Digital Communications (ELC 623)

© Samy S. Soliman

Electronics and Electrical Communications Engineering Department
Cairo University, Egypt

Email: *samy.soliman@cu.edu.eg*
Website: http://scholar.cu.edu.eg/samysoliman

Postgraduate Program

Communication over AWGN Channel

Equivalent Complex Baseband Representation: Signals

Consider a real-valued signal $s(t)$ with spectrum $S(f) = \mathcal{F}\{s(t)\}$,

Analytic Signal

$$S_+(f) = 2u(f)S(f) \Leftrightarrow s_+(t)$$

Show that:

$$\begin{aligned} s_+(t) &= s(t) + j\frac{1}{\pi t} * s(t) \\ &= s(t) + j\hat{s}(t) \\ &= s(t) + j\mathcal{H}\{s(t)\}, \quad \mathcal{H}\{.\} \text{ is the Hilbert Transform} \end{aligned}$$

Hilbert Filter

$$h(t) = \frac{1}{\pi t} \qquad H(f) = \begin{cases} j, & f < 0 \\ 0, & f = 0 \\ -j, & f > 0 \end{cases} \qquad \hat{S}(f) = H(f)S(f)$$

Equivalent Complex Baseband Representation: Signals

Baseband Signal = Complex Envelope

$$
\begin{aligned}
S_b(f) &= \frac{1}{\sqrt{2}} S_+(f + f_c) \\
s_b(t) &= \frac{1}{\sqrt{2}} s_+(t) e^{-j2\pi f_c t}
\end{aligned}
$$

With mathematical manipulation,

$$
\begin{aligned}
s(t) + j\hat{s}(t) &= \sqrt{2} s_b(t) e^{j2\pi f_c t} \\
s(t) &= \sqrt{2} \Re\{s_b(t) e^{j2\pi f_c t}\} \\
&= \sqrt{2} \Re\{s_b(t)\} \cos(2\pi f_c t) - \sqrt{2} \Im\{s_b(t)\} \sin(2\pi f_c t)
\end{aligned}
$$

Equivalent Complex Baseband Representation: Signals

Spectrum Analysis

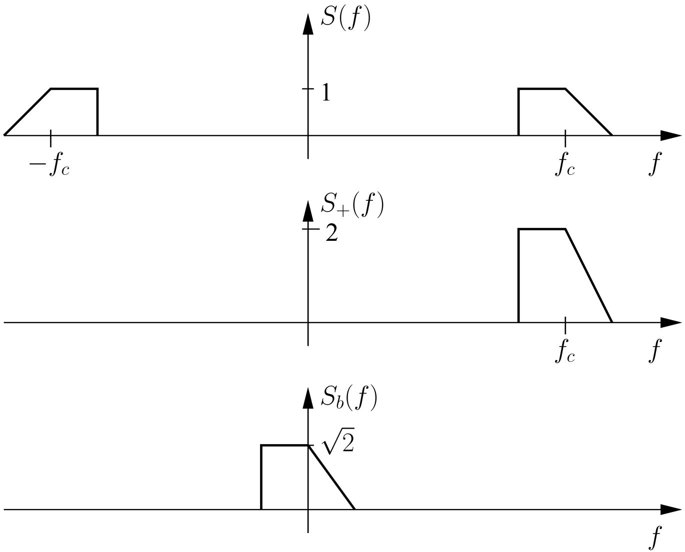
$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} \left[\sqrt{2}\Re\{s_b(t)e^{j2\pi f_c t}\} \right] e^{-j2\pi f t} dt \\ &= \frac{1}{\sqrt{2}} \left[S_b(f - f_c) + S^*(-f - f_c) \right] \end{aligned}$$

Energy

It can be shown that:

$$E = \int_{-\infty}^{\infty} s^2(t) dt = \int_{-\infty}^{\infty} |s_b(t)|^2 dt$$

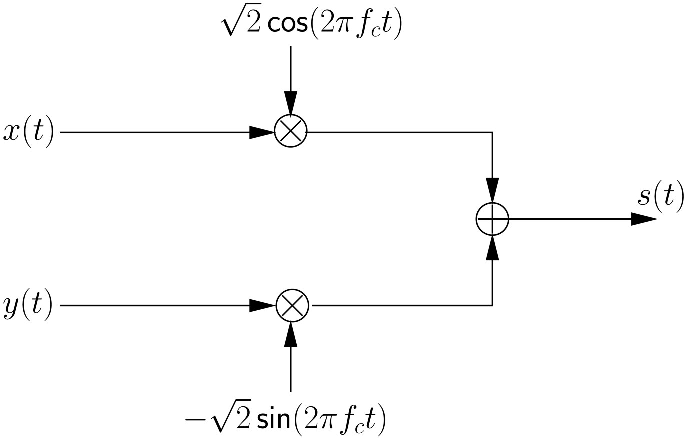
Equivalent Complex Baseband Representation: Signals



Equivalent Complex Baseband Representation: Signals

Quadrature Modulation

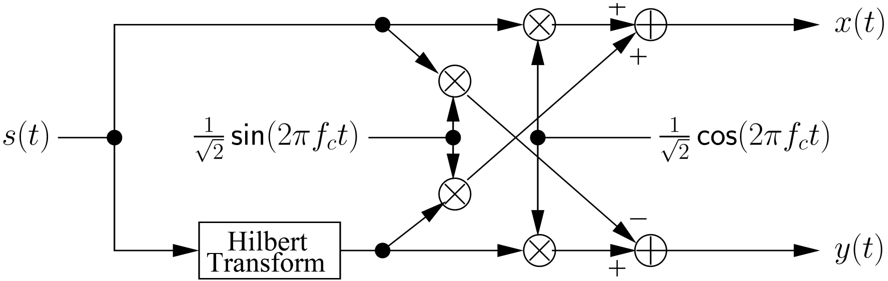
$$s(t) = \sqrt{2}x(t)\cos(2\pi f_c t) - \sqrt{2}y(t)\sin(2\pi f_c t)$$



