# Digital Communications (ELC 623)

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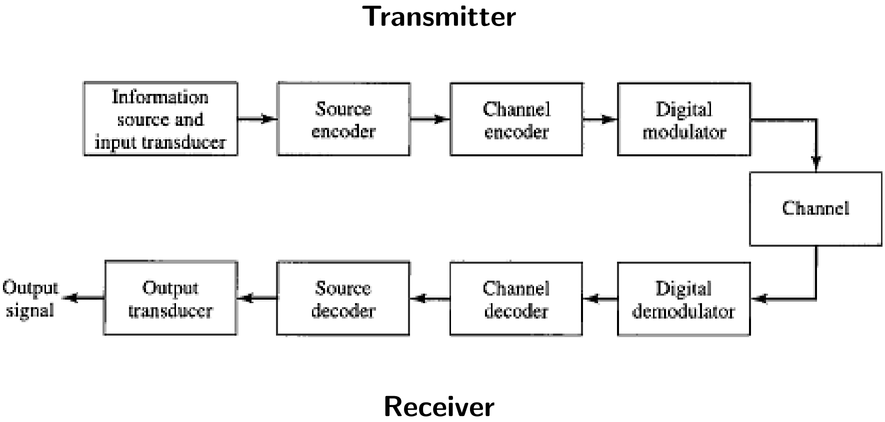
Postgraduate Program

# Introduction
# Statistical Decision Theory

# Elements of Communication Systems



# Elements of Communication Systems: Transmitter

## Transmitter

1. **Information Source**
   Analog - Digital
2. **Source Encoder**
   Represent the source signal as efficiently as possible (minimize the redundancy)
3. **Channel Encoder**
   Increase reliability of received data (add redundancy in a controlled manner to information bits)
4. **Digital Modulator**
   Transmit most efficiently over the (physical) transmission channel (map the input bit sequence to a signal waveform which is suitable for the transmission channel)

## Receiver

1. **Digital Demodulator**
   Reconstruct transmitted data symbols (binary or M-ary from channelcorrupted received signal
2. **Channel Decoder**
   Exploit redundancy introduced by channel encoder to increase reliability of information bits
3. **Source Decoder**
   Reconstruct original information signal from output of channel decoder

**Note:** In advanced receivers, demodulation and decoding are sometimes performed iteratively to enhance the receiver's performance.

# Elements of Communication Systems: Transmitter

## Channel

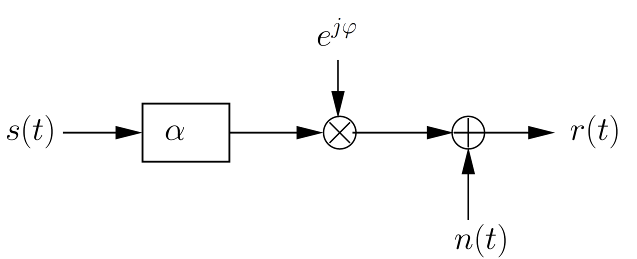
1. **Examples of Physical Channel**
   - Wireline - Optical Fiber
   - Wireless radio frequency (RF) channel - Optical Wireless Channel
   - Underwater Acoustic Channel - Storage Channel (CD, disc, etc.)
2. **Channel Impairments**
   - Noise (electronic, thermal, ...)
   - Nonlinearities, distortions, time-variance, ...
   - Interference

For the design of the transmitter and the receiver we need a simple mathematical model of the physical communication channel that captures its most important properties. This model will vary from one application to another.

# Channel Types: AWGN

## Additive White Gaussian Noise Channel - With Unknown Phase

$$r(t) = \alpha e^{j\phi} s(t) + n(t)$$

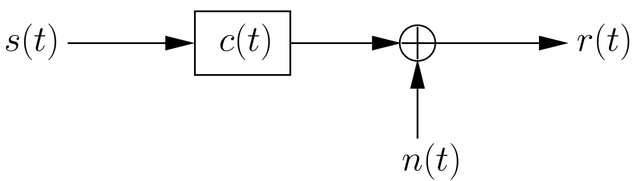



The transmitted signal experiences an unknown phase shift $\phi$, which is often modeled as a random variable, uniformly distributed in the interval $[-\pi, \pi)$. The transmitted signal is also attenuated by a factor of $\alpha$, and impaired by AWGN.

