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Numerical Study of Some Systems of Linear Algebraic Equations with Noise Related to Boundary-value Problems for Laplace's Equation in Rectangular Domains

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Abstract: We investigate two stable methods for solving the system of linear algebraic equations arising from plane, singular boundary-value problems for Laplace's equation for rectangular domains [1, 21]. The dynamical systems method and the variational regularization method are applied to derive stable solutions for these systems, and the results are compared with those obtained from the QR-factorization technique contaminated with noise. The results put in evidence the difficulties that may arise from the use of the QR-factorization method due to instability.

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INTRODUCTION

The two-dimensional boundary-value problems for Laplace's equation arise in various branches of Science, as useful models for the more realistic, three-dimensional problems. A tremendous number of applications of these problems may be found in the literature, for example in Engineering problems related to nondestructive testing techniques [2, 4, 5, 7, 8], in Geophysics and in Biomechanics [10, 35], in problems of heat transfer [11, 12]. Such problems are known to be generally ill-posed in the sense that a small change in the data may result in a dramatic change in the solution. Under an additional a priori boundedness condition, a continuous dependence of the solution on the data can be obtained. This is called conditional stability [24]. Other conditional stability conditions for Laplace's equation may be found in [9, 16]. Due to the severely ill-posedness of such problems, numerical computation becomes quite difficult. In order to obtain a stable numerical solution for these kinds of ill-posed problems, many regularization methods have been proposed, among which: The quasi-reversibility method [20, 22], the Tikhonov regularization method [3, 36], the boundary element method [17], the discretization method [13, 32], the moment problem method [6, 19], Fourier regularization method [14], the optimal regularization method [35], and others. The present paper is devoted to the study of systems of linear algebraic equations that arise from two plane singular boundary-value problems for Laplace's equation describing steady-state heat conduction in rectangular plates [1, 21]. The use of a modified variant of Trefftz's method for finding approximate solutions ultimately produces rectangular systems of linear algebraic equations. These are characterized by large condition numbers, due to the presence of boundary singularities for the solution functions, in addition to the effect of the corner boundary points. The QR-factorization method was used in [1] to obtain efficient approximate solutions for the considered boundary-value problems, in the sense that these solutions can be easily computed as finite linear combinations of harmonics, and the resulting error in satisfying the boundary conditions is relatively low as compared to that obtained by the use of the standard Boundary Collocation Method or by the standard Finite Element Method. It is our purpose here to compare the performance of the QR-factorization method contaminated with noise in dealing with such systems of linear algebraic equations, with two other methods which are known to be stable: The Dynamical Systems Method (DSM) and the Variational Regularization Method (VRM). The numerical results presented here below put in evidence the difficulties that may arise from the use of the QR-factorization method in solving the systems of linear algebraic equations due to instability. The method of solution as given in [1] is briefly presented.

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TREFFTZ'S METHOD

Let D be a simply connected region in the plane, bounded by a contour C of finite length L and let $t \in [0,T]$ be a real parameter characterizing the points of the contour C, starting from a point P_0 on C. In particular, t may be the arc length s measured on C anticlockwise, starting from P_0. Extension to multiply-connected domains is straightforward. Consider the following boundary-value problem for Laplace's equation for the unknown function U:

$$\Delta U(r) = 0 \text{ in } D \qquad (2.1)$$

$$WU(t) = f(t) \text{ on } C \qquad (2.2)$$

where r is the position vector of a general point $P \in D$, W is a linear partial differential operator and f is a given function on C. Trefftz's method relies on a choice of an expression for the approximate solution that exactly satisfies the Laplace's equation in D. The task then reduces to satisfying the boundary conditions approximately using well-known techniques (Galerkin-type Method, Boundary Collocation Method, Least Squares Method, Boundary Fourier Expansion Method, etc [1, 25]).

