, \hat{\nabla}$ and $\overset{\circ}{\nabla}$ are given respectively by:*

(a) $\tilde{R}(Y, Z)V = (\nabla_V T)(Y, Z). \tag{3.1}$

$$\begin{aligned}
\text{(b)} \quad \widehat{R}(Y, Z)V &= \frac{1}{2} \left((\nabla_Z T)(Y, V) - (\nabla_Y T)(Z, V) \right) - \frac{1}{2} T(T(Y, Z), V) \\
&\quad + \frac{1}{4} \left(T(Y, T(Z, V)) - T(Z, T(Y, V)) \right). \tag{3.2}
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad \mathring{R}(Y, Z)V &= (\nabla_Z C)(Y, V) - (\nabla_Y C)(Z, V) - C(T(Y, Z), V) \\
&\quad + C(Y, C(Z, V)) - C(Z, C(Y, V)). \tag{3.3}
\end{aligned}$$

Proof. We prove **(a)** only. The proof of the other parts can be carried out in the same manner. Using (2.4), we get

$$\begin{aligned}
\widetilde{\nabla}_Y \widetilde{\nabla}_Z V &= \widetilde{\nabla}_Y (\nabla_V Z + [Z, V]) = \widetilde{\nabla}_Y \nabla_V Z + \widetilde{\nabla}_Y [Z, V] \\
&= \nabla_{\nabla_V Z} Y + [Y, \nabla_V Z] + \nabla_{[Z, V]} Y + [Y, [Z, V]].
\end{aligned}$$

Similarly,

$$\widetilde{\nabla}_Z \widetilde{\nabla}_Y V = \nabla_{\nabla_V Y} Z + [Z, \nabla_V Y] + \nabla_{[Y, V]} Z + [Z, [Y, V]].$$

and

$$\widetilde{\nabla}_{[Y, Z]} V = \nabla_V [Y, Z] + [[Y, Z], V].$$

Using the above three identities, together with the Jacobi identity, we get

$$\begin{aligned}
\widetilde{R}(Y, Z)V &= \widetilde{\nabla}_Y \widetilde{\nabla}_Z V - \widetilde{\nabla}_Z \widetilde{\nabla}_Y V - \widetilde{\nabla}_{[Y, Z]} V \\
&= \nabla_{\nabla_V Z} Y + [Y, \nabla_V Z] - \nabla_{\nabla_V Y} Z - [Z, \nabla_V Y] \\
&\quad - \nabla_V [Y, Z] + \nabla_{[Z, V]} Y - \nabla_{[Y, V]} Z.
\end{aligned}$$

Using the fact that the curvature tensor of the canonical connection vanishes (Theorem 1.7), it follows that

$$\nabla_{[Y, Z]} V = \nabla_Y \nabla_Z V - \nabla_Z \nabla_Y V.$$

Using the above identity, we get

$$\begin{aligned}
\widetilde{R}(Y, Z)V &= \nabla_{\nabla_V Z} Y + [Y, \nabla_V Z] - \nabla_{\nabla_V Y} Z - [Z, \nabla_V Y] - \nabla_V [Y, Z] \\
&\quad + \nabla_Z \nabla_V Y - \nabla_V \nabla_Z Y - \nabla_Y \nabla_V Z + \nabla_V \nabla_Y Z \\
&= (\nabla_V \nabla_Y Z - \nabla_V \nabla_Z Y - \nabla_V [Y, Z]) \\
&\quad - (\nabla_{\nabla_V Y} Z - \nabla_Z \nabla_V Y - [\nabla_V Y, Z]) \\
&\quad - (\nabla_Y \nabla_V Z - \nabla_{\nabla_V Z} Y - [Y, \nabla_V Z]) \\
&= \nabla_V T(Y, Z) - T(\nabla_V Y, Z) - T(Y, \nabla_V Z) \\
&= (\nabla_V T)(Y, Z). \quad \square
\end{aligned}$$

The above theorem shows that the curvature tensors \widetilde{R} , \widehat{R} and \mathring{R} are expressible in terms of the torsion tensor of the space only. This proves that the geometry of an AP-space depends crucially on the torsion tensor. It is worth mentioning that the vanishing of that tensor implies that the four connections ∇ , $\widetilde{\nabla}$, $\widehat{\nabla}$ and $\mathring{\nabla}$ coincide and a trivial flat Riemannian space is achieved. Thus,

a sufficient condition for the non-vanishing of the torsion tensor is the non-vanishing of any one of the three curvature tensors \widetilde{R} , \widehat{R} or $\overset{\circ}{R}$.

The next result gives the expressions of the Ricci tensor $\overset{\circ}{Ric}$ of $\overset{\circ}{\nabla}$ and the Ricci-like tensors \widetilde{Ric} and \widehat{Ric} of $\widetilde{\nabla}$ and $\widehat{\nabla}$ together with their respective contractions (the scalar curvature $\overset{\circ}{Sc}$ and the curvature-like scalars \widetilde{Sc} and \widehat{Sc}). The orthonormality of the parallelization vector fields X_i plays an essential role in the proof.

Theorem 3.2. *In an AP-space (M, X) we have, for every $Y, Z \in \mathfrak{X}(M)$,*

(a) $\widetilde{Ric}(Y, Z) = -(\nabla_Z B)(Y).$

(b) $\widehat{Ric}(Y, Z) = \frac{1}{2}(\mathcal{L}_{X_i} T)(Y, Z, X_i) + \frac{1}{4}T(Y, T(Z, X_i), X_i) - \frac{1}{2}(\nabla_Y B)(Z) - \frac{1}{4}B(T(Y, Z)).$

(c) $\overset{\circ}{Ric}(Y, Z) = (\mathcal{L}_{X_i} C)(Y, Z, X_i) + C(Y, C(Z, X_i), X_i) - (\nabla_Y B)(Z) - B(C(Y, Z)).$

(a)' $\widetilde{Sc} = -X_i \cdot B(X_i).$

(b)' $\widehat{Sc} = -\frac{1}{2}X_i \cdot B(X_i) + \frac{1}{4}T(X_j, [X_i, X_j], X_i).$

(c)' $\overset{\circ}{Sc} = -2X_i \cdot B(X_i) + B(X_i)B(X_i) + C(T(X_i, X_j), X_j, X_i) + C(X_j, C(X_i, X_j), X_i).$

Proof. We prove (b) and (c)' only. The other identities can be proved similarly.