# Channel Types: Linear

## Linearly Dispersive Channel (Linear Filter Channel)

$$r(t) = c(t) * s(t) + n(t)$$




The transmitted signal is linearly distorted by $c(t)$ and impaired by AWGN.

# Channel Types

## Types of Channels

- Multiuser channels
- MIMO channels
- Relaying channels
- Fading channels
- $\cdots$

**Discussion:**

What tools do we need for the analysis/design of communication systems?

# The Theory of Probability: Random Experiments

An essential tool in the design of digital communication systems.

## Probability

- Experiment
- Outcome
- Event
- Probability - Axioms of Probability
- Union - Intersection
- Joint events and joint probabilities
- Conditional probability
- Statistical Independence

## Axioms of Probability

Assume events A and B are subsets of the sample space S, i.e. $A \subset S$ and $B \subset S$

1. $P(S) = 1$
2. $0 < P(A) < 1$
3. If $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$

**Note:** If $A \cap B \neq \phi$, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

# Joint Events and Joint Probability

Consider two experiments with outcomes $A_i, i = 1, 2, \cdots, n$ and $B_j, j = 1, 2, \cdots, m$
If both experiments are carried out, then the outcome $(A_i, B_j)$ is assigned the probability $P(A_i, B_j)$ with $0 \leq P(A_i, B_j) \leq 1$

1. If the outcome of $B_j, j = 1, 2, \cdots, m$ are mutually exclusive, then

$$\sum_{j=1}^{m} P(A_i, B_j) = P(A_i)$$

2. If all the outcome of both $A_i, i = 1, 2, \cdots, n$ and $B_j, j = 1, 2, \cdots, m$ are mutually exclusive, then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} P(A_i, B_j) = 1$$

# Conditional Probability

The conditional probability $P(A|B)$ is the probability of event A given that event B has already been observed.

## Conditional Probability

$$
\begin{aligned}
P(A|B) &= \frac{P(A,B)}{P(B)} \\
P(B|A) &= \frac{P(A,B)}{P(A)}
\end{aligned}
$$

## Bayes' Theorem

$$
\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\
P(B|A) &= \frac{P(A|B)P(B)}{P(A)}
\end{aligned}
$$

# Statistical Independence

If observing B does not change the probability of observing A, i.e., $P(A|B) = P(A)$, then A and B are statistically independent. In this case:

$$P(A, B) = P(A|B)P(B) = P(A)P(B)$$

## Statistically Independent Events

Two events A and B are statistically independent if and only if

$$P(A, B) = P(A)P(B)$$

# The Theory of Probability: Random Experiments

## Probability

- Experiment
- Outcome
- Event
- Probability - Axioms of Probability
- Union - Intersection
- Joint events and joint probabilities
- Conditional probability
- Statistical Independence

## Let's play

Where is the Prize?

## Random Variables

- Probability distributions
- Probability densities
- Joint probability distributions
- Conditional probability distributions
- Statistically independent random variables
- Statistical averages

## Transformation of Random Variables

$$Y = g(X)$$

# Random Variables: Cumulative Distribution Function

The CDF $F(x)$ denotes the probability that a random variable (RV) $X$ is smaller than or equal to a specific value $x$, i.e.

$$F(x) = P(X \leq x)$$

## Properties of CDF

$$\begin{aligned} 0 \leq \quad & F(x) \quad \leq 1 \\ \lim_{x \to -\infty} \quad & F(x) \quad = 0 \\ \lim_{x \to \infty} \quad & F(x) \quad = 1 \\ \frac{d}{dx} \quad & F(x) \quad \geq 0 \end{aligned}$$

# Random Variables: Probability Density Function

The PDF of a RV $X$ is defined as:

$$p(x) = \frac{dF(x)}{dx}, \qquad -\infty \leq x \leq \infty$$

## Properties of PDF

$$\begin{aligned} p(x) &\geq 0 \\ F(x) &= \int_{-\infty}^{x} p(u)du \\ \int_{-\infty}^{\infty} p(u)du &= 1 \end{aligned}$$

