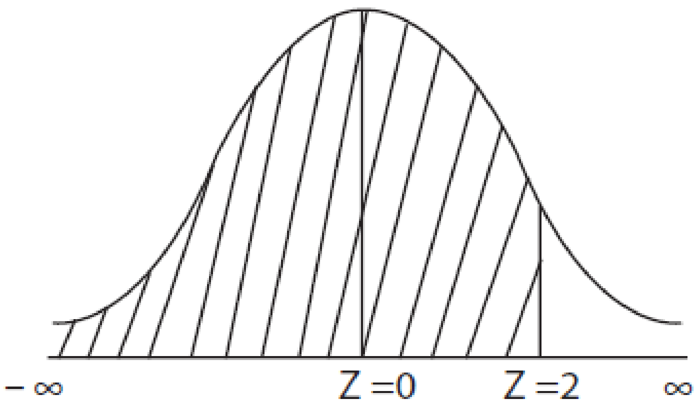
Probability Distribution

In the framework of the undergraduate course:
"Applied Mathematics"



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A. Probability Distribution

Trail and Event:

Consider an experiment of throwing a coin. When tossing a coin, we may get a head (H) or tail (T). Here tossing of a coin is a trail and getting a head or tail is an event.

From a pack of cards drawing any three cards is trail and getting a King or queen or a jack are events.

Throwing of a dice is a trail and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Sample Space:

The set of all possible cases of an experiment is called the sample space and is denoted by S.

Mathematical Definition of Probability:

$$\text{Probability of Event E} = \frac{\text{No. of favourable cases}}{\text{Total no. of outcomes}}$$

$$P(E) = \frac{m}{n}$$

Where 'm' is number of favourable cases = n (E) and 'n' is number of exhaustive cases = n(S).

A.1. Random Variable

Random Variable:

A function X which transforms events of a random experiment into real numbers is called random variable. It is denoted as $X : S \rightarrow R$, where S is sample space of random experiment and R is set of real numbers.

Example:

Two coins are tossed at a time.

Sample space is S = {HH, HT, TH, TT}. If we take X is the number of heads appearing then HH becomes 2. HT and TH becomes 1, and TT becomes 0.

$\therefore$ X (number of heads) is a random variable.

Types of Random Variables:

There are two types of random variables known as

(i) Discrete random variable

(ii) Continuous random variable

Discrete random variable:

If a random variable takes only a finite or a countable number of values, it is called a discrete random variable.

For example, when two coins are tossed the number of heads obtained is the random variable X. Where X assumes the values 0, 1, 2. Which is a countable set. Such a variable is called discrete random variable.

Definition:

Probability Mass Function:

Let X be a discrete random variable with values $x_1, x_2, x_3, \ldots\ldots\ldots x_n$. Let $p(x_i)$ be a number associated with each x_i.

Then the function p is called the probability mass function of X if it satisfies the conditions

(i) $p(x_i) \geq 0$ for $i = 1, 2, 3, \ldots\ldots\ldots n$

(ii) $\Sigma p(x_i) = 1$

The set of ordered pairs $[x_i, p(x_i)]$ is called the probability distribution of X.

Continuous Random Variable:

A random variable X is said to be continuous if it can take all possible values between certain limits.

Examples:

1. Life time of electric bulb in hours.
2. Height, weight, temperature, etc.

Definition:

Probability density function:

A function *f* is said to be probability density function (pdf) of the continuous random variable X if it satisfies the following condition:

1. $f(x) \geq 0$ for all $x \in R$
2. $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Definition:

Distribution function (Cumulative Distribution Function).

The function F(X) is said to be the distribution function of the random variable X, if

$F(X) = P(X \leq x)\ ;\ -\infty \leq x \leq \infty$

The distribution function F is also called cumulative distribution function.

Note:

1. If X is a discrete random variable then from the definition it follows that $F(X) = \Sigma p(x_i)$, where the summation is overall x_i, such that $x_i \leq x$.
2. If X is a continuous random variable, then from the definition it follows that

$$y = Ae^{x} + Be^{2x} + \frac{e^{-4x}}{10} + 4xe^{2x}\quad ,$$

where f(t) the value of the probability density function of X at t.

Examples

PART – A

1. Find the probability distribution of X when tossing a coin, when X is defined as getting a head.

Solution:

Let X denote getting a head.

Probability of getting a head $= \frac{1}{2}$

Probability of getting a tail $= \frac{1}{2}$

The probability distribution of X is given by

X :	0	1
P (X = x):	½	½

2. When throwing a die what is probability of getting a 4?

Solution:

Total number of cases n = 6 (1, 2, 3, 4, 5, 6)

Number of favourable cases = 1

$\therefore$ Probability of getting 4 $= \frac{m}{n} = \frac{1}{6}$

3. Verify that $f(x) = \begin{cases} \frac{2x}{9}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$ is a probability density function.