Demodulation: Using Hilbert Filter

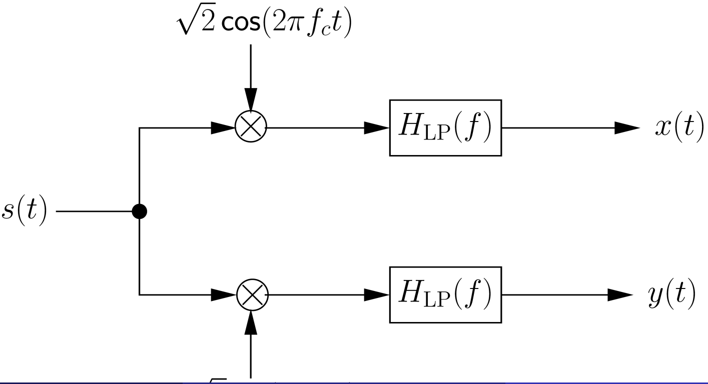
$$x(t) = \frac{1}{\sqrt{2}}\left[s(t)\cos(2\pi f_c t) + \hat{s}(t)\sin(2\pi f_c t)\right]$$

$$y(t) = \frac{1}{\sqrt{2}}\left[\hat{s}(t)\cos(2\pi f_c t) - s(t)\sin(2\pi f_c t)\right]$$



Demodulation: Using Low Pass Filter (Band-limited Signals)

$$x(t) = LPF\{\sqrt{2}s(t)\cos(2\pi f_c t)\}$$
$$y(t) = LPF\{-\sqrt{2}s(t)\sin(2\pi f_c t)\}$$



Equivalent Complex Baseband Representation: Systems

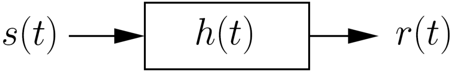
Analytic System

$$H_+(f) = 2u(f)H(f) \Leftrightarrow h_+(t)$$

Baseband System

$$\begin{aligned}
H_b(f) &= \frac{1}{2}H_+(f+f_c) \\
H_b(f-f_c) &= \begin{cases} H(f), & f \geq 0 \\ 0, & f < 0 \end{cases} \\
\mathbf{H(f)} &= \mathbf{H_b(f-f_c) + H_b^*(-f-f_c)} \\
\mathbf{h(t)} &= \mathbf{2\Re\{h_b(t)e^{j2\pi f_c t}\}}
\end{aligned}$$

Response of a Bandpass Systems to a Bandpass Signal



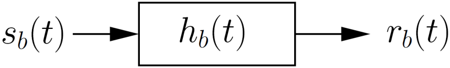
Show that:

$$R(f) = H(f)S(f) = \frac{1}{\sqrt{2}}\left[R_b(f - f_c) + R_b^*(-f - f_c)\right]$$

Note: Use the narrow-band assumptions, $S_b(f - f_c)H_b^*(-f - f_c) = 0$

Conclusion

We can perform linear filtering operations always in the equivalent baseband domain



Bandpass Noise Process

Assume a narrow-band bandpass WSS noise process, $n(t)$, with bandwidth B and center frequency f_c, i.e.,

$$\Phi_{NN}(f) \begin{cases} \neq 0, & f_c - B/2 \leq |f| \leq f_c + B/2 \\ = 0, & \text{otherwise} \end{cases}$$

Defining the equivalent complex baseband noise process as $z(t) = x(t) + jy(t)$, then

$$n(t) = \sqrt{2}\Re\{z(t)e^{j2\pi f_c t}\}$$

Show that: For $n(t)$ to be WSS,

$$\phi_{XX}(\tau) = \phi_{YY}(\tau) \quad (1)$$

$$\phi_{XY}(\tau) = -\phi_{YX}(\tau) \quad (2)$$

$$\phi_{NN}(\tau) = 2\left[\phi_{XX}(\tau)\cos(2\pi f_c\tau) + \phi_{XY}(\tau)\sin(2\pi f_c\tau)\right] \quad (3)$$

ACF and PSD

Show that: For the baseband equivalent, $z(t)$,

$$\phi_{zz}(\tau) = 2\left[\phi_{XX}(\tau) + j\phi_{XY}(\tau)\right] \quad (4)$$

$$\phi_{NN}(\tau) = \Re\{\phi_{zz}(\tau)e^{j2\pi f_c \tau}\} \quad (5)$$

$$\Phi_{NN}(f) = \frac{1}{2}\left[\Phi_{ZZ}(f - f_c) + \Phi_{ZZ}(-f - f_c)\right] \quad (6)$$

Show that:

1. $\phi_{XY}(\tau)$ is an odd function
2. If $x(t)$ and $y(t)$ are uncorrelated, then

$$\phi_{zz}(\tau) = 2\phi_{XX}(\tau)$$

$$\Phi_{ZZ}(f) = \Phi_{ZZ}(-f)$$

White Noise

If $\Phi_{NN}(f)$ can be approximated by a constant (flat) in the region of interest, i.e.

$$\Phi_{NN}(f) = \begin{cases} \frac{N_0}{2}, & f_c - B/2 \leq |f| \leq f_c + B/2 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\Phi_{ZZ}(f) = \begin{cases} N_0, & |f| \leq B/2 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\phi_{ZZ}(\tau) = N_0 \frac{\sin(\pi B \tau)}{\pi \tau}$$

White Noise

Definition

A noise process with flat spectrum for all frequencies is also called a white noise process. When $B \to \infty$,

$$\begin{aligned} \Phi_{ZZ}(f) &= N_0 \\ \phi_{zz}(\tau) &= N_0 \delta(\tau) \end{aligned}$$

Since $\Phi_{ZZ}(f)$ is an even function, the quadrature components of z(t) are uncorrelated, i.e., $\phi_{XY}(\tau) = 0$. Moreover,

$$\phi_{XX}(\tau) = \phi_{YY}(\tau) = \frac{1}{2}\phi_{ZZ}(\tau) = \frac{N_0}{2}\delta(\tau)$$

This means that $x(t)$ and $y(t)$ are mutually uncorrelated, white processes with equal infinite variances.

White Gaussian Noise

Definition

The quadrature components x(t) and y(t) of z(t) are mutually uncorrelated white Gaussian processes.

$$\sigma^2 = \phi_{XX}(0) = \phi_{YY}(0) = \frac{N_0}{2} B$$

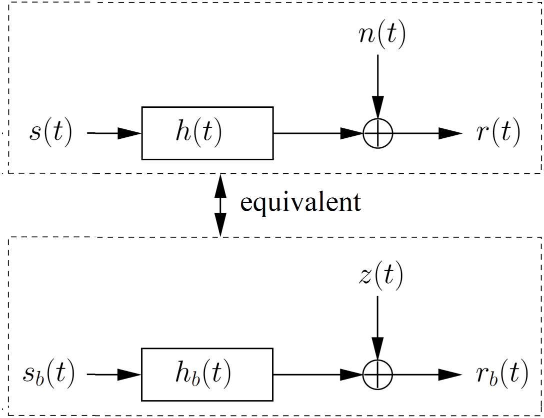
White Gaussian Noise through a LPF of bandwidth B, results in a filtered noise $\tilde{z} = \tilde{x} + j\tilde{y}$, such that

$$p_Z(\tilde{z}) = p_{XY}(\tilde{x}, \tilde{y}) = \frac{1}{\pi \sigma_Z^2} \exp\left[-\frac{|\tilde{z}|^2}{\sigma_Z^2} \right]$$

Since this PDF is rotationally symmetric, the corresponding equivalent baseband noise process is also referred to as **circularly symmetric complex Gaussian noise**.