A variant of Trefftz's method by boundary fourier expansion (BFEM): This method was developed in [1] to find approximate solutions of some two-dimensional regular or singular boundary-value problems for Laplace's equation. After expressing the approximate solution U_a as a finite linear combination of trial functions $\{ \varphi_i$: for i = 0,1, …, N } in the form :

$$U_a(r) = \sum_{i=0}^{N} a_i \varphi_i(r)$$

where a_i, i = 0,1,2,3,…,N are constant, according to this variant, verification of the boundary condition produces a "boundary function" that assumes, in principle, zero values on some interval on the real axis corresponding to the boundary. This function is then expanded in a Boundary Fourier Expansion (the method is subsequently denoted BFEM) in terms of a properly chosen set of orthogonal trigonometric functions. Equating to zero as many Fourier coefficients of this boundary function as needed, according to the level of approximation, yields a rectangular system of linear algebraic equations for the coefficients of the expansion as follows:

For the above defined problem, consider a set of harmonics $\{\varphi_i(r), i = 0,1,\ldots, N\}$. This set of linearly independent "trial functions" is required to generate the approximate solution $U_a(r)$ as a linear combination of the functions $\varphi_i(r)$ with a certain error tolerance. This subject, related to the completeness property, was investigated for special sets of trial functions, among which the set of polar harmonics

$$1, \ln r, \theta, r^{\pm\lambda_i} \cos\lambda_i\theta,\ r^{\pm\lambda_i} \sin\lambda_i\theta,\ i = 1,2,\ldots$$

where λ_i are constants, in which we are presently interested. An additional factor determining the choice of the trial functions would be the possibility of satisfaction of the boundary condition on certain parts of the boundary from the outset. Thus, the linear combination

$$U_a(r) = \sum_{i=0}^{N} a_i \varphi_i(r) \qquad (2.1.1)$$

rigorously satisfies Eq. (2.1) and, possibly, the boundary condition (2.2) on certain parts of the boundary. The number N is usually referred to as the number of degrees of freedom. The unknown coefficients will now be determined so as to enforce the boundary condition on the remaining part of the boundary.

The proposed method relies on the following idea: Substitution of (2.1.1) into (2.2) yields

$$ER(t) \equiv \sum_{n=0}^{N} a_n W\varphi_n(t) - f(t),\ t \in [0,T] \qquad (2.1.2)$$

Extending the function ER (t) evenly to the interval [-T, 0], one obtains a function that, hopefully, should vanish on [-T, T]. The Fourier coefficients of this function with respect to the orthonormal set of functions $\{1, \cos m\pi t / T\}$, m = 1,2, …, should then vanish.

Now, the Fourier expansion of ER (t) i

$$ER(t) = \frac{c_0}{2} + \sum_{j=1}^{\infty} c_j \cos\frac{j\pi t}{T}$$

where c_j are the Fourier coefficients given by

$$c_j = \frac{2}{T}\int_0^T ER(t)\cos\frac{j\pi t}{T}\,dt$$

Setting to zero the first M Fourier coefficients generates a rectangular system of linear algebraic equations of size M×N for the expansion coefficients $\{a_i, i = 0,1,\ldots, N\}$ in the form :

$$\sum_{n=0}^{N-1} A_{mn} a_n = B_m, \quad m = 0,1,\ldots,M-1 \qquad (2.1.3)$$

as follows:
since,

$$c_j = \frac{2}{T}\int_0^T ER(t)\cos\frac{j\pi t}{T}dt$$
$$= \frac{2}{T}\int_0^T \Big(\sum_{n=0}^{N} a_n W\varphi_n(t) - f(t)\Big)\cos\frac{j\pi t}{T}dt$$

therefore,

$$c_j = \frac{2}{T}\int_0^T \Big(\sum_{n=0}^{N} a_n W\varphi_n(t)\cos\frac{j\pi t}{T} - f(t)\cos\frac{j\pi t}{T}\Big)dt$$

Setting $c_0 = c_1 = \ldots = c_{M-1} = 0$ implies that:

$$\sum_{n=0}^{N}\Big[a_n \int_0^T W\varphi_n(t)\cos\frac{m\pi t}{T}dt\Big] = \int_0^T f(t)\cos\frac{m\pi t}{T}dt$$

for m = 0, 1,…, M-1, i.e.

$$\sum_{n=0}^{N-1} A_{mn} a_n = B_m$$

for m = 0, 1,…, M-1, where

$$A_{mn} = \int_0^T W\varphi_n(t)\cos\frac{m\pi t}{T}dt, \quad B_m = \int_0^T f(t)\cos\frac{m\pi t}{T}dt$$

for m = 0, 1,…, M-1, n = 0, 1,…, N-1.

