# PHYN001

## Cairo University, Faculty of Engineering

## Credit Hours System

### Fall 2016

# Unit 03 Universal Gravitation

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**Reference**

Serway, Raymond A. and Jewett, John W. *Physics for scientists and engineers with modern physics*. $9^{th}$ Ed.

# UNIVERSAL GRAVITATION

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# UNIVERSAL GRAVITATION

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# NEWTON'S LAW OF UNIVERSAL GRAVITATION

- Every particle ($m \neq 0$) in the universe attracts every other particle with force of magnitude proportional
  - directly to the product of masses
  - inversely to square of distance between them

$$F_g = G\frac{m_1 m_2}{r_{12}^2}$$

$G$: the universal gravitational constant

$G = 6.673 \times 10^{-11}$ Nm²/kg² measured experimentally

# NEWTON'S LAW OF UNIVERSAL GRAVITATION

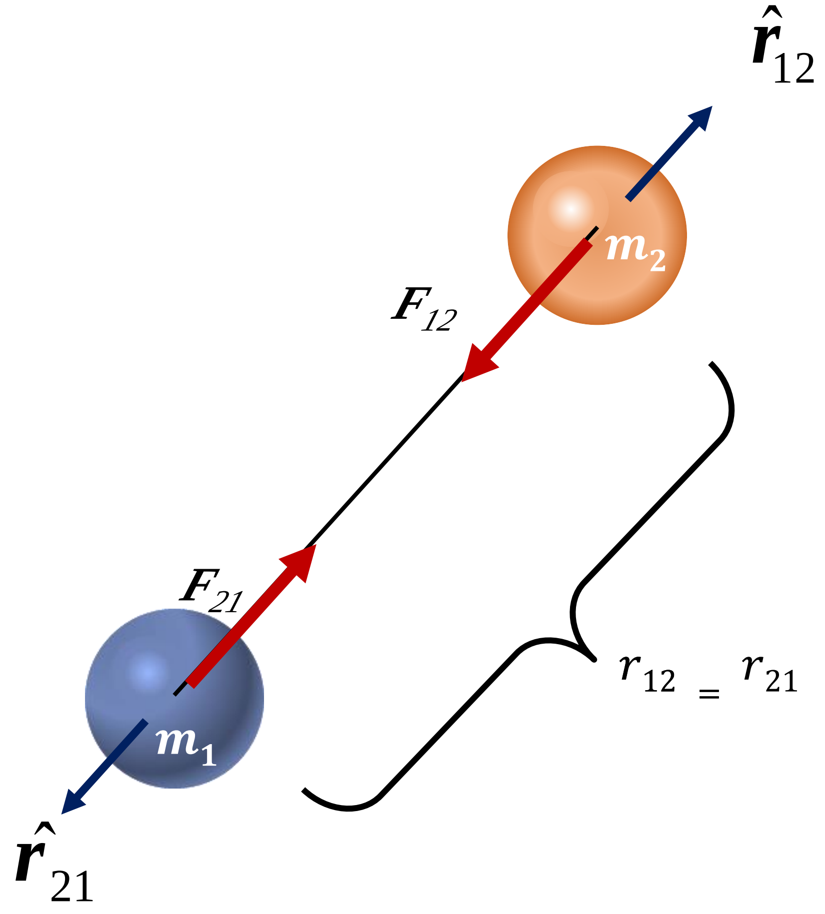
- The vector nature of the force can be written as

$$\underline{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\underline{\hat{r}}_{12}$$

$$\underline{F}_{21} = -G\frac{m_2 m_1}{r_{21}^2}\underline{\hat{r}}_{21}$$

- But $\underline{\hat{r}}_{12} = -\underline{\hat{r}}_{21}$

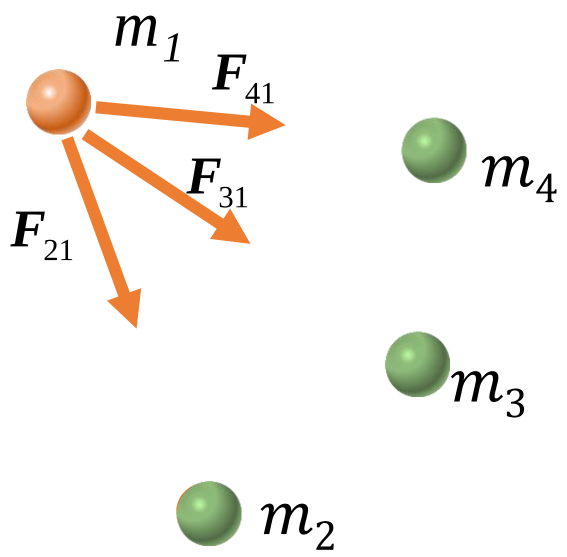
$$\therefore\ \underline{F}_{12} = -\underline{F}_{21}$$



# Newton's Law of Universal Gravitation

- Principle of Superposition of gravitational forces

$$\underline{F}_1 = \underline{F}_{21} + \underline{F}_{31} + \underline{F}_{41}$$