System Equivalence

A passband communication system that is impaired by stationary white Gaussian noise is equivalent to a baseband system that is impaired by circularly symmetric white Gaussian noise.



Signal Space Representation of Signals

Signal Space

- CauchySchwarz inequality
- GramSchmidt procedure
- Orthogonal expansion of signals
- Representation of signals by Basis Functions
- Euclidean distance between signals

Refer to [Proakis, Section 2.2] and [?, Section 2.6]

M-ary Pulse Amplitude Modulation

Definitions

- **Memoryless:** The transmitted waveform depends only on the current k bits but not on previous bits
- M-ary Pulse Amplitude Modulation (MPAM) = M-ary Amplitude Shift Keying (MASK).

M-PAM

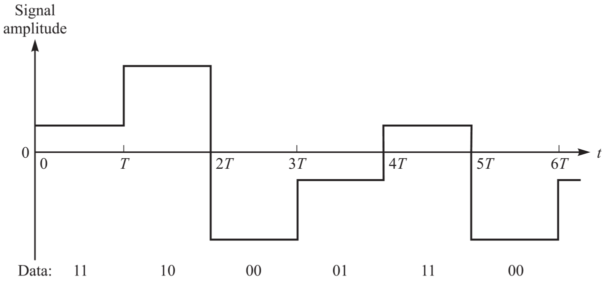
The M-PAM waveform in passband representation is given by

$$\begin{aligned} s_m(t) &= \sqrt{2}\Re\{A_m g(t) e^{j2\pi f_c t}\} \\ &= \sqrt{2} A_m g(t)\cos(2\pi f_c t), \qquad m = 1, 2, \cdots M \end{aligned}$$

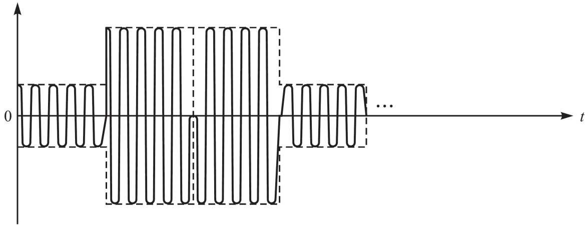
Baseband Representation

$$s_{bm}(t) = A_m g(t)$$

M-ary Pulse Amplitude Modulation



(a) Baseband PAM signal



(b) Bandpass PAM signal

M-PAM

- $A_m = (2m - 1 - M)d, m = 1, 2, \cdots, M$, are the M possible amplitudes or symbols, where $2d$ is the distance between two adjacent amplitudes.
- $g(t)$: real-valued signal pulse of duration T.
- Bit interval (duration) $T_b = 1/R$, where R is the bit rate to the modulator. The symbol duration is related to the bit duration by $T = kT_b$.
- PAM symbol rate $R_S = R/k$ symbols/s.

Transmitted Waveform

$$
\begin{aligned}
s(t) &= \sum_{k=-\infty}^{\infty} s_m(t - kT) \\
s_b(t) &= \sum_{k=-\infty}^{\infty} A_m[k]\, g(t - kT)
\end{aligned}
$$

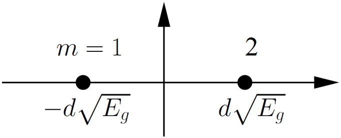
M-PAM

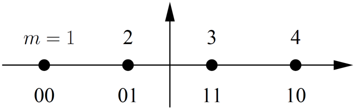
Energy

$$E_m = \int_0^T |s_{bm}(t)|^2 dt = A_m^2 E_g$$

Signal Space Representation

$$s_{bm}(t) = s_m f_b(t), \text{ where } f_b(t) = \frac{g(t)}{\sqrt{E_g}} \text{ , } s_m = \sqrt{E_g} A_m$$





M-PAM

Euclidean Distance Between Signal Points

$$\begin{aligned} d_{mn} &= \sqrt{(s_m - s_n)^2} \\ &= 2\sqrt{E_g}d|m-n| \\ d_{min} &= 2\sqrt{E_g}d \end{aligned}$$

Correlation

In general, the (cross)correlation between two signals allows to quantify the similarity of the signals. The correlation of two signals is defined as

$$\rho_{mn} = \frac{1}{\sqrt{E_m E_n}} \int_{-\infty}^{\infty} s_m(t) s_n(t) dt$$

Special Case: M=2

M-ary Phase Shift Keying

M-PSK

The M-PSK waveform in passband representation is given by

$$\begin{aligned} s_m(t) &= \sqrt{2}\Re\{e^{j2\pi\frac{m-1}{M}}g(t)e^{j2\pi f_c t}\} \\ &= \sqrt{2}g(t)\cos(2\pi f_c t + \Theta_m) \\ &= \sqrt{2}g(t)\cos(\Theta_m)\cos(2\pi f_c t) - \sqrt{2}g(t)\sin(\Theta_m)\sin(2\pi f_c t) \end{aligned}$$

Baseband Representation

$$s_{bm}(t) = e^{j2\pi\frac{m-1}{M}}g(t) = e^{j\Theta_m}g(t)$$

Signal Energy

$$E_m = \int_0^T |s_{bm}(t)|^2 dt = E_g$$

M-PSK

Signal Space Representation

$$s_{bm}(t) = s_{bm} f_b(t), \text{ where } f_b(t) = \frac{g(t)}{\sqrt{E_g}}, \; s_{bm} = \sqrt{E_g}\, e^{j\Theta_m}$$

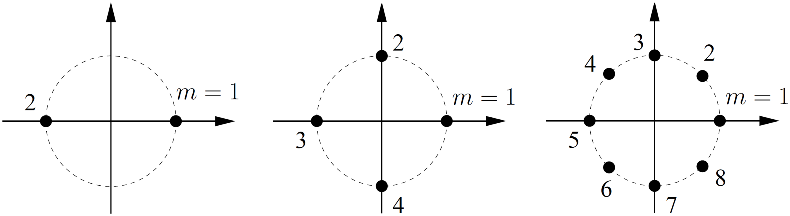
$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

where,

$$\begin{aligned} f_1(t) &= \sqrt{\frac{2}{E_g}}\, g(t)\cos(2\pi f_c t), \qquad s_{m1} = \sqrt{E_g}\cos(\Theta_m) \\ f_2(t) &= -\sqrt{\frac{2}{E_g}}\, g(t)\sin(2\pi f_c t), \qquad s_{m2} = \sqrt{E_g}\sin(\Theta_m) \end{aligned}$$

In the complex baseband, we get one basis function but the coefficients s_{bm} are complex valued. In the passband, we have two basis functions and the elements s_{m1}, s_{m2} are real valued.