## Discrete Random Variables

For discrete random variables, where $X \in \{x_1, x_2, \cdots, x_n\}$,

$$p(x) = \sum_{i=1}^{n} P(X = x_i)\delta(x - x_i)$$

**Note:** The probability that $x_1 \leq X \leq x_2$ is given as

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} p(u)du = F(x_2) - F(x_1)$$

## Joint CDF and PDF

Given two RVs, $X$ and $Y$,

$$\begin{aligned} F_{XY}(x,y) &= P(X \leq x, Y \leq y) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y} p_{XY}(u,v)dudv \\ p_{XY}(x,y) &= \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_{XY}(x,y) \end{aligned}$$

## Marginal Densities

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} p_{XY}(x,y)dy \\ p_Y(y) &= \int_{-\infty}^{\infty} p_{XY}(x,y)dx \end{aligned}$$

# Random Variables: Joint PDF and Joint CDF

## Properties of Joint CDF and PDF

$$
\begin{aligned}
F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = F_{XY}(-\infty, -\infty) &= 0 \\
F_{XY}(\infty, \infty) &= 1
\end{aligned}
$$

## Conditional PDF and CDF

$$
\begin{aligned}
p_{X|Y}(x|y) &= \frac{p_{XY}(x, y)}{p_Y(y)} \\
F_{X|Y}(x|y) &= \int_{-\infty}^{x} p_{X|Y}(u|y) du
\end{aligned}
$$

# Random Variables: Joint PDF and Joint CDF

## Statistical Independence

X and Y are statistically independent iff

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

## Complex Random Variables

For a complex RV $Z = X + jY$,
CDF:

$$F_Z(z) = P(X \leq x, Y \leq y) = F_{XY}(x, y)$$

PDF

$$p_Z(z) = p_{XY}(x, y)$$

# Functions of Random Variables

## Problem Statement

Given a RV $X$ with known PDF, $p_X(x)$, and CDF, $F_X(x)$, what are the PDF and the CDF of another RV $Y$, where $Y = g(X)$

## Transformation of RVs

$$p_Y(y) = \frac{p_X(f(y))}{J(x)}, \qquad \text{where} \qquad J(x) = \frac{dy}{dx}\Big|_{x=f(y)}$$

## Examples

1. $Y = aX + b$
2. $Y = aX^2 + b$
3. $Y = X_1 + X_2$

# Sum of Two Random Variables

## Problem

Given two random variables $X_1$ and $X_2$, with joint probability $p_{X_1,X_2}(x_1,x_2)$, what the PDF of $Y = X_1 + X_2$?

## Solution

Since $x_1 = y - x_2$, then the PDF of Y and $X_2$ is obtained as

$$p_{Y,X-2}(y,x_2) = p_{X_1,X_2}(x_1,x_2)|_{x_1=y-x_2} = p_{X_1,X_2}(y-x_2,x_2)$$

Then, using marginal densities, the PDF of $Y$ can be obtained as

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} p_{X_1,X_2}(y-x_2,x_2)dx_2 \\ &= \int_{-\infty}^{\infty} p_{X_1,X_2}(x_1,y-x_1)dx_1 \end{aligned}$$

## Special Case: Sum of Two SI RVs

If $X_1$ and $X_2$ are statistically independent, then

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

Then,

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} p_{X_1}(y-x_2)p_{X_2}(x_2)dx_2 \\ &= p_{X_1}(x_1) * p_{X_2}(x_2) \end{aligned}$$

The PDF of Y is the **convolution** of the PDFs of $X_1$ and $X_2$.

# Statistical Averages of RVs

## General Case: Statistical Averaging

$$\mu = \mathcal{E}\{g(X)\} = \int_{-\infty}^{\infty} g(X)p_X(x)dx$$

## Mean

$$\mathcal{E}\{X\} = \int_{-\infty}^{\infty} x \, p_X(x)dx$$

## nth Moment

$$\mathcal{E}\{X^n\} = \int_{-\infty}^{\infty} x^n \, p_X(x)dx$$

# Statistical Averages of RVs

## nth Central Moment

$$\mathcal{E}\{(X-\mu)^n\} = \int_{-\infty}^{\infty} (x-\mu)^n \, p_X(x) dx$$

## Variance = 2th Central Moment

$$\begin{aligned} \sigma^2 &= \mathcal{E}\{(X-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 \, p_X(x) dx \\ &= \mathcal{E}\{X^2\} - (\mathcal{E}\{X\})^2 \end{aligned}$$