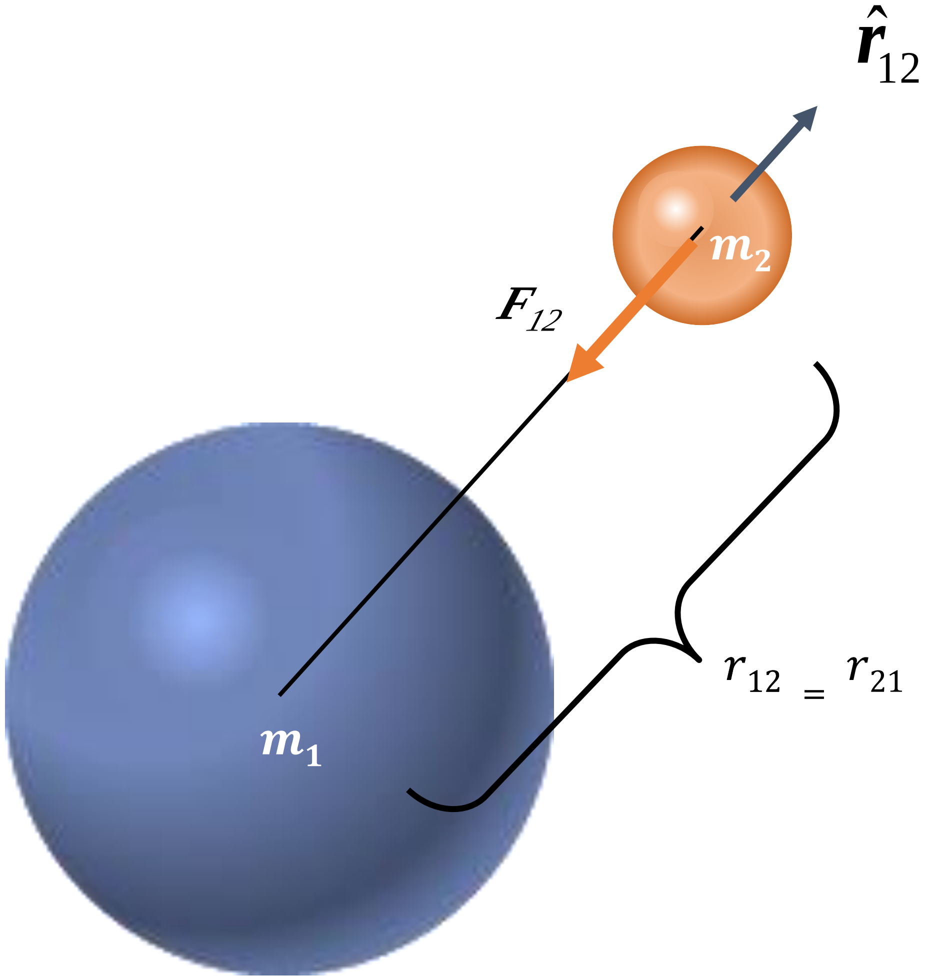


# Newton's Law of Universal Gravitation

- The Mutual force between a uniform sphere and a particle is given by the same equation

$$\underline{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\underline{\hat{r}}_{12}$$

$r_{12}$ is the distance between the particle and the center of the sphere.