The number M may be increased until some error criterion is satisfied. For our purposes, one of two measures of error will be considered hereafter:

(i) The maximal boundary error (ERB) measuring the largest error in satisfying the boundary conditions:

$$ERB = \sup_{t\in[0,T]} |ER(t)| \qquad (2.1.4)$$

(ii) The maximal solution error (ERS) measuring the largest error between the approximate solution $U_a(r)$ and the exact solution (assumed known) $U_e(r)$ at certain properly chosen set of points in the domain of the solution:

$$ERS = \max_k |U_a(r_k) - U_e(r_k)| \qquad (2.1.5)$$

The error may be improved only by increasing the number of degrees of freedom N. BFEM clearly belongs to the semi-analytic methods of approximation in the mean. It has the advantage of improving the accuracy for any fixed number of degrees of freedom N by increasing the number of zeroed Fourier coefficients and does not suffer from the well-known problem of crowdedness, one of the main drawbacks of the classical Trefftz's method. The solved problems in [1] have clearly shown that BFEM generally performs better than other numerical methods and allows dealing effectively with the singular behavior of the solution on the boundary.

Regularization methods are often used to obtain stable and smooth solutions to some ill-posed problems. In the following two sections, we will present two different stable regularization techniques for solving some ill-posed problems affected with noise. The first regularization technique is the dynamical systems methods (DSM) which is proposed by A. G. Ramm [27-31] and the references therein). The DSM is based on analysis of the solution of Cauchy problem for linear and nonlinear differential equations in Hilbert space. Such an analysis was done for well-posed and some ill-posed problems [27] and the references therein). The DSM has several attractive properties; it is fast convergent, can be easily designed and no need to calculate the inverse of large condition number matrices. The second regularization technique considered here is the varitional regularization method (VRM). This is the most common and well known techniques for regularizing ill-posed problems, this method was developed independently by Phillips [26], Tikhonov [15, 23, 36, 37.

THE DYNAMICAL SYSTEMS METHOD

The dsm is a stable regularized algorithm for solving linear or nonlinear operator equations of the form

$$F(u) = Au - f = 0, u \in H \qquad (3.1)$$

especially when f is replaced by noise data f_δ, where H is a Hilbert space and A is a linear operator in H which is not necessarily bounded but closed and densely defined. The idea of DSM for solving (3.1) is to construct a function $\Phi(t,u)$ and solve the following Cauchy problem:

$$\dot{u} = \Phi(t,u), \quad u(0) = u_0; \; \dot{u} = \frac{du}{dt} \qquad (3.2)$$

A. G. Ramm [27, 31] proves that the function $\Phi(t,u)$ takes the form

$$\Phi(t,u) = -u + (T + a(t))^{-1} A^* f$$

where $T := A^* A$ and $a(t) > 0$ is non increasing function such that $\lim_{t\to\infty} a(t) = 0$.

Theorem: Assume that $f = Ay, y \perp N(y), A$ is a linear operator, closed and densely defined in H. Consider the problem

$$u' = -u + T_{\in(t)}^{-1} A^* f, \; u(0) = 0 \qquad (3.3)$$

$$N(A) := \{u : Au - f = 0\}, u_0 \in H$$

is arbitrary,

$$T_{\in} = T + \in(t), \; T = A^* A, \in = \in(t)$$

is a continuous function monotonically decaying to zero at $t \to \infty$ and $\int_0^\infty \in(s) ds = \infty$. Then problem (3.3) has a unique solution u(t) defined on $[0,\infty)$, and the following limit exists :

$$\lim_{t\to\infty} u(t) := u(\infty) \text{ and } u(\infty) = y$$

It is pointed out that if f_δ is given in place of the exact solution f, calculate its solution $u_\delta(t)$ as $t = t_\delta$, it can be proved that $\lim_{\delta\to 0} \| u_\delta(t_\delta) - y \| = 0$ if t_δ is suitable chosen.