(b) Using (2.2), (3.2), (1.4) and (2.10), we have

$$\begin{aligned}
\widehat{Ric}(Y, Z) &= g(\widehat{R}(Y, X_i)Z, X_i) \\
&= \frac{1}{2} \left((\nabla_{X_i} T)(Y, Z, X_i) - (\nabla_Y T)(X_i, Z, X_i) - T(T(Y, X_i), Z, X_i) \right) \\
&\quad + \frac{1}{4} \left(T(Y, T(X_i, Z), X_i) - T(X_i, T(Y, Z), X_i) \right) \\
&= \frac{1}{4} \left(2X_i \cdot T(Y, Z, X_i) - 2T(\nabla_{X_i} Y, Z, X_i) - 2T(Y, \nabla_{X_i} Z, X_i) - 2(\nabla_Y B)(Z) \right. \\
&\quad \left. - 2T(T(Y, X_i), Z, X_i) + T(Y, T(X_i, Z), X_i) - B(T(Y, Z)) \right) \\
&= \frac{1}{4} \left(2X_i \cdot T(Y, Z, X_i) - T(Y, \nabla_{X_i} Z, X_i) - 2(\nabla_Y B)(Z) \right. \\
&\quad \left. - 2T([X_i, Y], Z, X_i) - T(Y, [X_i, Z], X_i) - B(T(Y, Z)) \right) \\
&= \frac{1}{4} \left(2(\mathcal{L}_{X_i} T)(Y, Z, X_i) - T(Y, \nabla_{X_i} Z, X_i) - T(Y, [Z, X_i], X_i) \right. \\
&\quad \left. - 2(\nabla_Y B)(Z) - B(T(Y, Z)) \right) \\
&= \frac{1}{2}(\mathcal{L}_{X_i} T)(Y, Z, X_i) + \frac{1}{4}T(Y, T(Z, X_i), X_i) - \frac{1}{2}(\nabla_Y B)(Z) - \frac{1}{4}B(T(Y, Z)).
\end{aligned}$$

(c)' Using (2.2), (c), (1.4), (2.8), Proposition 2.10(b) and the torsionless property of $\overset{\circ}{\nabla}$, we get

$$\begin{aligned}
\overset{\circ}{S}c &= \overset{\circ}{Ric}(X, X) \\
&= (\mathcal{L}_X C)(X, X, X) + C(X, C(X, X), X) - (\nabla_X B)(X) - B(C(X, X)) \\
&= X \cdot C(X, X, X) - C([X, X], X, X) - C(X, [X, X], X) - X \cdot B(X) \\
&\quad - C(X, \overset{\circ}{\nabla}_X X, X) + B(X)B(X)
\end{aligned}$$

$$\begin{aligned}
\overset{\circ}{S}c &= -2X \cdot B(X) + B(X)B(X) - C([X, X], X, X) \\
&\quad - \left(C(X, \overset{\circ}{\nabla}_X X, X) + C(X, [X, X], X) \right) \\
&= -2X \cdot B(X) + B(X)B(X) + C(T(X, X), X, X) - C(X, \overset{\circ}{\nabla}_X X, X) \\
&= -2X \cdot B(X) + B(X)B(X) + C(T(X, X), X, X) + C(X, C(X, X), X). \quad \square
\end{aligned}$$

Table1: Linear connections in AP-geometry

Connection	Symbol	Torsion	Curvature	Metricity
Canonical	∇	T	0	metric
Dual	$\tilde{\nabla}$	$-T$	\tilde{R}	nonmetric
Symmetric	$\hat{\nabla}$	0	\hat{R}	nonmetric
Riemannian	$\overset{\circ}{\nabla}$	0	$\overset{\circ}{R}$	metric

Let D be an arbitrary linear connection on M with torsion \mathbf{T} and curvature \mathbf{R} . Then the Bianchi identities are given, for all $Y, Z, V, U \in \mathfrak{X}(M)$, by [4]:

First Bianchi identity: $\mathfrak{S}_{Y,Z,V} \left\{ \mathbf{R}(Y, Z)V \right\} = \mathfrak{S}_{Y,Z,V} \left\{ (D_V \mathbf{T})(Y, Z) + \mathbf{T}(\mathbf{T}(Y, Z), V) \right\}.$

Second Bianchi identity: $\mathfrak{S}_{Y,Z,V} \left\{ (D_V \mathbf{R})(Y, Z)U - \mathbf{R}(V, \mathbf{T}(Y, Z))U \right\} = 0.$

In what follows, we derive some identities using the above Bianchi identities. Some of the derived identities will be used to simplify other formulae thus obtained.

Proposition 3.3. *The first Bianchi identity for the connections $\nabla, \tilde{\nabla}, \hat{\nabla}$ and $\overset{\circ}{\nabla}$ reads:*

- (a) $\mathfrak{S}_{Y,Z,V} \left\{ (\nabla_V T)(Y, Z) + T(T(Y, Z), V) \right\} = 0.$
- (b) $\mathfrak{S}_{Y,Z,V} \left\{ \tilde{R}(Y, Z)V \right\} = \mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) - (\tilde{\nabla}_V T)(Y, Z) \right\}.$
- (c) $\mathfrak{S}_{Y,Z,V} \left\{ \hat{R}(Y, Z)V \right\} = 0.$
- (d) $\mathfrak{S}_{Y,Z,V} \left\{ \mathring{R}(Y, Z)V \right\} = 0.$

The proof is straightforward. We have to use the relations $R = 0$, $\tilde{T} = -T$ and $\hat{T} = \mathring{T} = 0$.