Solution:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^3 \frac{2x}{9}dx = \frac{2}{9}\left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{9}\left[\frac{9}{2}\right]$$

$$= 1$$

$\Rightarrow$ f(x) is a probability density function.

PART – B

1. A random variable X has the following probability distribution:

X :	0	1	2	3	4
P(X) :	3a	4a	6a	7a	8a

Find (i) Value of a.

(ii) $P(X \le 2)$

Solution:

(i) Since $\Sigma P(X) = 1$

$$\Rightarrow 3a + 4a + 6a + 7a + 8a = 1$$

$$28a = 1$$

$$a = \frac{1}{28}$$

(ii)

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= 3a + 4a + 6a$$

$$= 13a$$

$$= 13\left(\frac{1}{28}\right)$$

$$= \frac{13}{28}$$

PART – C

1. If $f(x) = \begin{cases} kx^2, & 0 \le x \le 3 \\ 0 & \text{else where} \end{cases}$, is a pdf, find the value of k.

Solution:

Since f(x) is a pdf.

We have

$$\int_0^3 f(x)\ dx = 1$$

$$\int_{-\infty}^{+\infty} kx^2\ dx = 1$$

$$K\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$K\left[\frac{3^3}{3} - \frac{0^3}{3}\right] = 1$$

$$K\left[\frac{27}{3}\right] = 1$$

$$9K = 1$$

$$K = \frac{1}{9}$$

A.2. Mathematical Expectation of Discrete Variable

Expectation of a discrete random variable:

If X denotes a discrete random variable which can assume the value $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ with respective probabilities $p_1, p_2 \ldots\ldots\ldots\ldots p_n$ then the mathematical expectation of X, denoted by E(X) is defined by

$$E(X) = p_1x_1 + p_2x_2 + \ldots\ldots\ldots + p_nx_n$$

$$= \sum_{i=1}^{n} p_i x_i \quad \text{where} \sum_{i=1}^{n} p_i = 1$$

Thus E(X) is the weighted arthmetic mean of the values x_i with the weight to $p(x_i)$.

$\therefore$ Mean $\overline{X} = E(X)$.

Hence the mathematical expectation E(X) of a random variable is simply the arithmetic mean.

Result: If $\phi(x)$ is a function of a random variable X, then

$E[\phi(x)] = \Sigma P(X = x)\, \phi(x)$.

$E(X^2) = p_1x_1^{\,2} + p_2x_2^{\,2} + p_3x_3^{\,3} + \ldots\ldots\ldots\ldots$

Properties of mathematical expectation:

1. $E(C) = C$, where C is a constant.
2. $E(CX) = CE(X)$.
3. $E(ax + b) = aE(X) + b$, where a, b are constants.
4. Variance of X = $Var(X) = E\{X - E(X)\}^2$.
5. $Var(X) = E(X^2) - [E(X)]^2$
6. $Var (X \pm C) = Var (X)$ where C is a constant.
7. $Var (aX) = a^2 Var(X)$.
8. $Var (aX + b) = a^2 Var(X)$.
9. $Var (C) = 0$, where C is a constant.

Examples

PART – A

1. Find the expected value of the random variable X has the following probability distribution.

X :	1	2	3
P(X) :	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$

Solution:

Exepctation of X.

$$E(X) = \sum x_i P(X_i)$$

$$= 1 \times \frac{1}{6} + 2 \times \frac{4}{6} + 3 \times \frac{1}{6}$$

$$= \frac{1}{6} + \frac{8}{6} + \frac{3}{6}$$

$$= \frac{1+8+3}{6}$$

$$E(X) = \frac{12}{6} = 2$$

2. Evaluate Var (2X ± 3).

Solution:

We have $\text{Var}(ax \pm b) = a^2 \text{ Var}(X)$

$$\text{Var}(2X \pm 3) = 2^2 \text{ Var}(X)$$

$$= 4 \text{ Var}(X)$$

3. A random variable X has E(X) = 2 and $E(X^2) = 8$. Find its variance.

Solution:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 8 - [2]^2$$

$$= 8 - 4$$

$$\text{Var}(X) = 4$$

PART – B

1. A random variable X has the following probability distribution:

X :	-3	6	9
P(X) :	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find E (2X + 1).