The unique solution to (3.2) is given by [18] as:

$$u(t) = u_0 e^{-t} + e^{-t} \int_0^t e^s (T + a(s))^{-1} A^* f \, ds \qquad (3.4)$$

Equation (3.4) leads to the following iterative formula [18]:

$$u_{n+1} = e^{-h_n} u_n + (1 + e^{-h_n})(T + a_n)^{-1} A^* f_\delta, \; h_n = t_{n+1} - t_n \quad (3.5)$$

with a_0 satisfying

$$\delta < \| Au_{a_0} - f_\delta \| < 2\delta$$

Thus, u_n can be obtained iteratively if u_0, a_n and t_n are known. In our experiment; as suggested in [18], we choose

$$a_n = \frac{a_0}{1 + t_n}, h_n = q^n$$

where $1 \le q \le 2$, $t_0 = 0$ and

$$u_0 := u_{a_0} = (T + a_0 I)^{-1} A^* f_\delta$$

To improve the algorithm; we use q = 2 and the relaxed discrepancy principle: at each step iteration one checks if

$$0.9\delta < \| A_u - f_\delta \| < 1.001\delta \qquad (3.6)$$

The choice of a_0 satisfying (3.6) is carried out by iterations as follows [18]:

1. As an initial guess for a_0 one takes $a_0 = \frac{1}{3} \|A\|^2 \delta_{rel}$, where $\delta_{rel} = \frac{\delta}{\|f\|}$.
2. If $\frac{\| Au_{a_0} - f_\delta \|}{\delta} = c > 3$, then one takes $a_1 = \frac{a_0}{(c-1)}$ as the next guess and checks if condition (3.6) is satisfied. If 2<c = 3, then one takes $a_1 = \frac{a_0}{3}$.
3. If $\frac{\| Au_{a_0} - f_\delta \|}{\delta} = c < 1$, $a_1 = 3a_0$ is used as the next guess.
4. After a_0 is updated, one checks if (3.6) is satisfied. If (3.6) is not satisfied, one repeats steps 2 and 3 until a_0 are found to satisfy condition (3.6), for more details [18]. To find a solution of (3.1) we will apply the following algorithm :

Algorithm 1: DSM (A, f_δ, δ)

Step1: q:=2, The best value of q, (Note that $q^n = h_n$)

Step2: $g_\delta := A^* f_\delta$; $T := A^* A$;

Step3: $itermax := 30; u = (T + a_0)^{-1} g_\delta$;

Step 4: $i := 0$; $t = 1$; $h := 1$; halve:=0;

Step 5:

while $(1.001\delta \le \| Au - f_\delta \|$ and $(i < itermax)$ do

$i := i + 1; t = t + h; a = \frac{a_0}{t};$

$v := (T + aI)^{-1} g_\delta;$

$\tilde{u} = e^{-h} u + (1 - e^{-h})^{-1} v;$ The solution of ill-posed system

If $0.9\delta \le \| A\tilde{u} - f_\delta \|$ then

$u := \tilde{u};$

If halve = 0 then $h := hq$

elseif halve = 1; $t := t; h := \frac{h}{2};$

endif

endwhile

THE VARIATIONAL REGULARIZATION METHOD

This method attempts to provide a good estimate of:

$$\min\{\| Au - f_\delta \|^2 + \alpha \| u \|^2\} \qquad (4.1)$$

where f_δ is a noisy data and $\| f - f_\delta \| \le \delta$. The global minimizer of the quadratic functional (4.1) is the unique solution to the linear system

$$(A^* A + \alpha I) u_{\alpha,\delta} = A^* f_\delta$$

where I is the unit matrix. This system has a unique solution

$$u_{\alpha,\delta} = (A^* A + \alpha I)^{-1} A^* f_\delta$$

To determine the suitable α, let $u_{\alpha(\delta),\delta}$ be a solution of (4.1) and consider the equation

$$\| Au_{\alpha,\delta} - f_\delta \| = \tau\delta \qquad (4.2)$$

where $\tau \in]1,2[$. Equation (4.2) is the usual discrepancy principle. One can prove that equation (4.2) determines $\alpha = \alpha(\delta)$ uniquely,

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0, \text{and} u_\delta := u_{\alpha(\delta),\delta} \to y$$

where y is the minimal-norm solution to (3.1) as $\delta \to 0$. This justifies the usual discrepancy principle for choosing the regularization parameter [23]. For more details on the theory of Variational Regularization Method [27].