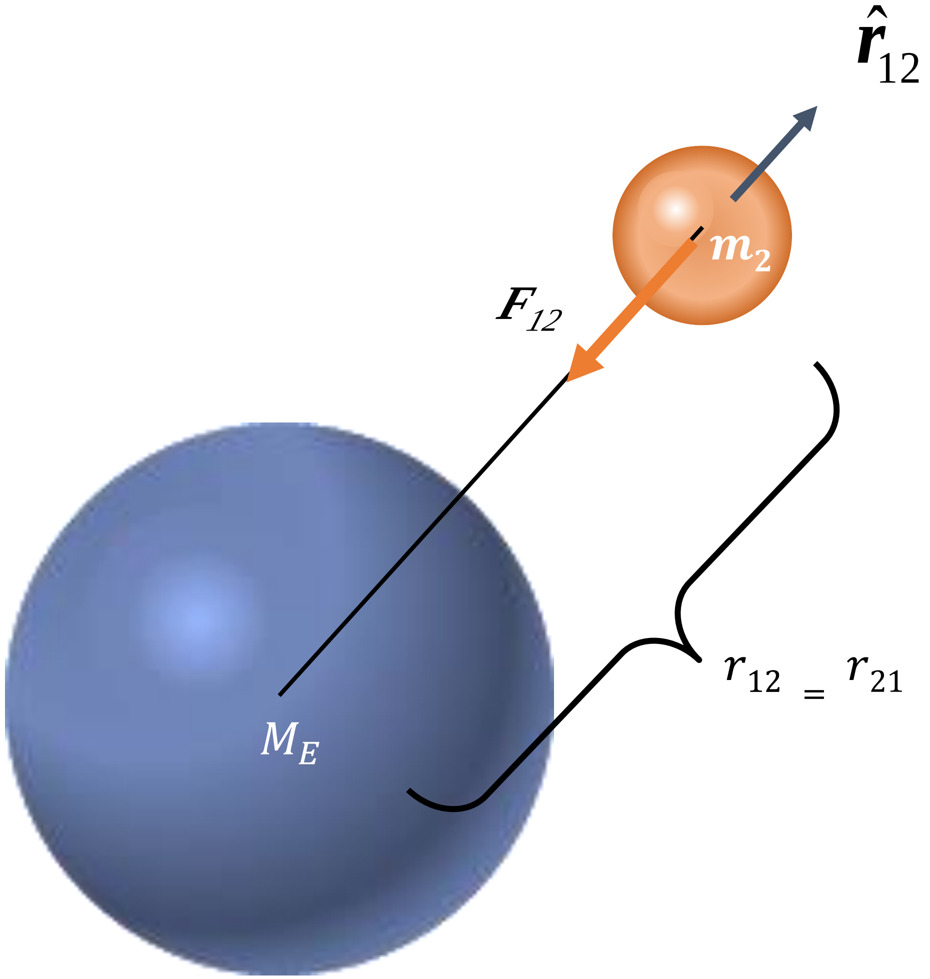


# NEWTON'S LAW OF UNIVERSAL GRAVITATION

- What about the magnitude of force between earth and any particle ???

$$F = G\frac{M_E m}{r^2}$$

  - $M_E$ is the mass of earth
  - $r$ is the distance between the particle $m$ and center of earth.



# UNIVERSAL GRAVITATION

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# Free-Fall Acceleration and the Gravitational Force

The weight of a particle of mass $m$ on the surface of the earth is given by

$$F = G\frac{M_E m}{R_E^2} = mg$$

$$\therefore g = G\frac{M_E}{R_E^2}$$

# Free-Fall Acceleration and the Gravitational Force

**Example:** Using the known Earth radius and $g$ = 9.80 m/s$^2$ at the Earth surface, find the <u>average density</u> of the Earth.

**Solution:**

$$g = G\frac{M_E}{R_E^2} \quad \rightarrow \quad \therefore M_E = \frac{gR_E^2}{G} = \rho_{Avg}V_E = \rho_{Avg}\left(\frac{4}{3}\pi R_E^3\right)$$

$$\therefore \rho_{Avg} = \frac{3\mathrm{g}\mathrm{R}_\mathrm{E}^2}{4\pi G R_E^3} = \frac{3g}{4\pi G R_E}$$

$$\therefore \rho_{Avg} = \frac{3(9.8)}{4\pi(6.67\times10^{-11})(6.37\times10^{6})} = 5.5\times10^3 \ \text{kg/m}^3$$

# FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

If the particle is at a distance $h$ above earth surface, how can you calculate $g$ ???

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

$$\therefore g = G\frac{M_E}{(R_E + h)^2}$$

$$g = G\frac{M_E}{R_E^2}(1 + h/R_E)^{-2}$$

If $h/R_E \ll 1$

$$\therefore g \cong G\frac{M_E}{R_E^2}(1 - 2h/R_E)$$

# FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

Variation of $g$ with altitude $h$

| Altitude $h$ (km) | $g$ (m/s$^2$) |
|---|---|
| 1 000 | 7.33 |
| 2 000 | 5.68 |
| 3 000 | 4.53 |
| 4 000 | 3.70 |
| 5 000 | 3.08 |
| 6 000 | 2.60 |
| 7 000 | 2.23 |
| 8 000 | 1.93 |
| 9 000 | 1.69 |
| 10 000 | 1.49 |
| 50 000 | 0.13 |
| $\infty$ | 0 |

# FREE-FALL ACCELERATION AND THE GRAVITATIONAL FORCE

Actual free fall acceleration near surface of earth is a little different than $g = G\, M_E / R_E^2$ due to three reasons:

1. Earth density is not uniform
2. Earth is not a perfect sphere
3. Earth is rotating

# UNIVERSAL GRAVITATION

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# Kepler's Laws and the Motion of Planets