M-PSK



Euclidean Distance Between Signal Points

$$
\begin{aligned}
d_{mn} &= \sqrt{(s_m - s_n)^2} \\
&= \sqrt{2E_g}\sqrt{1 - \cos\left(2\pi\frac{m-n}{M}\right)} \\
d_{min} &= \sqrt{2E_g}\sqrt{1 - \cos\left(\frac{2\pi}{M}\right)}
\end{aligned}
$$

M-QAM

The M-QAM waveform in passband representation is given by

$$\begin{aligned} s_m(t) &= \sqrt{2}\Re\{(A_{cm} + jA_{sm})g(t)e^{j2\pi f_c t}\} \\ &= \sqrt{2}\Re\{A_m g(t)e^{j2\pi f_c t}\} \\ &= \sqrt{2}A_{cm}g(t)\cos(2\pi f_c t) - \sqrt{2}A_{sm}g(t)\sin(2\pi f_c t) \end{aligned}$$

Baseband Representation

$$s_{bm}(t) = A_m g(t) = (A_{cm} + jA_{sm})g(t)$$

Signal Energy

$$E_m = \int_0^T |s_{bm}(t)|^2 dt = |A_m|^2 E_g$$

M-QAM

Signal Space Representation

$$s_{bm}(t) = s_{bm} f_b(t), \text{ where } f_b(t) = \frac{g(t)}{\sqrt{E_g}} \;, s_{bm} = \sqrt{E_g} A_m$$

$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

where,

$$f_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t), \qquad s_{m1} = \sqrt{E_g} A_{cm}$$

$$f_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t), \qquad s_{m2} = \sqrt{E_g} A_{sm}$$

In the complex baseband, we have onedimensional complex signal space.
In the passband, we have twodimensional real signal space.

M-QAM

Euclidean Distance Between Signal Points

$$
\begin{aligned}
d_{mn} &= \sqrt{(s_m - s_n)^2} \\
&= \sqrt{E_g}|A_m - A_n| \\
&= \sqrt{E_g}\sqrt{(A_{cm} - A_{cn})^2 + (A_{sm} - A_{sn})^2}
\end{aligned}
$$

For $A_{cm}, A_{sm} \in \{\pm d, \pm 3d, \cdots, \pm(\sqrt{M} - 1)d\}$,

$$d_{min} = 2\sqrt{E_g}d$$

M-ary Frequency Shift Keying

M-FSK

The M-FSK waveform in passband representation is given by

$$
\begin{aligned}
s_m(t) &= \sqrt{\frac{2E}{T}} \Re\{e^{j2\pi m\Delta f t} e^{j2\pi f_c t}\} \\
&= \sqrt{\frac{2E}{T}} \cos(2\pi(m\Delta f + f_c)t)
\end{aligned}
$$

Baseband Representation

$$
s_{bm}(t) = \sqrt{\frac{E}{T}} e^{j2\pi m\Delta f t}
$$

Signal Energy

$$
E_m = \int_0^T |s_{bm}(t)|^2 dt = E
$$

M-FSK

Correlation and Orthogonality

$$\begin{aligned}
\rho^b_{mn} &= \frac{1}{\sqrt{E_m E_n}} \int_{-\infty}^{\infty} s_{bm}(t) s^*_{bn}(t) dt \\
&= sinc\left((m-n)\Delta f T\right) e^{j\pi(m-n)\Delta f T} \\
\rho_{mn} &= \Re\{\rho^b_{mn}\} \\
&= sinc\left((m-n)\Delta f T\right) \cos(\pi(m-n)\Delta f T) \\
&= sinc\left(2(m-n)\Delta f T\right) \\
\rho_{mn} &= 0 \;\text{ for } \Delta f T = \frac{k}{2},\; k \in \{\pm 1, \pm 2, \cdots\}
\end{aligned}$$

Smallest frequency separation $\Delta f = 1/2T$, at which

$$\rho^b_{mn} = \begin{cases} 0, & (m-n) \text{ even} \\ \frac{2j}{\pi(m-n)}, & (m-n) \text{ odd} \end{cases}$$

Signal Space Representation $\Delta f T = 1/2$

$$f_m(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t + \pi m t/T)$$

$$\begin{aligned} \mathbf{s_1} &= \left[\sqrt{E}\, 0 \cdots 0\right]^T \\ \mathbf{s_2} &= \left[0\, \sqrt{E}\, 0 \cdots 0\right]^T \\ \cdots & \\ \mathbf{s_M} &= \left[0\, 0 \cdots \sqrt{E}\right]^T \end{aligned}$$

M-FSK

Euclidean Distance Between Signal Points

$$
\begin{aligned}
d_{mn} &= \sqrt{(s_m - s_n)^2} \\
&= \sqrt{2E} \\
d_{min} &= \sqrt{2E}
\end{aligned}
$$

Biorthogonal Signals

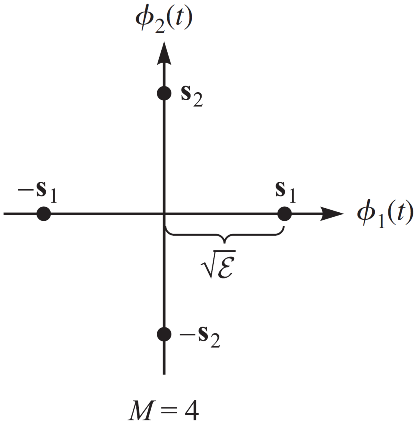
A set of $2M$ biorthogonal signals is derived from a set of M orthogonal signals $\{s_m(t)\}$ by including the negative signals $\{-s_m(t)\}$.
For biorthogonal signals, the Euclidean distance between pairs of signals is

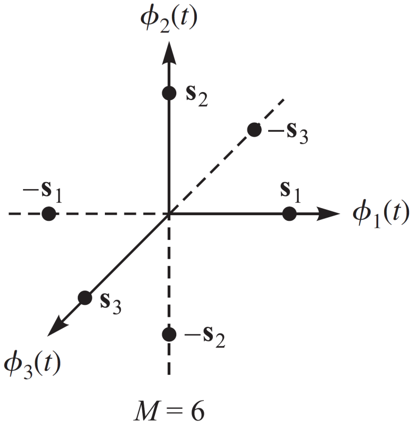
$$\begin{aligned} d_{mn} &= \sqrt{2E} \\ \text{or } d_{mn} &= 2\sqrt{E} \end{aligned}$$