Corollary 3.4. *The following identities hold:*

- (a) $\mathfrak{S}_{Y,Z,V} \left\{ (\tilde{\nabla}_V T)(Y, Z) \right\} = 2 \mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}.$
- (b) $\mathfrak{S}_{Y,Z,V} \left\{ (\hat{\nabla}_V T)(Y, Z) \right\} = \frac{1}{2} \mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}.$
- (c) $\mathfrak{S}_{Y,Z,V} \left\{ \tilde{R}(Y, Z)V \right\} = - \mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}.$

The proof follows from the above proposition together with Corollary 2.14 and (3.1).

Proposition 3.5. *The second Bianchi identity for the connections $\tilde{\nabla}$, $\hat{\nabla}$ and $\mathring{\nabla}$ reads:*

- (a) $\mathfrak{S}_{Y,Z,V} \left\{ (\tilde{\nabla}_V \tilde{R})(Y, Z)U \right\} = \mathfrak{S}_{Y,Z,V} \left\{ (\nabla_U T)(T(Y, Z), V) \right\}.$
- (b) $\mathfrak{S}_{Y,Z,V} \left\{ (\hat{\nabla}_V \hat{R})(Y, Z)U \right\} = 0.$
- (c) $\mathfrak{S}_{Y,Z,V} \left\{ (\mathring{\nabla}_V \mathring{R})(Y, Z)U \right\} = 0.$

The proof is straightforward making use of (3.1).

Now, we will give another formula for the curvature tensor \hat{R} of the symmetric connection $\hat{\nabla}$ which is more compact than (3.2).

Theorem 3.6. *The curvature tensor \hat{R} can be written in the form:*

$$\hat{R}(Y, Z)V = \frac{1}{2}(\nabla_V T)(Y, Z) - \frac{1}{4} \left(T(Y, T(Z, V)) + T(Z, T(V, Y)) \right).$$

Proof. Taking into account (3.2) and Proposition 3.3(a), one has

$$\begin{aligned} \hat{R}(Y, Z)V &= -\frac{1}{2} \mathfrak{S}_{Y,Z,V} \left\{ (\nabla_V T)(Y, Z) \right\} + \frac{1}{2}(\nabla_V T)(Y, Z) \\ &\quad + \frac{1}{4} \mathfrak{S}_{Y,Z,V} \left\{ T(V, T(Y, Z)) \right\} + \frac{1}{4}T(V, T(Y, Z)) \\ &= -\frac{1}{4} \mathfrak{S}_{Y,Z,V} \left\{ T(V, T(Y, Z)) \right\} + \frac{1}{4}T(V, T(Y, Z)) + \frac{1}{2}(\nabla_V T)(Y, Z) \\ &= \frac{1}{2}(\nabla_V T)(Y, Z) - \frac{1}{4} \left(T(Y, T(Z, V)) + T(Z, T(V, Y)) \right). \quad \square \end{aligned}$$

Corollary 3.7. *On an AP-space (M, X) the Ricci-like tensor \widehat{Ric} with respect to the symmetric connection $\widehat{\nabla}$ can be written as:*

$$\widehat{Ric}(Y, Z) = -\frac{1}{2}(\nabla_Z B)(Y) + \frac{1}{4}B(T(Y, Z)) - \frac{1}{4}T(Y, T(X, Z), X).$$

It is to be noted that the expression $\mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}$ appears in many of the identities obtained above. We discuss now the case in which this expression vanishes.

Let us write $[X, X] =: C_{ij}^h X$. The functions $C_{ij}^h \in \mathfrak{F}(M)$ are global functions on M and will be referred to as the global structure coefficients of the AP-space.

Theorem 3.8. *On an AP-space (M, X) the expression $\mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}$ vanishes if and only if, for all h , the expression $\mathfrak{S}_{i,j,k} \left\{ X \cdot C_{ij}^h \right\}$ vanishes.*

Consequently, if the global structure coefficients of the AP-space are constant functions on M , then $\mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\} = 0$.

Proof. Using the parallelization vector fields instead of Y, Z and V , we have:

$$\begin{aligned} \mathfrak{S}_{i,j,k} \left\{ T(T(X, X), X) \right\} = 0 &\iff -\mathfrak{S}_{i,j,k} \left\{ T([X, X], X) \right\} = 0, \text{ by (1.5)} \\ &\iff \mathfrak{S}_{i,j,k} \left\{ \nabla_X [X, X] + [[X, X], X] \right\} = 0, \text{ by (1.4)} \\ &\iff \mathfrak{S}_{i,j,k} \left\{ \nabla_X [X, X] \right\} = 0, \text{ by Jacobi identity} \\ &\iff \mathfrak{S}_{i,j,k} \left\{ \left(X \cdot \Omega([X, X]) \right)_h X \right\} = 0, \text{ by (1.3)} \\ &\iff \mathfrak{S}_{i,j,k} \left\{ \left(X \cdot \Omega([X, X]) \right)_h X \right\} = 0 \\ &\iff \mathfrak{S}_{i,j,k} \left\{ \left(X \cdot \Omega([X, X]) \right)_h \right\} = 0 \quad \forall h, \text{ by the independence of } X \\ &\iff \mathfrak{S}_{i,j,k} \left\{ X \cdot C_{ij}^h \right\} = 0 \quad \forall h, \text{ by (1.2).} \quad \square \end{aligned}$$

It should be noted that for the natural basis $\left\{ \frac{\partial}{\partial x^\alpha} : \alpha = 1, \dots, n \right\}$, the bracket $\left[\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right] = 0$ and so the structure coefficients associated with $\left(\frac{\partial}{\partial x^\alpha} \right)$ vanish. For this reason the *local* expression (in the natural basis) of the identity $\mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\} = 0$ is valid as is established in [14].