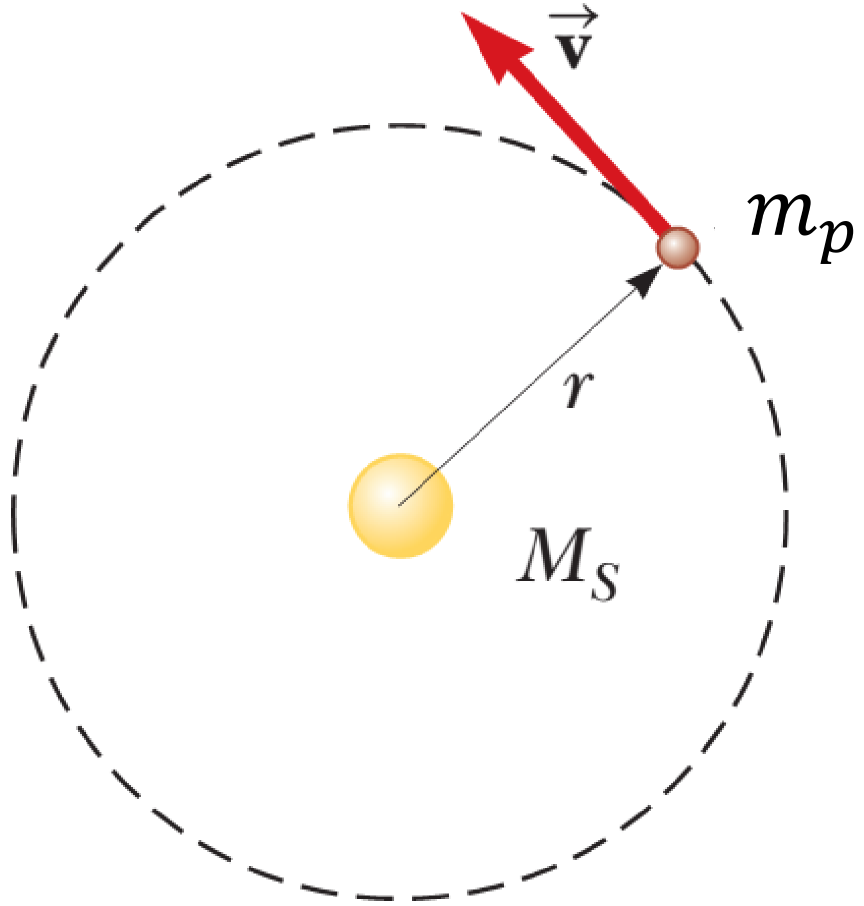
1. All planets move in elliptical orbits with the Sun at one focus.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Watch animation on
http://astro.unl.edu/naap/pos/animations/kepler.swf

# Kepler's Laws and the Motion of Planets

- If a small object $M_p$ is moving around a uniform sphere of $M_s \gg m_p$, the motion of the orbit is an ellipse.
- A special case of the ellipse is a circle.
- A particle moving in a circle of radius $R$ with constant speed $v$ will have a centripetal acceleration given by

$$a_c = v^2/R$$



# Kepler's Laws and the Motion of Planets

$$a_c = \frac{v^2}{R}$$

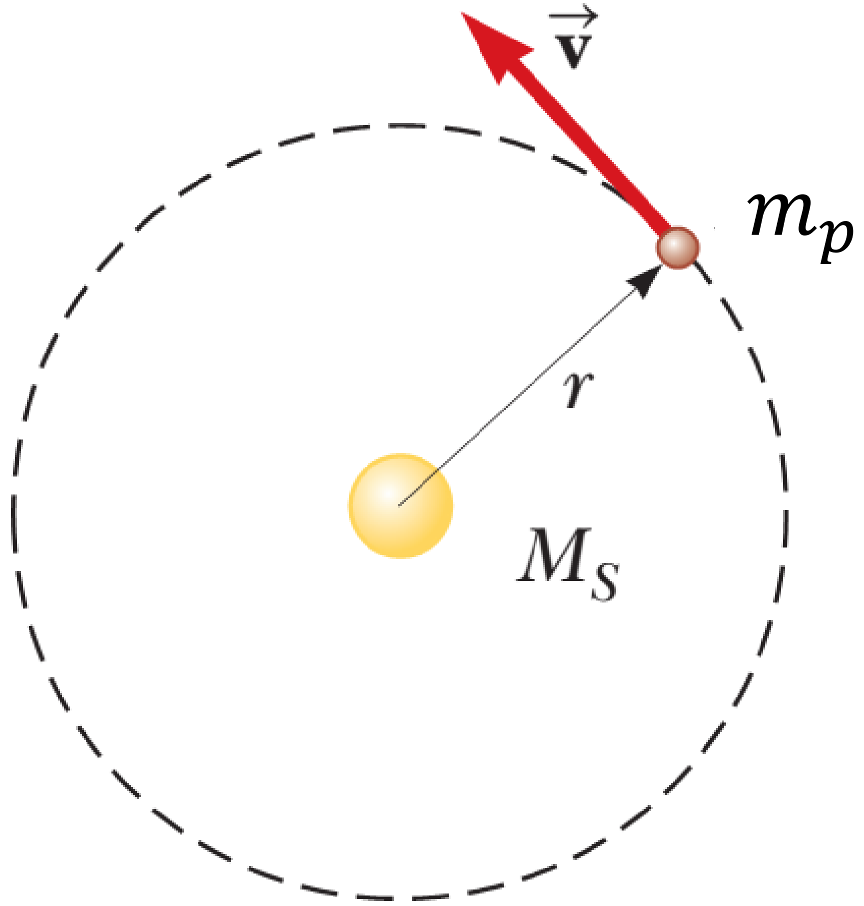
$$F_g = F_c$$

$$G\frac{M_s m_p}{R^2} = \frac{m_p v^2}{R}$$

$$G\frac{M_s m_p}{R^2} = \frac{m_p (2\pi R/T)^2}{R}$$

$$\therefore \quad \boxed{T = 2\pi\sqrt{\frac{R^3}{M_s G}}}$$

Kepler's Third Law



# KEPLER'S LAWS AND THE MOTION OF PLANETS

**Example**: Calculate the mass of Sun using the facts that the period of the Earth's orbit around the Sun is $3.156 \times 10^7\ s$ and its distance from the Sun is $1.496 \times 10^{11}$ m.