APPLICATIONS

Example 5.1: Consider the mixed boundary-value problem for Laplace's equation in a rectangular domain

$$\Delta U = 0, 0 < x < 1, 0 < y < E = 0.5 \qquad (5.1.1)$$

with the boundary conditions (Fig 5.1.1)

$$\begin{aligned} &\frac{\partial U}{\partial y} = 0, \quad 0 \le x \le 1, \; y = E \\ &U = 0, \quad 0 \le x \le 1, \; y = 0 \text{ and } x = 1, 0 \le y \le E \qquad (5.1.2) \\ &U = 1, \quad x = 0, \quad 0 \le y \le E \end{aligned}$$

The exact solution of the problem in terms of the orthogonal Cartesian coordinates (x, y) as suggested in [1] is

$$U_e = 1 - x - \frac{2}{p} \sum_{k=1}^{\infty} \frac{1}{K} \frac{\cosh[kp(\frac{1}{2} - y)]}{\cosh \frac{1}{2} kp} \sin(kpx) .$$

As seen from the boundary conditions, the solution of this problem has a singularity at the left lower corner of the domain. Consequently, the approximate solution is taken to be

$$U_a(r,q) = \frac{2}{p} q + \sum_{k=0}^{N-1} a_k \, r^{2k+2} \sin[(2k+2)q].$$

This satisfies the boundary conditions on the two sides of the rectangle lying on the coordinate axes Ox, Oy. Satisfaction of the other two boundary conditions yield

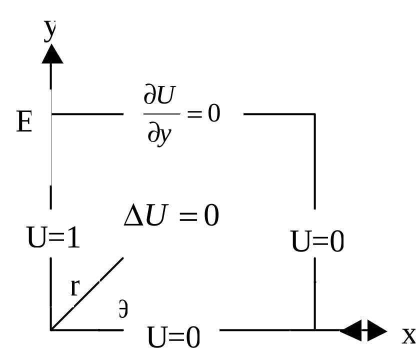

Fig. 5.1.1

$$0 = ER(\theta) = \begin{cases} U_a, & 0 \le \theta \le p, \\ \frac{\partial U_a}{\partial y} & p < \theta \le \frac{\pi}{2} \end{cases}, \quad p = \tan^{-1}\left(\frac{1}{2}\right) = \begin{cases} \frac{2\theta}{\pi} + \sum_{k=0}^{N-1} a_k (\sec\theta)^{2k+2} \sin(2k+2)\theta, & 0 \le \theta \le p, \\ \frac{2}{\pi}\sin\theta\cos\theta + \sum_{k=0}^{N-1} a_k (k+1)(\csc\theta)^{2k+1}(\tfrac{1}{2})^{2k+1}\cos(2k+1)\theta, & p < \theta \le \frac{\pi}{2} \end{cases}$$

where r must now be considered as a function of θ:

$$r(\theta) = \begin{cases} \sec\theta, & 0 \le \theta \le p \\ \frac{1}{2}\csc\theta, & p \le \theta \le \frac{\pi}{2} \end{cases}$$

Extending function ER (θ) symmetrically to [-π/2, 0], and equating to zero its first M Fourier coefficients leads to the system of linear algebraic equations (2.1.3) with

$$A_{mn} = \int_0^p (\sec t)^{2n+2} \sin[(2n+2)t] \cos 2mt \, dt + \tfrac{n+1}{2^{2n+1}} \int_p^{p/2} (\csc t)^{2n+1} \cos[(2n+1)t] \cos 2mt \, dt$$

$$m = 0,1,2,\ldots,M-1, \qquad n = 0,1,2,\ldots,N-1 \qquad (5.1.3)$$

and

$$B_m = -\tfrac{2}{p}\int_0^p t \cos 2mt \, dt - \tfrac{2}{p}\int_p^{p/2} \cos t \sin t \cos 2mt \; dt, \qquad m = 0,1,2,\ldots,M-1.$$

We have introduced a noise term in the last entry of the right-hand side of the considered systems of linear algebraic equations.