The last proposition gives rise to the following interesting formulae.

Corollary 3.9. *In an AP-space (M, X) , if the global structure coefficient of the AP-space are constant functions on M , then the next formulae hold:*

(a) $(\nabla_V T)(Y, Z) = (\widetilde{\nabla}_V T)(Y, Z) = (\widehat{\nabla}_V T)(Y, Z).$

(b) $\mathfrak{S}_{Y,Z,V} \left\{ (\nabla_V T)(Y, Z) \right\} = 0.$

(c) $\mathfrak{S}_{Y,Z,V} \left\{ (\widetilde{\nabla}_V T)(Y, Z) \right\} = 0.$

$$(d) \quad \mathfrak{S}_{Y,Z,V} \left\{ (\widehat{\nabla}_V T)(Y, Z) \right\} = 0.$$

$$(e) \quad \mathfrak{S}_{Y,Z,V} \left\{ \widetilde{R}(Y, Z)V \right\} = 0.$$

$$(f) \quad \widehat{R}(Y, Z)V = \frac{1}{2}(\nabla_V T)(Y, Z) - \frac{1}{4}T(T(Y, Z), V).$$

$$(h) \quad \mathfrak{S}_{Y,Z,V} \left\{ (\nabla_V C)(Y, Z) \right\} = \mathfrak{S}_{Y,Z,V} \left\{ (\nabla_V C)(Z, Y) \right\}.$$

4. Wanas Tensor

The Wanas tensor was first defined locally by M. I. Wanas in 1975 [10]. It has been used by F. Mikhail and M. Wanas [6] to construct a pure geometric theory unifying gravity and electromagnetism. In this section, we introduce the global definition of the Wanas tensor and investigate it.

Definition 4.1. *Let (M, X) be an AP-space. The tensor field W of type $(1, 3)$ on M defined by the formula*

$$W(Y, Z)X = \widetilde{\nabla}_{Y,Z}^2 X - \widetilde{\nabla}_{Z,Y}^2 X,$$

where $\widetilde{\nabla}_{Y,Z}^2 = \widetilde{\nabla}_Y \widetilde{\nabla}_Z - \widetilde{\nabla}_{\widetilde{\nabla}_Y Z}$, is called the Wanas tensor, or simply the W -tensor, of (M, X) .

Using (1.2), for every $Y, Z, V \in \mathfrak{X}(M)$, we get

$$W(Y, Z)V = (\widetilde{\nabla}_{Y,Z}^2 X - \widetilde{\nabla}_{Z,Y}^2 X) \Omega(V). \quad (4.1)$$

The next result gives a quite simple expression for a such tensor.

Theorem 4.2. *The W -tensor satisfies the following identity*

$$W(Y, Z)V = \widetilde{R}(Y, Z)V - T(T(Y, Z), V). \quad (4.2)$$

Proof. Consider the commutation formula for the parallelization vector field X with respect to $\widetilde{\nabla}$:

$$\widetilde{\nabla}_{Y,Z}^2 X - \widetilde{\nabla}_{Z,Y}^2 X = \widetilde{R}(Y, Z)X - \widetilde{\nabla}_{\widetilde{\nabla}_T(Y,Z)} X$$

Consequently,

$$\begin{aligned} W(Y, Z)V &\stackrel{(4.1)}{=} \Omega(V) \widetilde{R}(Y, Z)X - \Omega(V) \widetilde{\nabla}_{\widetilde{\nabla}_T(Y,Z)} X \\ &= \widetilde{R}(Y, Z)V + \Omega(V) \widetilde{\nabla}_{T(Y,Z)} X, \text{ by (1.2)} \\ &= \widetilde{R}(Y, Z)V + \widetilde{\nabla}_{T(Y,Z)} \Omega(V)X - (T(Y, Z) \cdot \Omega(V))X \\ &= \widetilde{R}(Y, Z)V + \widetilde{\nabla}_{T(Y,Z)} V - \nabla_{T(Y,Z)} V, \text{ by (1.2) and (1.3)} \\ &= \widetilde{R}(Y, Z)V - T(T(Y, Z), V), \text{ by Proposition 2.13.} \quad \square \end{aligned}$$

Corollary 4.3. *The W-tensor can be expressed in the form:*

$$W(Y, Z)V = (\nabla_V T)(Y, Z) - T(T(Y, Z), V). \quad (4.3)$$

In fact, this expression follows from (3.1). This shows that the W-tensor is expressed in terms of the torsion tensor of the AP-space only.

Proposition 4.4. *The Wanas tensor has the following properties:*

(a) $W(Y, Z)V$ is skew symmetric in the first two arguments Y, Z .

(b) $\mathfrak{S}_{Y,Z,V} \left\{ W(Y, Z)V \right\} = -2 \mathfrak{S}_{Y,Z,V} \left\{ T(T(Y, Z), V) \right\}.$

(c) $\mathfrak{S}_{Y,Z,V} \left\{ W(Y, Z)V \right\} = - \mathfrak{S}_{Y,Z,V} \left\{ (\tilde{\nabla}_V T)(Y, Z) \right\}.$

Proof. Property (a) is trivial, (b) follows from Proposition 3.3(a) and (4.3), (c) follows from (b) and Corollary 3.4(a). \square

The identity satisfied by the W-tensor in Proposition 4.4(b) is the same as the first Bianchi identity (Corollary 3.4(c)) of the dual curvature tensor up to a constant. The identity corresponding to the second Bianchi identity is given by:

Proposition 4.5. *The W-tensor satisfies the following identity:*

$$\begin{aligned} & \mathfrak{S}_{V,Y,Z} \left\{ (\tilde{\nabla}_V W)(Y, Z)U \right\} \\ &= - \mathfrak{S}_{V,Y,Z} \left\{ T\left(T(T(Y, Z), V), U\right) + T\left(T(T(Y, Z), U), V\right) + T(T(U, V), T(Y, Z)) \right\} \\ & \quad + \mathfrak{S}_{V,Y,Z} \left\{ (\nabla_U T)(T(Y, Z), V) - (\nabla_V T)(T(Y, Z), U) \right\} \end{aligned} \quad (4.4)$$

Proof. Taking into account (4.2) together with Proposition 3.5(a), Corollary 3.4(a) and Corollary 2.14(a)

$$\begin{aligned} & \mathfrak{S}_{V,Y,Z} \left\{ (\tilde{\nabla}_V W)(Y, Z)U \right\} \\ &= \mathfrak{S}_{V,Y,Z} \left\{ \left((\tilde{\nabla}_V \tilde{R})(Y, Z)U - (\tilde{\nabla}_V T)(T(Y, Z), U) - T((\tilde{\nabla}_V T)(Y, Z), U) \right) \right\} \\ &= \mathfrak{S}_{V,Y,Z} \left\{ (\nabla_U T)(T(Y, Z), V) - (\tilde{\nabla}_V T)(T(Y, Z), U) - 2T\left(T(T(Y, Z), V), U\right) \right\} \\ &= \mathfrak{S}_{V,Y,Z} \left\{ (\nabla_U T)(T(Y, Z), V) - (\nabla_V T)(T(Y, Z), U) \right\} \\ & \quad - \mathfrak{S}_{V,Y,Z} \left\{ 2T\left(T(T(Y, Z), V), U\right) + \mathfrak{S}_{V,T(Y,Z),U} T\left(T(T(Y, Z), U), V\right) \right\} \\ &= \mathfrak{S}_{V,Y,Z} \left\{ (\nabla_U T)(T(Y, Z), V) - (\nabla_V T)(T(Y, Z), U) \right\} \\ & \quad - \mathfrak{S}_{V,Y,Z} \left\{ T\left(T(T(Y, Z), V), U\right) + T\left(T(T(Y, Z), U), V\right) + T(T(U, V), T(Y, Z)) \right\}. \quad \square \end{aligned}$$

Corollary 4.6. *In an AP-space (M, X) , if the global structure coefficient of the AP-space are constant, we have*

$$(a) \quad \mathfrak{S}_{V,Y,Z} \left\{ W(Y, Z)V \right\} = 0.$$

$$(b) \quad \mathfrak{S}_{V,Y,Z} \left\{ (\tilde{\nabla}_V W)(Y, Z)U \right\} = \mathfrak{S}_{V,Y,Z} \left\{ (\nabla_U T)(T(Y, Z), V) - (\nabla_V T)(T(Y, Z), U) \right\}$$

The proof is straightforward from Theorem 3.8 and Corollary 3.9.

We end this section by the following comments and remarks on Wanas tensor.

- The W-tensor is defined by using the commutation formula with respect to the dual connection $\tilde{\nabla}$. Nothing new arose from the same definition if we use the three other connections (∇ , $\widehat{\nabla}$ and $\overset{\circ}{\nabla}$).
- Using the commutation formula for the parallelization form Ω_i instead of the parallelization vector field X_i in the definition of the W-tensor:

$$W(Y, Z)V = \left((\tilde{\nabla}_{Z,Y}^2 \Omega_i)(V) - (\tilde{\nabla}_{Y,Z}^2 \Omega_i)(V) \right) X_i$$

gives the same formula (4.2) for the W-tensor and consequently the same properties.

- Being defined by using the parallelization vector fields X_i , the Wanas tensor is defined only in AP-geometry. It has no analogue in other geometries.
- Although the W-tensor and the dual curvature tensor have some common properties (for example, Proposition 4.4(b)), there are significantly different properties (for example, (4.4)). In the case of constant global structure coefficients, the W-tensor has some properties common with the Riemannian curvature $\overset{\circ}{R}$.
- For a physical discussion concerning the W-tensor we refer to [14].

5. Parallelization basis versus natural basis

This section is devoted to a double-view for the fundamental geometric objects of AP-geometry. On one hand, we consider the local expressions of these geometric objects in the natural basis [14] and, on the other hand, we compute their expressions in the parallelization basis, giving rise to a concise table expressing this double-view.

Let $(U, (x^\alpha))$ be a local coordinate system of M . At each point $x \in U$, we have two distinguished bases of $T_x M$, namely, the natural basis $\{\partial_\mu := \frac{\partial}{\partial x^\mu} : \mu = 1, \dots, n\}$ and the parallelization basis $\{X_i(x) : i = 1, \dots, n\}$. These two bases are fundamentally different. The parallelization vector fields X_i are defined globally on the manifold M whereas the natural basis vector fields ∂_μ are defined only on the coordinate neighborhood U . Consequently, the natural basis vector fields depend crucially on coordinate systems whereas the parallelization vector fields do not.

Greek (world) indices are related to the natural basis and Latin (mesh) indices are related to the parallelization basis. Einstein summation convention will be applied as usual on Greek indices.

It will also be applied on Latin indices whatever their position is (even if the two repeated indices are upward or downward).

A tensor field H of type (r, s) on M is written in the natural basis in the form:

$$H = H_{\mu_1 \dots \mu_s}^{\alpha_1 \dots \alpha_r} \partial_{\alpha_1} \otimes \dots \otimes \partial_{\alpha_r} \otimes dx^{\mu_1} \otimes \dots \otimes dx^{\mu_s}, \text{ on } U$$

and in the parallelization basis in the form:

$$H = H_{j_1 \dots j_s}^{i_1 \dots i_r} X_{i_1} \otimes \dots \otimes X_{i_r} \otimes \Omega_{j_1} \otimes \dots \otimes \Omega_{j_s}, \text{ on } M,$$

where $H_{\mu_1 \dots \mu_s}^{\alpha_1 \dots \alpha_r} \in \mathfrak{F}(U)$ and $H_{j_1 \dots j_s}^{i_1 \dots i_r} \in \mathfrak{F}(M)$.