The correlation is

$$\begin{aligned} \rho_{mn} &= 0 \\ \text{or } \rho_{mn} &= -1 \end{aligned}$$

Biorthogonal Signals





Simplex Signals

Usually, zero-mean waveforms are preferred. Therefore, it is desirable to modify the orthogonal signal set to a signal set with zero mean. This can be done by modifying the signal set to become

$$\acute{\mathbf{s}}_{\mathbf{m}} = \mathbf{s}_{\mathbf{m}} - \bar{\mathbf{s}},$$

where,

$$\bar{\mathbf{s}} = \frac{1}{M}\sum_{1}^{M}\mathbf{s}_{\mathbf{m}} = \frac{\sqrt{E}}{M}\mathbf{1}_{\mathbf{M}}$$

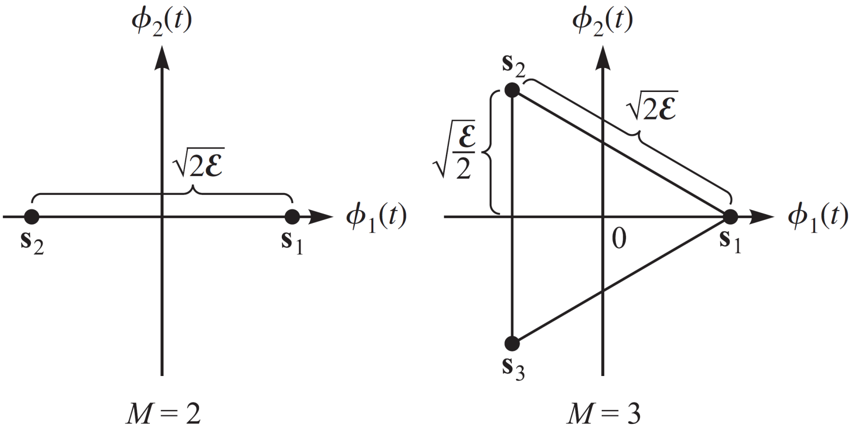
Energy

$$E_m = E\left(1 - \frac{1}{M}\right)$$

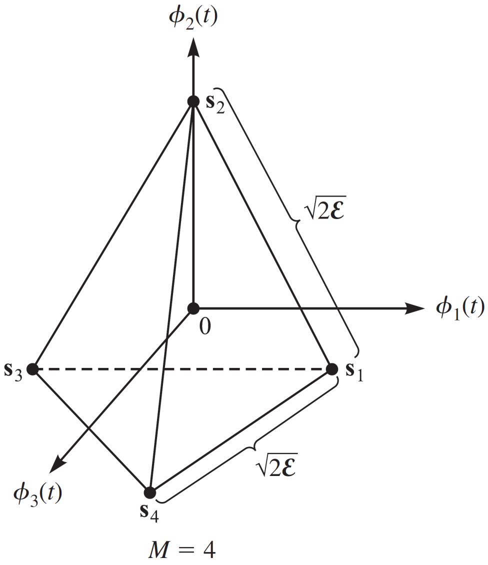
Correlation

$$\rho_{mn} = -\frac{1}{M-1}$$

Simplex Signals



Simplex Signals



Modulation With Memory

Refer to [Proakis, Section 3.3]

Optimum Receivers for AWGN Channel

Problem Statement

In symbol interval $0 \leq t \leq T$, information is transmitted using one of M possible waveforms $s_m(t), 1 \leq m \leq M$.

The received passband signal $r(t)$ is corrupted by real-valued AWGN, $n(t)$, such that

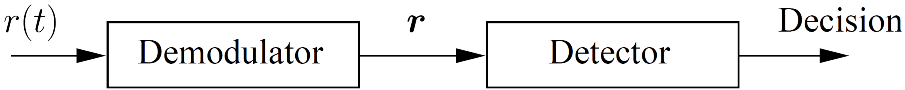
$$\begin{aligned} r(t) &= s_m(t) + n(t), \quad & 0 \leq t \leq T \\ r_b(t) &= s_{bm}(t) + z(t), \quad & 0 \leq t \leq T \end{aligned}$$

The receiver's task is to make the best decision to determine $s_m(t)$ from observing $r(t)$.

Note:

$$\begin{aligned} \Phi_{NN}(f) &= \frac{N_0}{2} \\ \Phi_{ZZ}(f) &= N_0 \end{aligned}$$

Optimum Receiver for AWGN Channel



1. **Demodulation**
   Transform the received signal into N-dimensional vector, forming a *sufficient statistic*

$$\mathbf{r} = [r_1 \ r_2 \ \cdots \ r_N]^T$$

   Note:
   - The transmit waveforms can be represented by a set of N orthogonal basis function
   - The noise require infinite number of basis functions, however, only those in the signal space of the N basis functions are relevant

2. **Detection**
   Determine an estimate $s_m(t)$ based on the vector $\mathbf{r}$

Correlation Demodulation

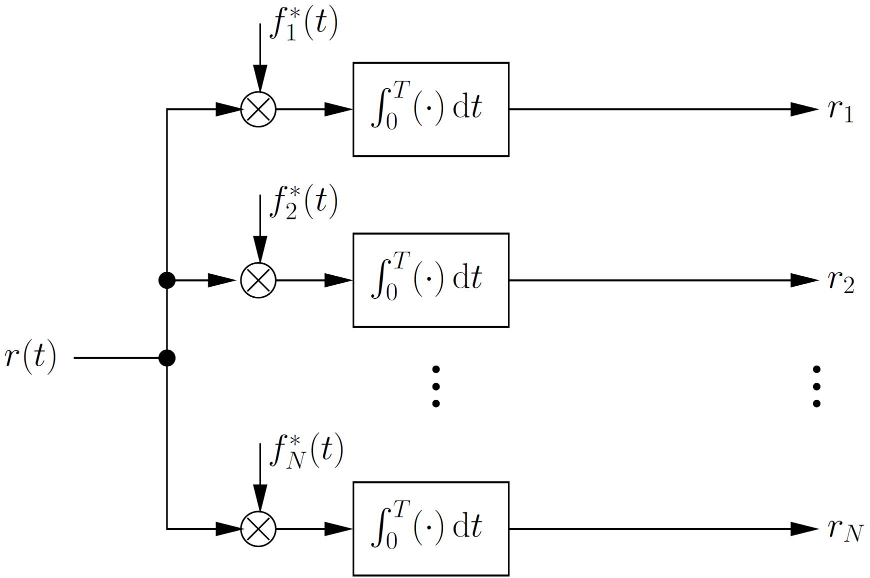
The elements of the sufficient statistic **r** are obtained by correlating the received signal $r(t)$ with the basis functions, $f_k(t)$

$$\begin{aligned} r_k &= \int_0^T r(t) f_k^*(t) dt \\ &= \int_0^T s_m(t) f_k^*(t) dt + \int_0^T n(t) f_k^*(t) dt \\ &= s_{mk} + n_k \end{aligned}$$