Example 5.2: Consider the mixed boundary-value problem for Laplace's equation in a rectangular region:

$$\Delta U = 0, \; 0<x<1, \; 0<y<E \qquad (5.2.1)$$

With the boundary conditions

$$\frac{\partial U}{\partial q} = 0, \; q = 0, \; 0 \le r \le 1 \; \textit{and} \; q = \frac{p}{2}, \; 0 \le r \le E, \qquad (5.2.2)$$

$$U = -\frac{1}{2}r^2, \; x = 1, \; 0 \le y \le E \; \textit{and} \; y = E, \; 0 \le x \le 1,$$

(r, θ) being the usual polar coordinates placed as shown in Fig 5.2.1

The exact solution of the problem in terms of the orthogonal Cartesian coordinates (x, y) as suggested in [21] is

$$U_e(x,y) = -1 + \frac{1}{2}(x^2 - y^2) + \frac{32}{p^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\,\frac{\cosh[(2k+1)\frac{p\,y}{2}]}{\cosh[(2k+1)\frac{p\,E}{2}]}\cos[(2k+1)\frac{p\,x}{2}] \qquad (5.2.3)$$

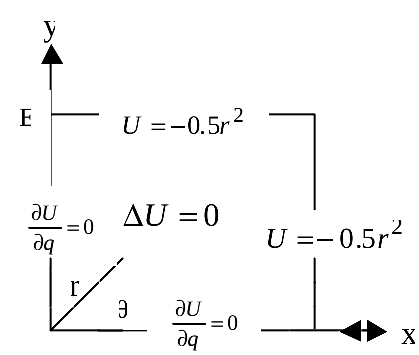

Fig. 5.2.1

It is not difficult to show [1] that the mixed second derivative of the exact solution has a singularity at the upper right corner of the domain. It behaves like (4/π)ln(1-x) as this point is approached along the upper boundary. This problem will be numerically solved for three cases, E = 0.5, 0.25 and 0.10.

The domain here is the region included in a rectangle of sides lengths 2,2E, by extending both the domain and the unknown function U symmetrically with respect to the coordinate axes Ox and Oy and in view of the existing symmetry, one may represent the approximate solution of these three cases as [1]

$$U_a(r,\theta)=\sum_{n=0}^{N-1}a_n r^{2n}\cos 2n\theta + a_s f(x,y)$$

$$\text{with}\quad r(\theta)=\begin{cases}\sec\theta, & 0\le\theta\le\alpha,\\ E\sec\theta, & \alpha<\theta\le\dfrac{\pi}{2}\end{cases}\qquad \alpha=\tan^{-1}E \tag{5.2.4}$$

where

$$f(x,y)=\xi(x,y)+\xi(-x,y)+\xi(x,-y)+\xi(-x,-y) \tag{5.2.5}$$

and

$$\xi(x,y)=\frac{2}{\pi}(\rho^2\ln\rho\sin 2\phi + \rho^2\phi\cos 2\phi) \tag{5.2.6}$$

is harmonic in the rectangle and has the required type of singularity at the upper right corner. Here, (ρ,φ) are polar coordinates centered at the above-mentioned corner, with initial line taken along the upper boundary. The case $a_S = 0$ corresponds to the expansion used in [21]. Here, for the improved version of the solution, this coefficient is taken equal to unity to take into account the singular behavior of the solution at the upper right corner of the domain. The proposed formula for the approximate solution satisfies identically the first of boundary conditions on two sides of the rectangles. Satisfaction of the remaining boundary conditions yields

$$ER(\theta)=\sum_{n=0}^{N-1}a_n r^{2n}\cos 2n\theta + f(r\cos\theta, r\sin\theta) + \frac{1}{2}r^2=0,\ 0\le\theta\le\frac{\pi}{2} \tag{5.2.7}$$

where r must now be considered as a function of θ:

$$r(\theta)=\begin{cases}\sec\theta, & 0\le\theta\le\beta\\ E\csc\theta, & \beta\le\theta\le\dfrac{\pi}{2}\end{cases} \tag{5.2.8}$$

Extending function ER (θ) symmetrically to [-π/2, 0] and equating to zero its first M Fourier coefficients leads to the system of linear algebraic equations (2.1.3) with

$$A_{mn}=\int_0^{\alpha}(\sec t)^{2n}\cos 2nt\cos 2mt\,dt + E^{2n}\int_{\alpha}^{\pi/2}(\csc t)^{2n}\cos 2nt\cos 2mt\,dt$$

$$m=0,1,\ldots,M-1,\quad n=0,1,\ldots,N-1$$

and

$$B_m=-\int_0^{\alpha}\left[\tfrac{1}{2}(\sec t)^2+f(1,\tan t)\right]\cos 2mt\,dt - \int_0^{\pi/2}\left[\tfrac{1}{2}E^{2n}(\csc t)^2+f(E\cot t,E)\right]\cos 2mt\,dt \tag{5.2.9}$$