A vector field $Y \in \mathfrak{X}(M)$ is written in the natural basis in the form $Y = Y^\alpha \partial_\alpha$ and in the parallelization basis in the form $Y = Y^i X_i$. In particular, $X_i = X_i^\alpha \partial_\alpha$ and $\partial_\alpha = \Omega_i(\partial_\alpha) X_i = \Omega_{\alpha i} X_i$. Hence $(X_i^\alpha)_{1 \leq \alpha, i \leq n}$ is the matrix of change of bases and $(\Omega_{\alpha i})_{1 \leq \alpha, i \leq n}$ is the inverse matrix.

We use the following notations (with similar notations with respect to mesh indices):

- $\tilde{\Gamma}_{\mu\nu}^\alpha, \tilde{\Gamma}_{\mu\nu}^\alpha, \hat{\Gamma}_{\mu\nu}^\alpha, \overset{\circ}{\Gamma}_{\mu\nu}^\alpha$: the coefficients of the linear connections $\nabla, \tilde{\nabla}, \hat{\nabla}, \overset{\circ}{\nabla}$ respectively,
- $\tilde{|\cdot|}$: the covariant derivative with respect to the dual connection $\tilde{\nabla}$,
- $g_{\mu\nu}$ (resp. $g^{\mu\nu}$): the covariant (resp. contravariant) components of the metric tensor g ,
- $\Lambda_{\mu\nu}^\alpha$: the components of the torsion tensor T ,
- B_α : the components of the basic form B ,
- $\gamma_{\mu\nu}^\alpha$: the components of the contortion tensor C ,
- $W_{\sigma\mu\nu}^\alpha$: the components of the Wanas tensor W .

Let D be an arbitrary connection on M with torsion tensor T and curvature tensor R . We use the following conventions: $D_{\partial_\mu} \partial_\nu = D_{\nu\mu}^\alpha \partial_\alpha$, $T(\partial_\mu, \partial_\nu) = T_{\nu\mu}^\alpha \partial_\alpha$, $R(\partial_\mu, \partial_\nu) \partial_\sigma = R_{\sigma\mu\nu}^\alpha \partial_\alpha$, with similar conventions with respect to Latin indices.

The next table gives a comparison between the most important geometric objects of AP-geometry expressed in the natural basis and in the parallelization basis. Geometric objects, equations or identities having the same form in the two bases are not mentioned in that table. However, if a geometric object has the same form in the two bases, that is, if its expressions in world indices and mesh indices are similar, this does not mean that the geometric meaning of these two expressions is the same.

Table2: Parallelization basis versus natural basis

Geometric object	Local form In the natural basis (world indices)	Global form In the parallelization basis (mesh indices)
Parallelization vector field, parallelization forms	$X_i^\alpha \Omega_j^\alpha = \delta_{ij}, \quad X_i^\alpha \Omega_i^\alpha = \delta_\mu^\alpha$	$X_j^k = \delta_j^k, \quad \Omega_j^k = \delta_{jk}$
Metric tensor	$g_{\mu\nu} = \Omega_\mu^\alpha \Omega_\nu^\alpha$	$g_{jk} = \delta_{jk}$
Canonical connection	$\Gamma_{\nu\mu}^\alpha = X_i^\alpha \Omega_{\nu,\mu}^\alpha$ where $_{,\mu}$ denotes ∂_μ	$\Gamma_{jk}^h = 0$
Dual connection	$\tilde{\Gamma}_{\nu\mu}^\alpha = \Gamma_{\mu\nu}^\alpha$	$\tilde{\Gamma}_{jk}^h = C_{kj}^h$
Symmetric connection	$\hat{\Gamma}_{\nu\mu}^\alpha = \frac{1}{2}(\Gamma_{\nu\mu}^\alpha + \Gamma_{\mu\nu}^\alpha)$	$\hat{\Gamma}_{jk}^h = \frac{1}{2}C_{kj}^h$
Riemannian connection	$\overset{\circ}{\Gamma}_{\nu\mu}^\alpha = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\nu\mu,\sigma})$	$\overset{\circ}{\Gamma}_{jk}^h = \frac{1}{2}(C_{kj}^h + C_{hk}^j + C_{hj}^k)$
Torsion tensor	$\Lambda_{\nu\mu}^\alpha = \Gamma_{\nu\mu}^\alpha - \Gamma_{\mu\nu}^\alpha$	$\Lambda_{jk}^h = C_{jk}^h$
Contortion tensor	$\gamma_{\nu\mu}^\alpha = \Gamma_{\nu\mu}^\alpha - \overset{\circ}{\Gamma}_{\nu\mu}^\alpha$	$\gamma_{jk}^h = -\overset{\circ}{\Gamma}_{jk}^h$
Torsion in terms of contortion	$\Lambda_{\nu\mu}^\alpha = \gamma_{\nu\mu}^\alpha - \gamma_{\mu\nu}^\alpha$ $\Lambda_{\sigma\nu\mu} = \gamma_{\sigma\nu\mu} - \gamma_{\sigma\mu\nu}$ where $\Lambda_{\mu\nu\sigma} = g_{\epsilon\mu}\Lambda_{\nu\sigma}^\epsilon$ and $\gamma_{\mu\nu\sigma} = g_{\epsilon\mu}\gamma_{\nu\sigma}^\epsilon$	$\Lambda_{jk}^h = \gamma_{jk}^h - \gamma_{kj}^h$
Contortion in terms of torsion	$\gamma_{\nu\mu}^\alpha = \frac{1}{2}(\Lambda_{\nu\mu}^\alpha + (\Lambda_{\mu\nu\epsilon} + \Lambda_{\nu\mu\epsilon})g^{\alpha\epsilon})$ $\gamma_{\mu\nu\sigma} = \frac{1}{2}(\Lambda_{\sigma\nu\mu} + \Lambda_{\mu\nu\sigma} + \Lambda_{\nu\sigma\mu})$	$\gamma_{jk}^h = \frac{1}{2}(C_{jk}^h + C_{jh}^k + C_{kh}^j)$
Basic form	$B_\mu = \Lambda_{\mu\alpha}^\alpha = \gamma_{\mu\alpha}^\alpha$	$B_j = \Lambda_{jk}^k = \gamma_{jk}^k = C_{jk}^k$
Wanas tensor	$W_{\sigma\mu\nu}^\alpha = (X_i^\alpha \tilde{\Omega}_{\nu\mu}^\alpha - X_i^\alpha \tilde{\Omega}_{\mu\nu}^\alpha) \Omega_i^\alpha$ $W_{\sigma\mu\nu}^\alpha = (\Omega_i^\alpha \tilde{\Omega}_{\mu\nu}^\alpha - \Omega_i^\alpha \tilde{\Omega}_{\nu\mu}^\alpha) X_i^\alpha$	$W_{kij}^h = X_k^h \tilde{\Omega}_{ji}^h - X_k^h \tilde{\Omega}_{ij}^h$ $W_{kij}^h = \Omega_k^h \tilde{\Omega}_{ij}^h - \Omega_k^h \tilde{\Omega}_{ji}^h$