The received signal can be represented as

$$\begin{aligned} r(t) &= \sum_{k=1}^{N} r_k f_k(t) + \acute{n}(t) \\ \text{where } \acute{n}(t) &= n(t) - \sum_{k=1}^{N} n_k f_k(t) \end{aligned}$$

Correlation Demodulation



Correlation Demodulation

Properties of n_k

1. $n(t)$ is a Gaussian process, then n_k is a Gaussian RV
2. Mean
$$\mathcal{E}\{n_k\} = 0$$
3. Covariance
$$\mathcal{E}\{n_k n_m^*\} = \frac{N_0}{2}\delta[k-m]$$
4. Noise components are zero-mean, mutually uncorrelated Gaussian RVs

Effect of $\acute{n}(t)$

$$\mathcal{E}\{\acute{n}(t) r_k^*\} = 0$$

Then, $\mathbf{r}$ and $\acute{n}(t)$ are uncorrelated $\Rightarrow$ statistically independent (why?)
Then $\mathbf{r}$ is a sufficient statistic for the detection of $s_m(t)$.

Matched Filter

Instead of the correlators, we can use linear filters with impulse responses

$$
\begin{aligned}
h_k(t) &= f_k^*(T-t), \qquad 0 \le t \le T \\
H_k(f) &= e^{-j2\pi fT} F_k^*(f)
\end{aligned}
$$

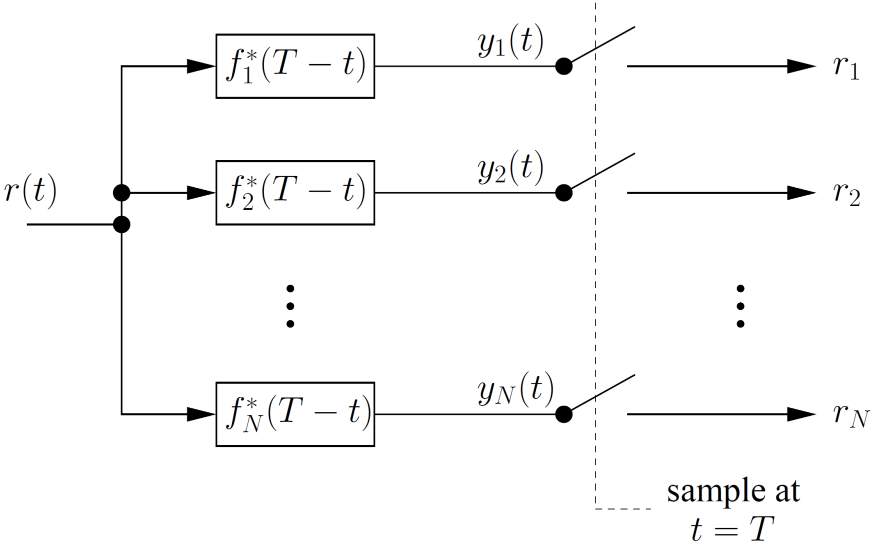
The filter output, sampled at $t = T$ is given by

$$
y_k(T) = \int_0^T r(\tau) f_k^*(\tau) d\tau = r_k
$$

Use the Cauchy-Schwartz inequality to show that the Matched Filter maximizes the SNR, such that

$$
\text{SNR} = \frac{E}{N_0/2}
$$

Matched Filter



Optimal Detection

Problem Statement

Using the sufficient statistic obtained from the demodulation step, we need to find the optimum detector, where the optimality criterion is taken as the probability for correct detection shall be maximized, i.e., probability of error shall be minimized.

Posteriori Probability

$$P\left(\mathbf{s}_{\tilde{m}} | \mathbf{r}\right), \qquad \tilde{m} = 1, 2, \cdots, M$$

The probability of error is minimized if the chosen symbol $\mathbf{s}_{\tilde{m}}$ maximizes the posteriori probability.

Maximum a Posteriori (MAP) Decision Rule

MAP

$$\hat{m} = \text{argmax}_{\tilde{m}}\{P(\mathbf{s}_{\tilde{m}}|\mathbf{r})\}$$

Using Bayes rule, the MAP rule can be simplified to

$$\hat{m} = \text{argmax}_{\tilde{m}}\{p(\mathbf{r}|\mathbf{s}_{\tilde{m}})\, P(\mathbf{s}_{\tilde{m}})\}$$

Note:

$$p(\mathbf{r}) = \sum_{m=1}^{M} p(\mathbf{r}|\mathbf{s}_{\tilde{m}})\, P(\mathbf{s}_{\tilde{m}})$$

Maximum Likelihood (ML) Decision Rule

In some applications, $P(\mathbf{s}_{\tilde{m}})$ is unknown at the receiver. Neglecting the influence of this *priori probability*, the ML rule is obtained

ML

$$\hat{m} = \text{argmax}_{\tilde{m}}\{p(\mathbf{r}|\mathbf{s}_{\tilde{m}})\}$$

Note: For equiprobable *priori probabilities*, the MAP and ML rules are identical.

ML Rule for AWGN Channel

For AWGN channel,

$$p\left(\mathbf{r}|\mathbf{s}_{\tilde{m}}\right) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0}\sum_{k=1}^{N}|r_k - s_{\tilde{m}k}|^2\right]$$