$$m=0,1,\ldots,m-1$$

NUMERICAL RESULTS

In the following, the DSM and the VRM are applied to derive stable solutions for the two systems of linear equations arising from Example. 5.1 and Example. 5.2, when the right hand side B is perturbed by adding a noise term δ only to the last row. We will calculate the solution $a=(a_n)^T$ of the noisy system $Aa = B_\delta$, then substitute into (2.1.4),(2.1.5) to obtain DSM-ERB, DSM-ERS, VRM-ERB and VRM-ERS. These results are compared with QR-ERB and QR-ERS obtained by the QR-factorization technique contaminated with the same noise in the following tables; α represents the regularization parameter in VRM.

Numerical results of example 5.1: The results in Table 6.1.1 are obtained when M=60 and N=20, where the condition number of the matrix A is 4.6×10^6, E=0.5.

Table 6.1.1:

Noise	QR-ERB	DSM-ERB	VRM-ERB	QR-ERS	DSM-ERS	VRM-ERS	α
0.06	0.156	0.580×10^{-1}	0.460×10^{-1}	0.631×10^{-1}	0.556×10^{-1}	0.405×10^{-1}	0.02
0.9	2.342	0.433	0.339	0.947	0.609	0.493	0.46

Table 6.1.2:

Noise	QR ERB	DSM ERB	VRM ERB	QR ERS	DSM ERS	VRM ERS	α
0.0006	0.146×10^{-2}	0.813×10^{-3}	0.777×10^{-3}	0.192×10^{-2}	0.714×10^{-3}	0.741×10^{-3}	0.000022
0.09	0.220	0.930×10^{-1}	0.900×10^{-1}	0.287	0.821×10^{-1}	0.806×10^{-1}	0.015
0.9	2.195	0.678	0.661	2.875	0.716	0.672	0.2

Table 6.1.3:

Noise	QR-ERB	DSM ERB	VRM-ERB	QR-ERS	DSM-ERS	VRM-ERS	α
0.001	0.404×10^{-2}	0.936×10^{-3}	0.851×10^{-3}	0.426×10^{-2}	0.966×10^{-3}	0.916×10^{-3}	0.000027
0.005	0.200×10^{-1}	0.396×10^{-2}	0.363×10^{-2}	0.213×10^{-1}	0.391×10^{-2}	0.426×10^{-2}	0.000057
0.06	0.242	0.470×10^{-1}	0.400×10^{-1}	0.256	0.448×10^{-1}	0.459×10^{-1}	0.001
0.7	2.829	0.768	0.459	2.984	0.449	0.532	0.001

Table 6.1.4:

Noise	QR-ERB	DSMERB	VRM-ERB	QR-ERS	DSM-ERS	VRM-ERS	α
0.00009	0.106×10^{-2}	0.396×10^{-3}	0.234×10^{-3}	0.208×10^{-2}	0.621×10^{-3}	0.190×10^{-3}	0.0000028
0.0009	0.110×10^{-1}	0.235×10^{-2}	0.235×10^{-2}	0.208×10^{-1}	0.190×10^{-2}	0.190×10^{-2}	0.000022
0.3	3.522	0.420	0.415	6.929	0.414	0.402	0.08
0.9	10.567	0.933	0.961	20.787	1.115	1.071	0.45

Table 6.2.1:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.05	0.432	0.980×10^{-1}	0.780×10^{-1}	0.01
0.4	3.459	0.421	0.328	0.1
0.9	7.783	0.873	0.568	0.22

Table 6.2.2:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.005	0.520×10^{-1}	0.763×10^{-2}	0.437×10^{-2}	0.000318
0.01	0.104	0.120×10^{-1}	0.695×10^{-2}	0.000546
0.1	1.040	0.550×10^{-1}	0.520×10^{-1}	0.0023
0.3	3.121	0.128	0.128	0.0049
0.9	9.363	0.328	0.302	0.01

The results in Table 6.1.2 are obtained when M=72 and N=25, where the condition number of A is 3.7×10^{8}, E=0.5.

The results in Table 6.1.3 are obtained when M=80 and N=30, where the condition number of A is 3.4×10^{10}, E=0.5.