The above table merits some comments. We conclude this section and the paper by the following remarks and comments.

- The third column of the above table is obtained by computing the expression of the geometric objects in the parallelization basis. For example, to compute the coefficients of the Riemannian connection $\overset{\circ}{\Gamma}_{jk}^h$, set $Y = \underset{k}{X}$, $Z = \underset{j}{X}$, $V = \underset{h}{X}$ in (2.6). Then, we get

$$\begin{aligned} 2g(\nabla_{\underset{k}{X}} \underset{j}{X}, \underset{h}{X}) &= \underset{k}{X} \cdot g(\underset{j}{X}, \underset{h}{X}) + \underset{j}{X} \cdot g(\underset{h}{X}, \underset{k}{X}) - \underset{h}{X} \cdot g(\underset{k}{X}, \underset{j}{X}) \\ &\quad - g(\underset{k}{X}, [\underset{j}{X}, \underset{h}{X}]) + g(\underset{j}{X}, [\underset{h}{X}, \underset{k}{X}]) + g(\underset{h}{X}, [\underset{k}{X}, \underset{j}{X}]). \end{aligned}$$

For the left-hand side (LHS),

$$LHS = 2g(\overset{\circ}{\Gamma}_{jk}^l \underset{h}{X}, \underset{h}{X}) = 2g_{lh} \overset{\circ}{\Gamma}_{jk}^l = 2\delta_{lh} \overset{\circ}{\Gamma}_{jk}^l = 2\overset{\circ}{\Gamma}_{jk}^h.$$

As $\underset{k}{X} \cdot g(\underset{j}{X}, \underset{h}{X}) = \underset{k}{X} \cdot g_{jh} = \underset{k}{X} \cdot \delta_{jh} = 0$, the first three terms of the right-hand side (RHS) vanish. Hence,

$$\begin{aligned} RHS &= -g(\underset{k}{X}, C_{jh}^l \underset{l}{X}) + g(\underset{j}{X}, C_{hk}^l \underset{l}{X}) + g(\underset{h}{X}, C_{kj}^l \underset{l}{X}) \\ &= -g_{kl} C_{jh}^l + g_{jl} C_{hk}^l + g_{hl} C_{kj}^l \\ &= -C_{jh}^k + C_{hk}^j + C_{kj}^h. \end{aligned}$$

Accordingly,

$$\overset{\circ}{\Gamma}_{jk}^h = \frac{1}{2}(C_{hk}^j + C_{kj}^h - C_{jh}^k).$$

- It is clear from the third column that almost all geometric objects of AP-geometry are expressed in terms of the global structure coefficients C_{jk}^h . The global structure coefficients thus play a dominant role in AP-geometry formulated in mesh indices. Its role is similar to, and even more important than, the role played by the torsion tensor Λ_{jk}^h in AP-geometry formulated in world indices.
- The structure coefficients $C_{\mu\nu}^\alpha$, with respect to an arbitrary basis (e_α) are not the components of a $(1,2)$ -tensor field. In fact, let $e_{\alpha'} = A_{\alpha'}^\alpha e_\alpha$ under a change of local coordinates from (x^α) to $(x^{\alpha'})$ and let $[e_\mu, e_\nu] = C_{\mu\nu}^\alpha e_\alpha$ and $[e_{\mu'}, e_{\nu'}] = C_{\mu'\nu'}^{\alpha'} e_{\alpha'}$. Then, one can easily show that the transformation formula for $C_{\mu\nu}^\alpha$ has the form:

$$C_{\mu'\nu'}^{\alpha'} = A_{\alpha'}^\alpha A_{\mu'}^\mu A_{\nu'}^\nu C_{\mu\nu}^\alpha + K_{\mu'\nu'}^{\alpha'} - K_{\nu'\mu'}^{\alpha'},$$

where $K_{\mu\nu}^\alpha = A_{\mu'}^\mu A_{\nu'}^\nu (e_{\mu'} \cdot A_{\nu'}^{\alpha'})$. Thus, $C_{\mu\nu}^\alpha$ are not the components of a tensor field of type $(1,2)$ unless $e_{\mu'} \cdot A_{\nu'}^{\alpha'} = 0$ (that is, the matrix of change of bases $A^{\alpha'}_\alpha$ is a constant matrix) or $K_{\mu\nu}^\alpha$ is symmetric with respect to μ and ν . Also, the global structure coefficients C_{jk}^h are not the components of a $(1,2)$ -tensor field (they are n^3 functions defined globally on M and having certain properties). Nevertheless, for fixed j and k , C_{jk}^h are the components of the $(1,0)$ -tensor field $[\underset{j}{X}, \underset{k}{X}]$.