After simplification,

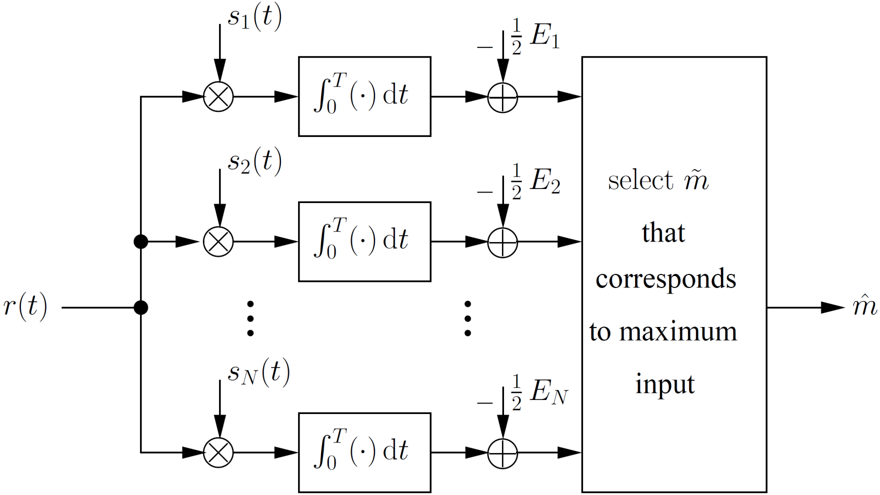
ML for AWGN

$$\hat{m} = \text{argmin}_{\tilde{m}}\{||\mathbf{r} - \mathbf{s}_{\hat{m}}||^2\}$$

This is equivalent to choosing the vector $\mathbf{s}_{\hat{m}}$ with the minimum Euclidean distance to the received vector $\Rightarrow$ Decision Regions in signal space.
It can be alternatively represented as

$$\hat{m} = \text{argmax}_{\tilde{m}}\{\int_0^T r(t)s_{\tilde{m}}(t)dt - \frac{1}{2}E_{\tilde{m}}\}$$

ML Rule for AWGN Channel



Optimum Receiver Example - PAM

Assume a rectangular pulse shape $g(t) = a, \quad 0 \leq t \leq T$

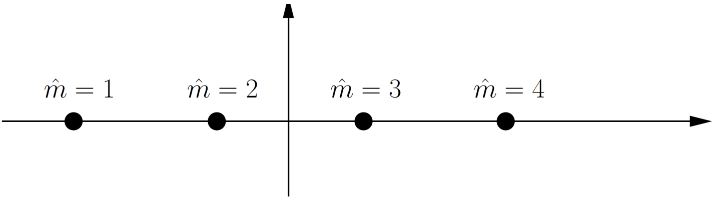
1. Demodulator

$E_g = a^2 T$ – $f(t) = \frac{1}{\sqrt{T}}, \quad 0 \leq t \leq T$ – $r_b = \sqrt{E_g} A_m + z$ – $\sigma_z^2 = N_0$

$$p\left(r_b / A_m\right) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0} |r_b - \sqrt{E_g} A_m|^2\right]$$

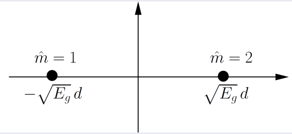
2. Detector

$$\hat{m} = \text{argmin}_{\tilde{m}} \{|r_b - \sqrt{E_g} A_{\tilde{m}}|^2\}$$



Binary PAM

$$r_b = \sqrt{E_g} A_m + z$$



Decision is based on the real part of r_b

$$r_R = \Re\{r_b\} = \sqrt{E_g} A_m + \Re\{z\}$$

Conditional error probability

$$P(e|s_-) = \int_0^\infty p_{r_R}(r_R|s_-) dr_R = Q\left(\sqrt{\frac{2E_g}{N_0}} d\right)$$

$$P(e|s_+) = \int_{-\infty}^0 p_{r_R}(r_R|s_+) dr_R = Q\left(\sqrt{\frac{2E_g}{N_0}} d\right)$$

Performance of Optimum Receivers: Binary Modulation

Binary PAM

Bit Error Probability (BEP) = Symbol Error Probability (SER)

$$
\begin{aligned}
P_b &= P(s_-)P(e|s_-) + P(s_+)P(e|s_+) \\
&= P(e|s_-) \\
P_b &= Q\left(\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}
$$

Note: $E_b = \mathcal{E}\{|\sqrt{E_g}A_m|^2\} = E_g d^2$

Binary PSK

BPSK is similar to Binary PAM

Binary Orthogonal Modulation (BFSK)

- Transmitted signals

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} \sqrt{E_g} & 0 \end{bmatrix}^T \\ \mathbf{s}_2 &= \begin{bmatrix} 0 & \sqrt{E_g} \end{bmatrix}^T \end{aligned}$$

- Demodulated received signals

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} \sqrt{E_g} + n_1 & n_2 \end{bmatrix}^T \\ \text{or } \mathbf{r} &= \begin{bmatrix} n_1 & \sqrt{E_g} + n_2 \end{bmatrix}^T \end{aligned}$$

where, $\mathcal{E}\{n_1^2\} = \mathcal{E}\{n_2^2\} = N_0/2$

Binary Orthogonal Modulation (BFSK)

- Decision Rule

$$\begin{aligned} \hat{m} &= \text{argmin}_{\tilde{m}}\{||\mathbf{r} - \mathbf{s}_{\hat{m}}||^2\} \\ &= \text{argmax}_{\tilde{m}}\{\mathbf{r} \bullet \mathbf{s}_{\tilde{m}}\} \end{aligned}$$

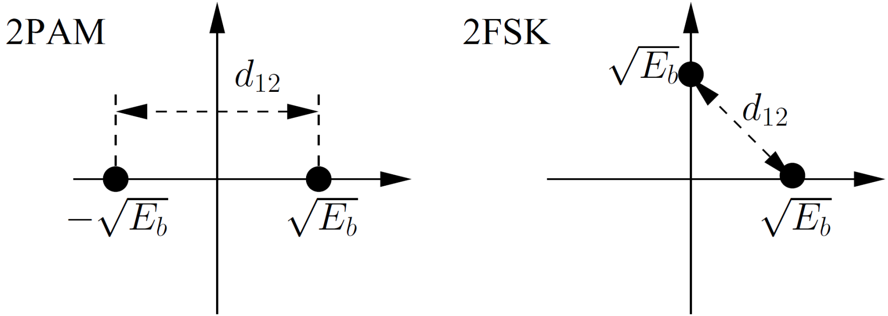
- Error Probabilities

$$\begin{aligned} P(e|s_1) &= P\left(\mathbf{r} \bullet \mathbf{s}_2 > \mathbf{r} \bullet \mathbf{s}_1\right) \\ &= P\left(n_2 - n_1 > \sqrt{E_b}\right) \end{aligned}$$

where, $n_2 - n_1 = X$ is a Gaussian RV with variance $\mathcal{E}\{|n_2 - n_1|^2\} = N_0$

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Performance of Optimum Receivers: BPSK vs BFSK