The results in Table 6.1.4 are obtained when M=80 and N=35, where the condition number of A is 3.4×10^{12}, E=0.5

Numerical results of example 5.2: The results in Table 6.2.1 are obtained when M=23 and N=14, where the condition number of A is 1.99×10^{4}, E=0.5.

The results in Table 6.2.2 are obtained when M=25 and N=11, where the condition number of A is 1.35×10^{5}, E=0.25.

The results in Table 6.2.3 are obtained when M=30 and N=15, where the condition number of A is 3.2×10^{7}, E=0.25

The results in Table 6.2.4 are obtained when M=25 and N=8, where the condition number of A is 2.5×10^{5}, E=0.1.

Table 6.2.3:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.00001	0.129×10^{-2}	0.449×10^{-4}	0.422×10^{-4}	0.0000008
0.0001	0.130×10^{-1}	0.386×10^{-3}	0.263×10^{-3}	0.000016
0.005	0.646	0.923×10^{-2}	0.657×10^{-2}	0.00036
0.01	1.292	0.150×10^{-1}	0.110×10^{-1}	0.0006
0.05	6.462	0.460×10^{-1}	0.310×10^{-1}	0.0022
0.2	25.850	0.810×10^{-1}	0.800×10^{-1}	0.01
0.9	116.323	0.311	0.307	0.05

Table 6.2.4:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.005	0.390×10^{-1}	0.918×10^{-3}	0.550×10^{-3}	0.000017
0.005	0.389	0.551×10^{-2}	0.522×10^{-2}	0.00013
0.05	3.893	0.400×10^{-1}	0.360×10^{-1}	0.0007
0.1	7.787	0.67×10^{-1}	0.590×10^{-1}	0.001
0.2	15.573	0.107	0.970×10^{-1}	0.0015
0.9	70.080	0.312	0.237	0.01

Table 6.2.5:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.000008	0.640×10^{-1}	0.612×10^{-4}	0.169×10^{-4}	0.00000022
0.00005	0.397	0.749×10^{-3}	0.840×10^{-3}	0.00000072
0.001	7.945	0.461×10^{-2}	0.163×10^{-2}	0.0000555
0.005	39.727	0.917×10^{-2}	0.763×10^{-2}	0.00024
0.05	397.266	0.690×10^{-1}	0.660×10^{-1}	0.002
0.1	794.532	0.119	0.105	0.004
0.9	7.151×10^{3}	0.322	0.242	0.03

Table 6.2.6:

Noise	QR-ERB	DSM-ERB	VRM-ERB	α
0.00004	0.489	0.187×10^{-3}	0.403×10^{-3}	0.000001
0.0001	1.221	0.951×10^{-3}	0.932×10^{-3}	0.000002
0.005	61.067	0.816×10^{-2}	0.459×10^{-2}	0.0002
0.01	122.130	0.110×10^{-2}	0.743×10^{-2}	0.00034
0.05	610.97	0.460×10^{-1}	0.360×10^{-1}	0.001
0.1	1.221×10^{3}	0.710×10^{-1}	0.640×10^{-1}	0.0018
0.9	1.099×10^{4}	0.324	0.289	0.014

The results in Table 6.2.5 are obtained when M=25 and N=11, where the condition number of A is 4.4×10^{8}, E=0.1.

The results in Table 6.2.6 are obtained when M=30 and N=12, where the condition number of A is 1.7×10^{9}, E=0.1.

DISCUSSION

We have investigated two systems of linear algebraic equations arising from the use of a variant of Trefftz's method to obtain approximate solutions for two planes, singular boundary-value problems for Laplace's equation in rectangular domains. Three methods of solution were used: QR-factorization contaminated with noise, DSM and VRM. Different sizes of the coefficient matrices were considered. Our results show that DSM and VRM yield relatively similar results all the time for both examples, irrespective of the type of boundary singularity of the solution function. Both methods clearly perform better than the QR-factorization method in what concerns the efficiency of the approximation, especially for relatively higher values of the noise. It is also noticed that the efficiency of DSM and VRM are diminished when the rectangular domain tends to a narrow strip, in which case the performance of the QR-factorization method deteriorates rapidly . It is thus recommended to use any one of DSM or VRM when dealing with thin rectangular regions. Our investigations are planned to extend to deal with some other more complicated boundary conditions, for example the radiation-type boundary condition on one side of the domain.

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