- In the parallelization basis, although the coefficients of the canonical connection ∇ vanish: $\Gamma_{jk}^h = 0$ ($\nabla_{X_j} X_k = 0$ because of the AP-condition), its torsion tensor T does not vanish: $\Lambda_{jk}^h = C_{jk}^h$; a phenomenon that never exist in natural local coordinates. This is due to the non-vanishing of the bracket $[X_j, X_k]$ in the expression of the torsion tensor: $T(X_j, X_k) = \nabla_{X_j} X_k - \nabla_{X_k} X_j - [X_j, X_k]$. For the same reason the dual connection $\tilde{\nabla}$ has also non-vanishing coefficients: $\tilde{\Gamma}_{jk}^h = C_{jk}^h$.
- From the table we have $\tilde{\Gamma}_{jk}^h = 2\hat{\Gamma}_{jk}^h = -C_{jk}^h$ and $\overset{\circ}{\Gamma}_{jk}^h = -\gamma_{jk}^h = \frac{1}{2}(C_{kj}^h + C_{hk}^j + C_{hj}^k)$. This means that the dual connection coefficients and the symmetric connection coefficients coincide (up to a constant) and are both equal to the global structure coefficients (up to a constant). On the other hand, the Riemannian connection coefficients coincide with the contortion coefficients (up to a sign). This shows again that everything in AP-geometry is expressible in terms of the global structure coefficients. Also, all surviving connections in the space may be represented by only one of them, say the Riemannian connection.
- A quick look at the third column of the above table may deceive and lead to erroneous conclusions: the symmetric connection is skew-symmetric and the Riemannian connection is non-symmetric. This is by no means true. The formulation of the notion of symmetry of connections using indices is not applied any more in this context. In fact, a linear connection is symmetric if and only if it coincides with its dual connection, and this is the case for both the symmetric and Riemannian connections. Another example: although the symmetric and dual connections coincide (up to a constant) in the parallelization basis, the symmetric connection has no torsion while the dual connection has a surviving torsion. This is, once more, due to the fact that the torsion expression has a bracket term which does not depend on the connection.
- The torsion and contortion tensors of type $(0, 3)$ are present in the natural basis while they are not in the parallelization basis. This is because the metric matrix is the identity matrix (δ_{jk}) . Consequently, mesh indices can not be raised or lowered using the metric g_{jk} .
- In local coordinates the structure coefficients vanish: $[\partial_\mu, \partial_\nu] = 0$ (while in the parallelization basis the global structure coefficients are alive: $[X_j, X_k] = C_{jk}^h X_h$). For this reason the structure coefficients, in local coordinates, have no effect and the second column of the above table give thus the usual expressions we are accustomed to [14]. As an example, as the connection coefficients depend on coordinate systems, the canonical connection coefficients do not vanish in the natural basis (while they vanish in the parallelization basis).
- For physical applications, especially in general relativity and gravitation, one can assign a signature to the positive definite metric g defined by (2.1). This can be achieved, for $n = 4$, by writing $g = \eta_{ij} \Omega_i \otimes \Omega_j$, where $\eta_{ij} = 0$ for $i \neq j$, $\eta_{ij} = -1$ for $i = j = 0$, $\eta_{ij} = +1$ for $i = j = 1, 2, 3$. The metric g is thus nomore positive definite but rather nondegenerate.

References

- [1] F. Brickell and R. S. Clark, *Differentiable manifolds*, Van Nostrand Reinhold Co., 1970.
- [2] A. Einstein, *Unified field theory based on Riemannian metrics and distant parallelism*, Math. Annal., **102** (1930), 685-697.
- [3] A. Einstein, *The meaning of relativity*, 5th. Ed., Princeton Univ. Press, 1955.
- [4] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I. Interscience Publ., 1963.
- [5] W. Kühnel, *Differential geometry, Curves-surfaces-manifolds*, Amer. Math. Soc., 2nd. Ed., 2006.
- [6] F. I. Mikhail and M. I. Wanas, *A generalized field theory, I. Field equations*, Proc. R. Soc. London, **356** (1977), 471-481.
- [7] C. Moller, *Conservation laws and absolute parallelism in general relativity*, Mat. Fys. Skr. Dan. Vid. Selsk., **39** (1961), 1-50.
- [8] Gamal G. L. Nashed, *Vacuum non singular black hole solutions in tetrad theory of gravitation*, Gen. Rel. Grav., **34** (2002), 1047-1058.
- [9] Gamal G. L. Nashed, *Charged axially symmetric solution, energy and angular momentum in tetrad theory of gravitation*, Int. J. Mod. Phys. A, **21** (2006), 3181-3197.
- [10] M. I. Wanas, *A generalized field theory and its applications in cosmology*, Ph. D. Thesis, Cairo Univ., 1975.
- [11] M. I. Wanas, *A self-consistent world model*, Astrophys. Space Sci., **154** (1989), 165-177.
- [12] M. I. Wanas, *Absolute parallelism geometry: developments, applications and problems*, Stud. Cercet. Stiin. Ser. Mat. Univ. Bacau, **10** (2001), 297-309.
- [13] M. I. Wanas, *The accelerating expansion of the universe and torsion energy*, Int. J. Mod. Phys. A, **22** (2007), 5709-5716.
- [14] Nabil L. Youssef and Amr M. Sid-Ahmed, *Linear connections and curvature tensors in the geometry of parallelizabl manifolds*, Rept. Math. Phys., **60** (2007), 39-53.