**Solution**

$$T = 2\pi\sqrt{R^3/(M_S G)}$$

$$\therefore M_S = 4\pi^2\, R^3/(T^2 G)$$

$$\therefore M_S = 4\pi^2 \frac{(1.496 \times 10^{11})^3}{(3.156 \times 10^7)^2 (6.67 \times 10^{-11})}$$

$$\therefore M_S = 1.99 \times 10^{30} \quad \text{Kg}$$

# Kepler's Laws and the Motion of Planets

**Example**: Consider a satellite of mass m moving in a circular orbit around the Earth at a constant speed $v$ and at an altitude $h$ above the Earth's surface.

- Determine the speed of the satellite in terms of $G$, $h$, $R_E$ (the radius of the Earth), and $M_E$ (the mass of the Earth).

**Solution**:

$$G\frac{Mm}{(R+h)^2} = \frac{mv^2}{R+h}$$

$$\therefore v = \sqrt{MG/(R+h)}$$

# KEPLER'S LAWS AND THE MOTION OF PLANETS

**Example:** If the satellite is geosynchronous (appearing to remain over a fixed position on the Earth), how fast is it moving through space?

**Solution**

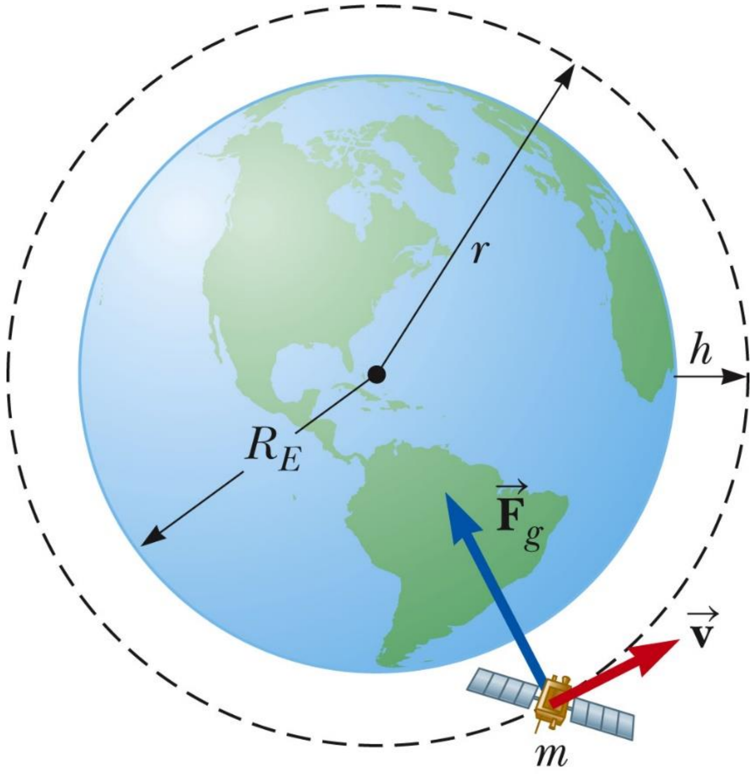
$$T_{Sat} = T_E = 24 \times 60 \times 60$$

$$T_{Sat} = 2\pi\sqrt{\frac{(R+h)^3}{MG}} = 2\pi\sqrt{\frac{(R+h)^3}{(5.98 \times 10^{24})(6.67 \times 10^{-11})}}$$

$$\therefore R + h = 4.23 \times 10^7 \text{ m}$$

$$v = \sqrt{MG/(R+h)}$$

$$\therefore v = 3.07 \times 10^3 \text{ m/s}$$



# UNIVERSAL GRAVITATION

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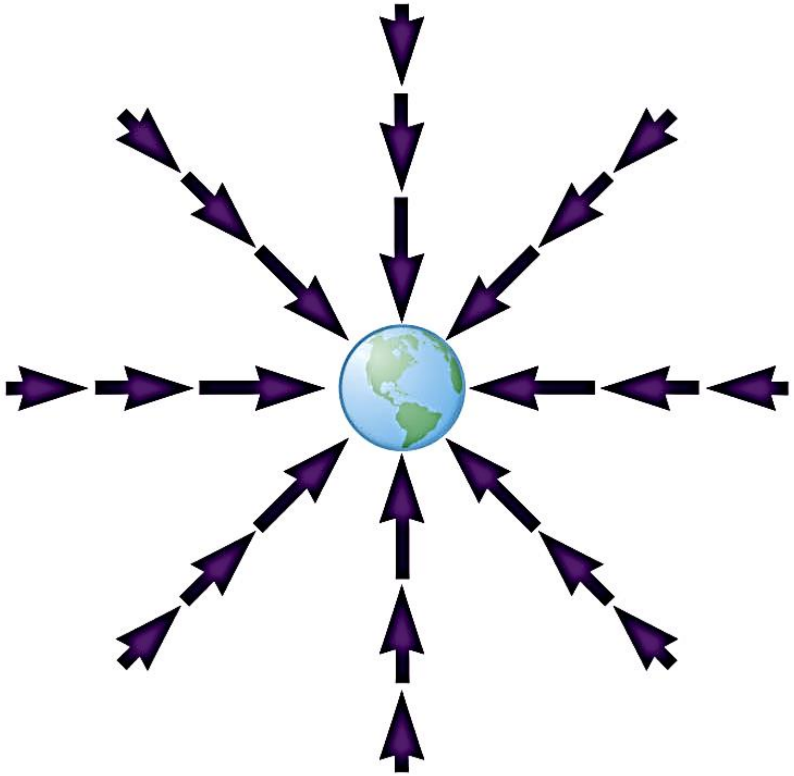
# The Gravitational Field