Performance of Optimum Receivers: M-PAM

1. The Transmitted Signal

$$s_{bm} = \sqrt{E_g} A_m, \qquad A_m = (2m - 1 - M)d$$

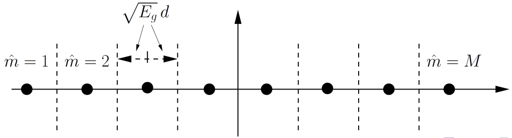
2. Average Signal Energy

$$E_s = \frac{1}{M} \sum_{m=1}^{M} E_m = \frac{M^2 - 1}{3} d^2 E_g$$

3. Baseband Received Signal

$$r_b = s_{bm} + z, \qquad \sigma_z^2 = N_0, \sigma_{\Re\{z\}}^2 = N_0/2$$

4. Decision Regions (Outer and Inner Signal Points)



Symbol Error Probability

$$\begin{aligned}
P_M &= \frac{1}{M}\left[(M-2)+2.\frac{1}{2}\right] P\left(\left|\Re\{r_b\}-s_{bm}\right| > d\sqrt{E_g}\right) \\
&= 2\frac{M-1}{M} Q\left(\sqrt{2d^2\frac{E_g}{N_0}}\right) \\
&= 2\frac{M-1}{M} Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right) \\
&= 2\frac{M-1}{M} Q\left(\sqrt{\frac{6\log_2(M)E_b}{(M^2-1)N_0}}\right) \\
P_b &\approx \frac{1}{\log_2(M)} P_M
\end{aligned}$$

Performance of Optimum Receivers: M-PSK

2-PSK

4-PSK

$$P_4 = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

M-PSK

$$P_M \approx 2Q\left(\sqrt{\frac{2\log_2(M)E_b}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

Performance of Optimum Receivers: M-QAM

4-QAM

M-QAM

$$P_M \leq 4Q\left(\sqrt{\frac{3\log_2(M)E_b}{(M-1)N_0}}\right)$$

Upper Bound for Arbitrary Signal Constellations

1. Pairwise Error Probabilities

$$PEP(s_\mu \to s_\nu) = P\left(s_\mu \text{ transmitted}, s_\nu \text{ detected}\right)$$

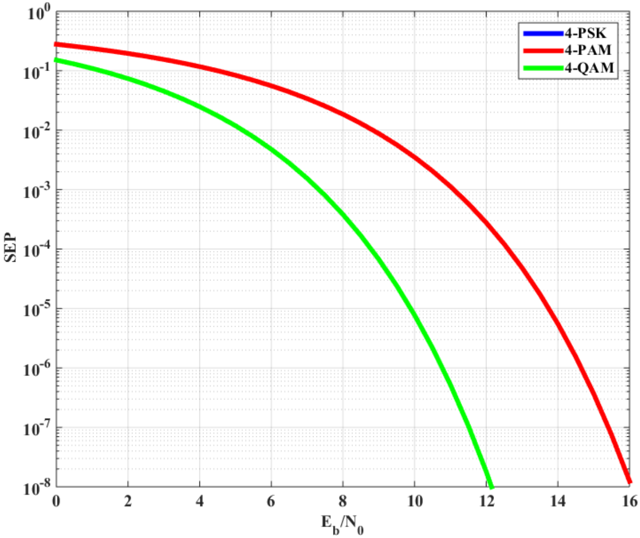
2. Union Bound

$$P_M \leq \frac{1}{M}\sum_{\mu=1}^{M}\sum_{\nu=1,\nu\neq\mu}^{M} PEP(s_\mu \to s_\nu)$$

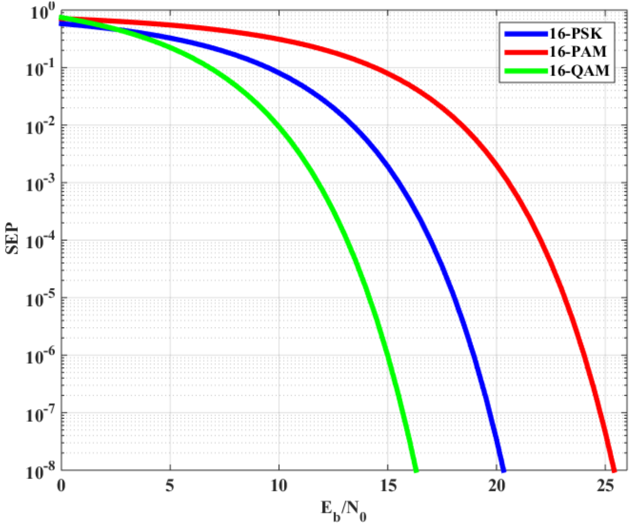
3. For Gaussian noise

$$\begin{aligned} P_M &\leq \frac{1}{M}\sum_{\mu=1}^{M}\sum_{\nu=1,\nu\neq\mu}^{M} Q\left(\sqrt{\frac{d_{\mu\nu}^2}{2N_0}}\right) \\ &\approx C_M Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \end{aligned}$$

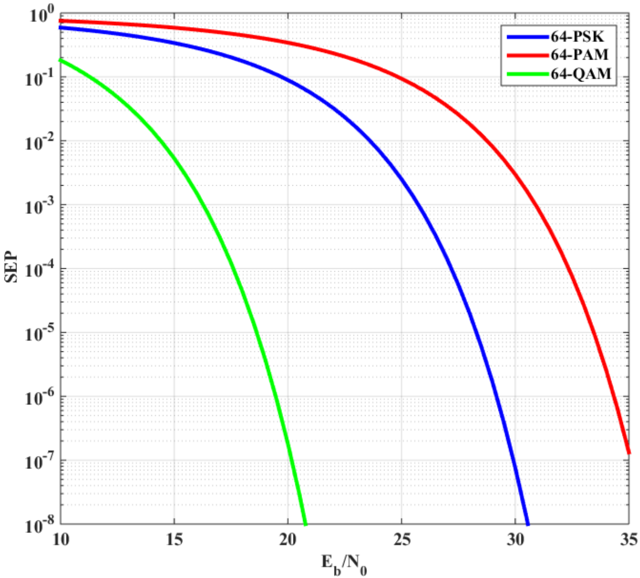
Comparison of Linear Modulation Techniques



Comparison of Linear Modulation Techniques



Comparison of Linear Modulation Techniques



Comparison of Linear Modulation Techniques

- As M increases, PAM and PSK become less favorable and the gap to QAM increases.
  The reason for this behavior is the smaller minimum Euclidean distance $d_{\min}$ of PAM and PSK
- For a given transmit energy, $d_{\min}$ of PAM and PSK is smaller since the signal points are confined to a line and a circle, respectively
- For QAM on the other hand, the signal points are on a rectangular grid, which guarantees a comparatively large $d_{\min}$

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Thank You

Questions?

samy.soliman@cu.edu.eg

http://scholar.cu.edu.eg/samysoliman