## Characteristic Function

$$\psi(jt) = \mathcal{E}\{e^{jtx}\} = \int_{-\infty}^{\infty} e^{jtx} \, p_X(x) dx$$

# Charachteristic Function

## Characteristic Function: Properties

$$\psi(jt) = \mathcal{F}_{p_X(x)}(-jt)$$

$$p_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(jt)e^{-jtx}dt$$

$$\mathcal{E}\{X^n\} = (-j)^n \frac{d^n}{dt^n}\psi(jt)|_{t=0}$$

## Sum of Two SI RVs: $Y = X_1 + X_2$

$$
\begin{aligned}
\psi_Y(jt) &= \mathcal{E}\{e^{jtY}\} \\
&= \mathcal{E}\{e^{jt(X_1+X_2)}\} \\
&= \mathcal{E}\{e^{jt(X_1)}e^{jt(X_2)}\} \\
&= \mathcal{E}\{e^{jt(X_1)}\}\mathcal{E}\{e^{jt(X_2)}\} \\
&= \psi_{X_1}(jt)\psi_{X_2}(jt)
\end{aligned}
$$

## Random Variables

- Probability distributions
- Probability densities
- Joint probability distributions
- Conditional probability distributions
- Statistically independent random variables
- Statistical averages

## Transformation of Random Variables

$$Y = g(X)$$

# The Theory of Probability: Distributions

## Useful Probability Distributions

- Uniform distribution
- **Gaussian (normal) distribution**
- Rayleigh distribution
- Nakagami-$m$ distribution
- Rician distribution
- Chi-square distribution

## Bounding

- Chernoff bound
- Central Limit Theorem

# Gaussian Distribution

The Gaussian distribution is an important probability distribution in practice because many physical phenomena can be described by a Gaussian distribution, e.g. AWGN

## One-Dimentional Gaussian RV

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F(x) = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

**Note:** Useful definitions

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$$

## One-Dimentional Gaussian RV

$$\psi(jt) = e^{jt\mu - t^2\sigma^2/2}$$

$$\mu_k = \mathcal{E}\{(x-\mu)^k\} = \begin{cases} 1.3.5.\cdots.(k-1)\sigma^k, & \text{even k} \\ 0, & \text{odd k} \end{cases}$$

$$\mathcal{E}\{x^k\} = \sum_{i=0}^{k} \binom{k}{i} \mu^i \mu_{k-i}$$

## Sum of n SI Gaussian RVs

$$Y = \sum_{i=1}^{n} X_i$$

$$\psi_Y(jt) = \cdots$$

# The Theory of Probability: Distributions

## Useful Probability Distributions

- Uniform distribution
- Rayleigh distribution
- Nakagami-$m$ distribution
- Rician distribution
- Chi-square distribution

**Refer to [Proakis, Section 2.3]**

# Tail Probability and Chernoff Bound

## Tail Probability

The tail probability (area under the tail of PDF) arises often to determine the error probability of digital communication systems, and is given by

$$\begin{aligned} P(x \geq \delta) &= \int_{\delta}^{\infty} p(x)dx \\ &= \int_{-\infty}^{\infty} U(x-\delta)p(x)dx = \mathcal{E}\{U(x-\delta)\} \end{aligned}$$

Note that, for any $\alpha \geq 0$,

$$U(x-\delta) \leq e^{\alpha(x-\delta)}$$

## Chernoff Bound

The tail probability (area under the tail of PDF) arises often to determine the error probability of digital communication systems, and is given by

$$\begin{aligned} P\left(x \geq \delta\right) &= \mathcal{E}\{U(x-\delta)\} \\ &\leq \mathcal{E}\{e^{\alpha(x-\delta)}\} \\ &= e^{-\alpha\delta}\mathcal{E}\{e^{\alpha x}\} \end{aligned}$$