- A new approach came after Newton's death describing interactions between not-in-contact objects: Gravitational Field that exists at every point in space.
- When a particle of mass $m$ is placed at a point where the gravitational field is $g$, the particle experiences a force $F_g = mg$
- The gravitational field $\underline{g}$ is defined as $\underline{g} = \underline{F}_g/m$
- We call the object creating the field the source particle.

# The Gravitational Field

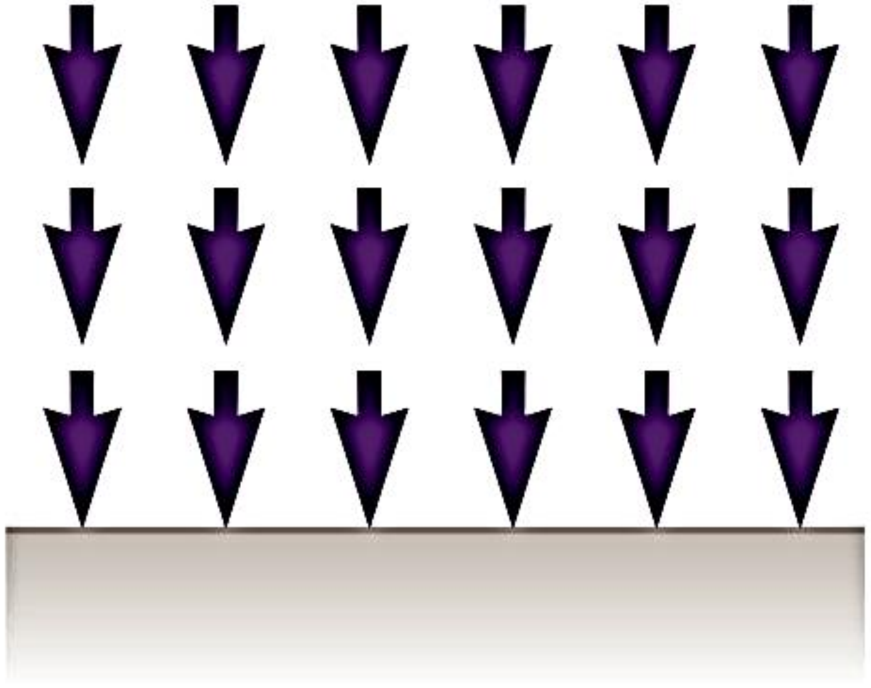
$\underline{g}$ in the vicinity of a uniform spherical $m$ varys in direction and magnitude.

- $\left|\underline{g}\right|$ at any point is the magnitude of the free-fall acceleration at that point.



# THE GRAVITATIONAL FIELD

$\underline{g}$ near the Earth's surface is uniform in both direction and magnitude.



# The Gravitational Field

- The Earth's gravitational field as a vector:

$$\underline{g} = -\frac{GM_E}{r^2}\,\underline{\hat{r}}$$

  $\underline{\hat{r}}$ : unit vector pointing from center of Earth to where we are calculating the field.

- The negative sign indicates that the field points toward the center of the Earth

# UNIVERSAL GRAVITATION

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# GRAVITATIONAL POTENTIAL ENERGY

Gravitational force is conservative and a potential energy can be defined for it

**The change in gravitational potential energy of a system**

- Work done by external force against gravitational force $\underline{F}_g$ to move a particle from $r_i$ to $r_f$