In order to obtain the tightest Chernoff bound, $\alpha$ should be optimized such that

$$\frac{d}{d\alpha} e^{-\alpha\delta}\mathcal{E}\{e^{\alpha x}\} = 0$$

# Central Limit Theorem

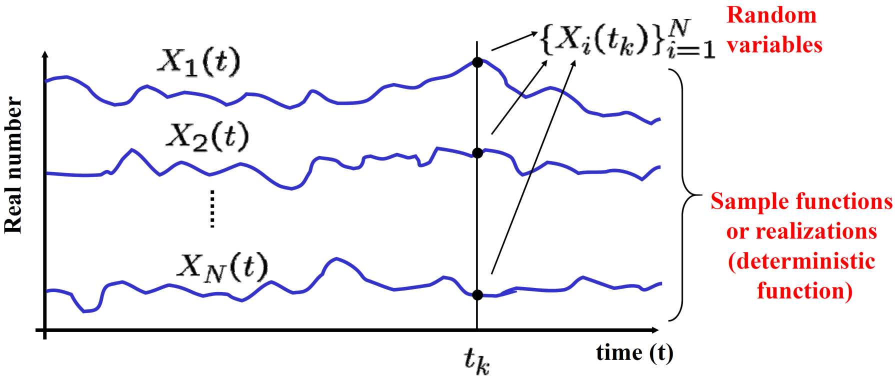
**Refer to [Proakis, Section 2.5]**

Stochastic processes arise whenever a random phenomenon is a function of time

## Stochastic Processes

- Stationary processes - Wide sense stationary processes
- Statistical averages
- Power Spectral Density
- Ergodic processes
- Cyclo-stationary processes

**Refer to [Proakis, Section 2.7]**




**Random Process - Sample Function - Ensemble**

# Strict Sense Stationary Stochastic Process

## RP $\rightarrow$ RV

Considering specific time instants $t_1 > t_2 > ... > t_n$ with the arbitrary positive integer index n, the random variables $X_{t_i} = X(t_i), i = 1, 2, ..., n$, are fully characterized by their joint PDF $p(x_{t_1}, x_{t_2}, ..., x_{t_n})$

## Definition

If $X_{t_i}$ and $X_{t_i+\tau}$ have the same statistical properties, $X(t)$ is stationary in the strict sense.

$$p(x_{t_1}, x_{t_2}, ..., x_{t_n}) = p(x_{t_1+\tau}, x_{t_2+\tau}, ..., x_{t_n+\tau})$$

# Statistical Averages - Ensemble Averages

## First-Order Moment = Mean

$$m(t_i) = \mathcal{E}\{X_{t_i}\} = \int_{-\infty}^{\infty} x_{t_i} p(x_{t_i}) dx_{t_i}$$

For SSS RP, $m(t_i) = m$

## Second-Order Moment = Autocorrelation Function

$$\phi(t_1, t_2) = \mathcal{E}\{X_{t_1} X_{t_2}\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_{t_1} x_{t_2} p(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2}$$

For SSS RP, $\phi(t_1, t_2) = \phi(t_1 - t_2) = \phi(\tau) \Rightarrow$ **Average Power =** $\phi(0)$

## Central Second-Order Moment = Covariance Function

$$\mu(t_1, t_2) = \mathcal{E}\{(X_{t_1} - m(t_1))(X_{t_2} - m(t_2))\} = \phi(t_1, t_2) - m(t_1)m(t_2)$$

For SSS RP, $\mu(t_1, t_2) = \mu(\tau) = \phi(\tau) - m^2 \Rightarrow$ **Variance =** $\mu(0)$

# Wide Sense Stationary Random Process

**Definition**

If the first and second order moments of a stochastic process are invariant to any time shift $\tau$ , the process is referred to as wide sense stationary process.
Wide sense stationary processes are not necessarily strict sense stationary.

# Ergodic Random Process

## Definition

A process $X(t)$ is ergodic if its statistical averages can be calculated as timeaverages of sample functions.
Only wide sense stationary processes can be ergodic.

$$
\begin{aligned}
m &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \\
\phi(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt
\end{aligned}
$$

where $x(t)$ is any of the sample functions.

# Jointly Stochastic Processes

## Joint Stationarity

$X(t)$ and $Y(t)$ are jointly stationary if their joint PDF is invariant to time shifts, $\tau$.