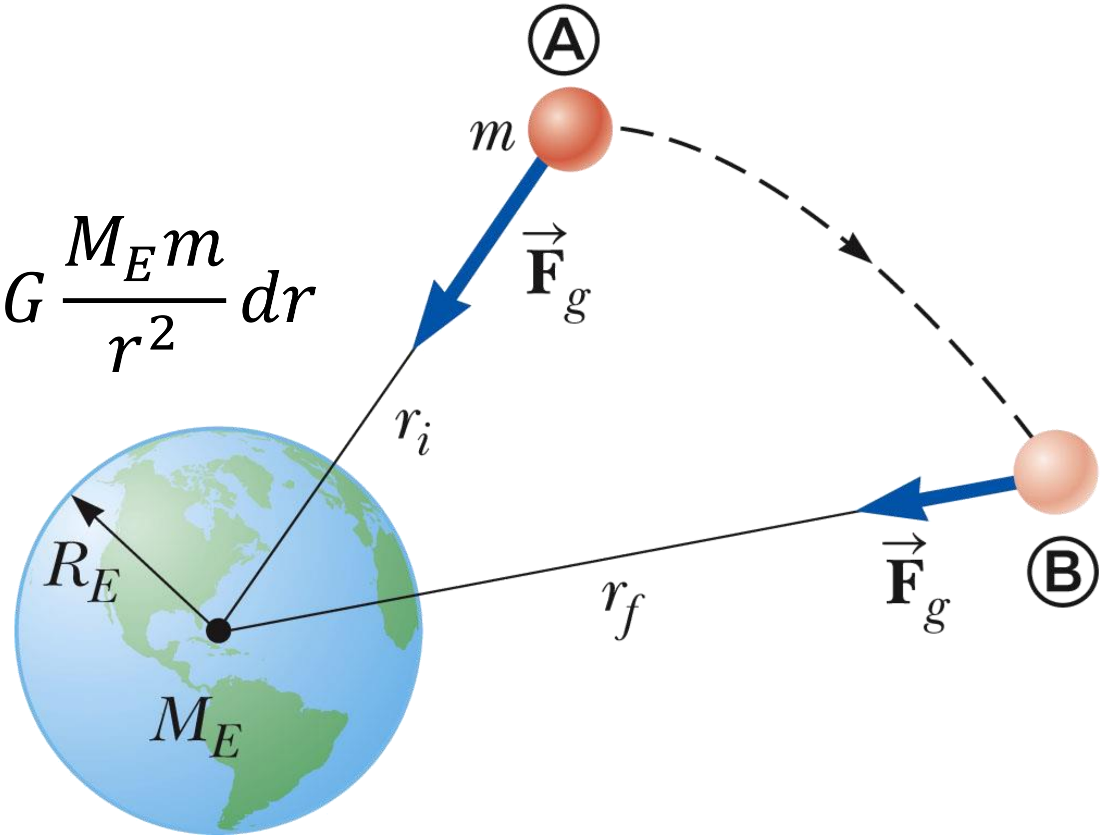
$$\boldsymbol{\Delta U} = U_f - U_i = -\int_{r_i}^{r_f} F_g \, dr$$

# Gravitational Potential Energy

$$U_f - U_i = -\int_{r_i}^{r_f} F_g \, dr$$

$$U_f - U_i = -\int_{r_i}^{r_f} -G\frac{M_E m}{r^2} dr = \int_{r_i}^{r_f} G\frac{M_E m}{r^2} dr$$

$$\therefore U_f - U_i = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$



# Gravitational Potential Energy

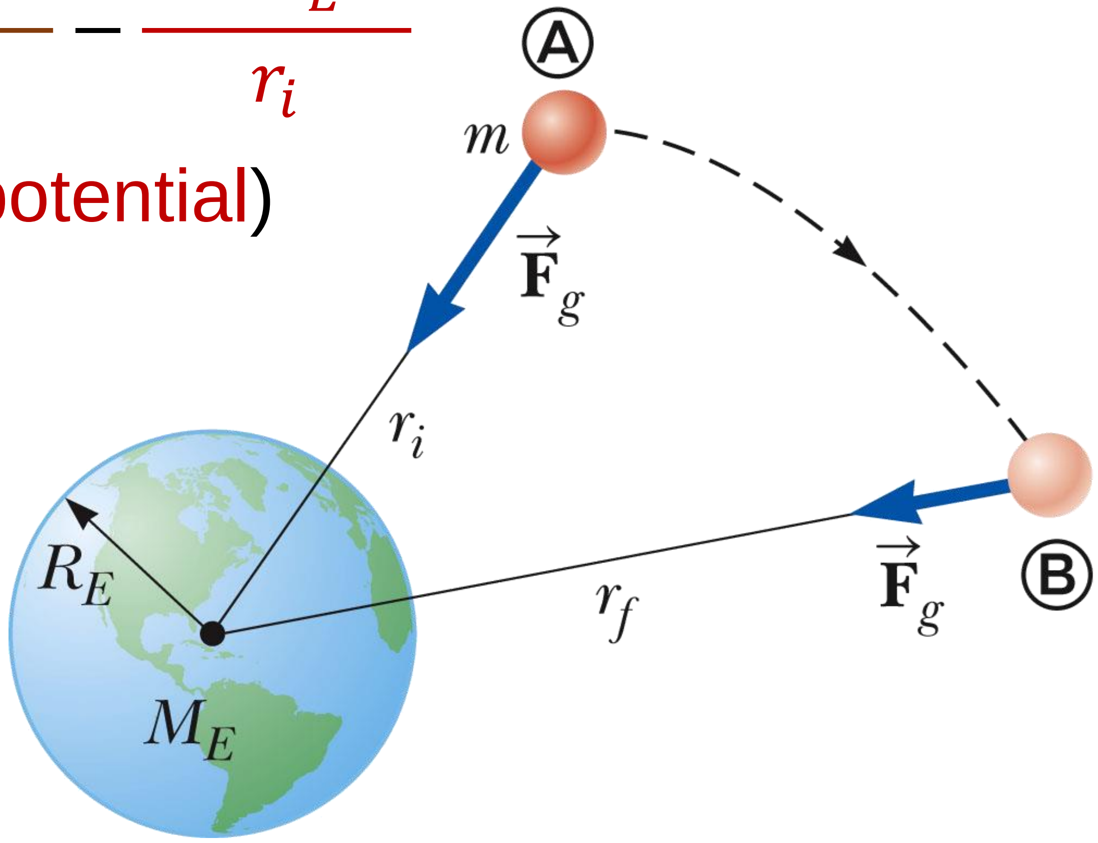
$$U_f - U_i = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = \frac{-GM_E m}{r_f} - \frac{-GM_E m}{r_i}$$

Taking $U_i \rightarrow 0$ at $r_i \rightarrow \infty$ (reference potential)

$$\therefore U_f = -G\frac{M_E m}{r_f}$$

$$\boxed{U(r) = -G\frac{M_E m}{r}}$$ Valid for $r \geq R_E$

The negative sign is because of our choice of $U_{ref}$



# Gravitational Potential Energy

For small vertical displacement $\Delta h$ near the Earth's surface

$$\Delta U = -GM_E m\left(\frac{1}{r+\Delta h} - \frac{1}{r}\right)$$

$$\therefore \Delta U = -GM_E m\,\frac{r-(r+\Delta h)}{r(r+\Delta h)} \simeq -GM_E m\,\frac{-\Delta h}{r^2}$$

$$\because g(r) = G\frac{M_E}{r^2} \approx g$$

$$\boxed{\therefore \Delta U = mg\Delta h}$$

# GRAVITATIONAL POTENTIAL ENERGY

**Binding Energy**

- The absolute value of the potential energy can be thought of as the binding energy.
- If an external agent applies a force larger than the binding energy, the excess energy will be in the form of kinetic energy of the particles when they are at infinite separation.

# GRAVITATIONAL POTENTIAL ENERGY

**Example**

How much work is done by the Moon's gravitational field on a 1000 kg meteor as it comes in from outer space and impacts on the Moon's surface? $G = 6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
$M_{Moon} = 7.36 \times 10^{22}\ \mathrm{Kg}\ \ R_{Moon} = 1.74 \times 10^{6}\ \mathrm{m}$

**Solution**

$$W_g = -\Delta U = +\frac{GM_{Moon}m}{R_{Moon}}$$

$$W = \frac{6.67 \times 10^{-11} \cdot 7.36 \times 10^{22} \cdot 1000}{1.74 \times 10^{6}} = 2.9 \times 10^{9}\ \mathrm{Joule}$$

# UNIVERSAL GRAVITATION

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# Energy Considerations in Planetary and Satellite Motion

Since gravitational forces are conservative, in the absence of any other forces the total mechanical energy will be conserved

$$E = K + U = \frac{1}{2}m\upsilon^2 - G\frac{Mm}{r} = \text{Constant}$$

$$\therefore \frac{1}{2}m\upsilon_f^2 - G\frac{Mm}{r_f} = \frac{1}{2}m\upsilon_i^2 - G\frac{Mm}{r_i}$$

$E$ may be positive, negative, or zero, depending on $\upsilon$.

However $E$ is necessarily less than zero in a bound system

# Energy Considerations in Planetary and Satellite Motion

For an orbiting satellite

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = \frac{mv^2}{r}$$

$$\therefore K = \frac{1}{2}mv^2 = \frac{1}{2}G\frac{Mm}{r} = -\frac{1}{2}U$$

$$\therefore E = K + U = \frac{1}{2}U = -\frac{1}{2}G\frac{Mm}{r}$$

Negative (Bound)

# Energy Considerations in Planetary and Satellite Motion

**Example**

A 1000 kg satellite orbits the Earth at an altitude of 100 km. (a) How much energy must be added to the system to move the satellite into a circular orbit with altitude 200 km? (b) What are the changes in the system's kinetic energy and potential energy?

**Solution**

$$E = \frac{1}{2}U = -\frac{GM_E m_s}{2r} \quad \rightarrow \quad \Delta E = E_f - E_i = -\frac{GM_E m_s}{2}\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = 469\ \text{MJ}$$

$$\Delta U = 2\Delta E = 938\ \text{MJ} \qquad \Delta K = -\Delta E = -469\ \text{MJ}$$

# UNIVERSAL GRAVITATION

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# Universal Gravitation (Summary)

$$\underline{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\underline{\hat{r}}_{12} = -\underline{F}_{21}$$

$$\underline{F} = m\underline{g}$$

$$\underline{g} = -\frac{GM_E}{r^2}\underline{\hat{r}}$$

$$U(r) = -G\frac{M_E m}{r}$$

**In Orbit**

$$G\frac{mM_E}{r^2} = m\frac{v^2}{r}$$

$$T^2 \propto R^3$$

$$E = K + U = \frac{1}{2}U = -K < 0$$