## Cross Correlation Function

$$\phi_{XY}(t_1, t_2) = \mathcal{E}\{X_{t_1} Y_{t_2}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_{t_1} y_{t_2} p(x_{t_1}, y_{t_2}) dx_{t_1} dy_{t_2}$$

If $X(t)$ and $Y(t)$ are jointly and individually SSS, $\phi_{XY}(t_1, t_2) = \phi_{XY}(\tau)$

## Cross Covariance Function

$$\begin{aligned} \mu_{XY}(t_1, t_2) &= \mathcal{E}\{(X_{t_1} - m_X(t_1))(Y_{t_2} - m_Y(t_2))\} \\ &= \phi_{XY}(t_1, t_2) - m_X(t_1) m_Y(t_2) \end{aligned}$$

If $X(t)$ and $Y(t)$ are jointly and individually SSS, $\mu_{XY}(t_1, t_2) = \mu_{XY}(\tau)$

## Statistical Independence

$X(t)$ and $Y(t)$ are SI iff for every $n$ and $m$

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t_1}, y_{t_2}, \dots, y_{t_m}) = p(x_{t_1}, x_{t_2}, \dots, x_{t_n})p(y_{t_1}, y_{t_2}, \dots, y_{t_m})$$

## Uncorrelation

$X(t)$ and $Y(t)$ are uncorrelated iff

$$\mu_{XY}(t_1, t_2) = 0$$

# Power Spectral Density

## Definition

Power spectral density is defined for stationary RP as the Fourier Transform of the ACF

$$\begin{aligned}\Phi(f) &= \mathcal{F}\{\phi(\tau)\} = \int_{-\infty}^{\infty} \phi(\tau) e^{-j2\pi f \tau} d\tau \\ \phi(\tau) &= \mathcal{F}^{-1}\{\Phi(f)\} = \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi f \tau} df\end{aligned}$$

## Uncorrelated Stationary Process

$$\begin{aligned}\phi(\tau) &= \delta(\tau) \\ \Phi(f) &= 1\end{aligned}$$

# Power Spectral Density

## Average Power of a Stationary RP

$$\phi(0) = \mathcal{E}\{|X_t|^2\} = \int_{-\infty}^{\infty} \Phi(f) df$$

## Symmetry of PSD

**Show that:** $\Phi^*(f) = \Phi(f)$

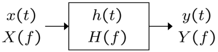
## Cross Correlation Spectrum

$$\Phi_{XY}(f) = \mathcal{F}\{\phi_{XY}(\tau)\} = \int_{-\infty}^{\infty} \phi_{XY}(\tau) e^{-j2\pi f \tau} d\tau$$

**Show that:**

$$\begin{aligned} \Phi^*_{XY}(f) &= \Phi_{YX}(f) \\ \Phi_{YX}(f) &= \Phi_{XY}(-f), \end{aligned}$$

For real X(t) and Y(t)




$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ \phi_{hh}(\tau) &= \int_{-\infty}^{\infty} h(\tau)h(t+\tau)d\tau \\ \Phi_{hh}(f) &= \end{aligned}$$

## Mean, ACF and PSD of $Y(t)$

$$\begin{aligned} m_Y &= m_X H(0) \\ \phi_{YY}(\tau) &= \mathcal{E}\{Y_{t_1} Y^*_{t_1-\tau}\} \\ &= \phi_{hh}(\tau) * \phi_{XX}(\tau) \\ \Phi_{YY}(f) &= |H(f)|^2 \Phi_{XX}(f) \\ \Phi_{YX}(f) &= H(f)\Phi_{XX}(f) \end{aligned}$$

**Show that: Power spectral densities are Non-negative**

# Cyclostationary Stochastic Processes

## Definition

Cyclo-stationary processes are non-stationary processes whose statistical averages are periodic.

Assume a digital communication signal expressed as

$$X(t) = \sum_{n=-\infty}^{\infty} a[n]g(t - nT)$$

## Mean of $X(t)$: **Show that the mean is periodic**

$$m_X(t) = \sum_n \mathcal{E}\{a[n]\}g(t - nT) = m_a \sum_n g(t - nT)$$

## ACF of $X(t)$: **Show that the ACF is periodic**

$$\phi_{XX}(t + \tau, t) = \mathcal{E}\{X(t + \tau)X^*(t)\} = \cdots$$

# Bandpass and Lowpass Random Processes

## Bandpass and Lowpass Random Processes

**Refer to [Proakis, Section 2.9]**

# References

J. Proakis
Digital Communications, 5th Edition.
*McGraw Hill.*

# Thank You

Questions?

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