Solution:

$$E(2X+1) = E(2X) + E(1)$$

$$E(2x+1) = 2E(X) + 1$$

Where $E(X) = \Sigma x_i P(x_i)$

$$= (-3)\left(\frac{1}{6}\right) + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$= -\frac{3}{6} + \frac{6}{2} + \frac{9}{3}$$

$$= \frac{-3+18+18}{6}$$

$$= \frac{-3+36}{6} = \frac{33}{6} = \frac{11}{2}$$

$$E(2X+1) = 2\left(\frac{11}{2}\right) + 1$$

$$= 11 + 1$$

$$E(2X+1) = 12$$

PART – C

1. The monthly demand for ladies hand bags is known to have the following distribution.

Demand	1	2	3	4	5	6
Probability	0.1	0.15	0.20	0.25	0.18	0.12

Determine the expected demand for ladies hand bags. Also obtain variance.

Solution:

(x^2)	1	4	9	16	25	36
Demand (x)	1	2	3	4	5	6
Probability P(x)	0.1	0.15	0.20	0.25	0.18	0.12

Expected demand, $E(x) = \sum_{i=1}^{n} x_i p_i$

$$= 1\,(0.1) + 2\,(0.15) + 3\,(0.20) + 4\,(0.25) + 5\,(0.18) + 6\,(0.12)$$

$$= 0.1 + 0.3 + 0.6 + 1.0 + 0.9 + 0.72$$

$$= 3.62$$

$$E(x^2) = \sum_{i=1}^{n} x_i^2 p_i^2$$

$$= 1\,(0.1) + 4\,(0.15) + 9\,(0.2) + 16\,(0.25) + 25\,(0.18) + 35\,(0.12)$$

$$= 0.1 + 0.6 + 1.8 + 4.0 + 4.5 + 4.32$$

$$= 15.32$$

$$\therefore \text{Variance} = E(x)^2 - [E(x)]^2$$

$$= 15.32 - (3.62)^2$$

$$= 15.32 - 13.10$$

$$= 2.22$$

A.3. Binomial Distribution

An experiment which has two mutually disjoint outcomes is called a Bernoulli trail. The two outcomes are usually called "success" and "failure".

An experiment consisting of repeated number of Bernoulli trails is called Binomial experiment. A Binomial distribution can be used under the followiong conditions.

(i) The number of trials is finite.

(ii) The trials are independent of each other.

(iii) The probability of success is constant for each trial.

Probability funciton of Binomial Distribution:

Let X denotes the number of success in n trial satisfying binomial distribution conditions. X is a random variable which can take the values 0, 1, 2, ...n, since we get no success, one success or all n success.

The general expression for the probability of x success is given by

$$P(X = x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, 3, \ldots\ldots\ldots n.$$

Where P = probability of success. In each trial and $q = 1 - p$.

Definition:

A random variable X is sai dto follow binomial distribution, if the probability mass funciton is given by

$$P(X = x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 3, \ldots\ldots\ldots n.$$
$$= 0 \qquad \text{otherwise}$$

Where n, p are called parameter of the distribution.

Constants of the binomial distribution:

Mean = np

Variance = npq

Standard Deviation = $\sqrt{npq}$

Note:

(i) $0 \le p \le 1,\ 0 \le q \le 1$ and $p + q = 1$.

(ii) In binomial distribution mean is always greater than variance.

(iii) To denote the random variable X which follows binomial distribution with parameters n and p is $X \sim B(n, p)$.

Examples

PART – A

1. Find n and p in the binomial distribution whose mean is 3 and variance is 2.

Solution:

Given, mean = 3

i.e $np = 3$ and Variance $npq = 2$

$$\frac{npq}{np} = \frac{2}{3}$$

$$q = \frac{2}{3} \qquad \therefore p = 1 - q$$

$$p = 1 - \frac{2}{3}$$

$$p = \frac{1}{3}$$

2. In a binomial distribution if $n = 9$ and $p = \frac{1}{3}$, what is the value of variance?

Solution:

Given: $n = 9, \quad p = \frac{1}{3}$

$$q = 1 - p$$

$$q = 1 - \frac{1}{3}$$

$$q = \frac{2}{3}$$

$$\therefore \text{Variance} = npq$$

$$= 9 \times \frac{1}{3} \times \frac{2}{3}$$

$$\text{Variance} = 2$$

PART – B

1. If the sum of mean and variance of a binomial distribution is 4.8 for 5 trials, find the distribution.

Solution:

Mean = np ; Variance = npq

Given sum of mean and variance is 4.8.

i.e $np + npq = 4.8$

$np(1 + q) = 4.8$

$np(1 + 1 - p) = 4.8$

$5p(2 - p) = 4.8$

$10p - 5p^2 - 4.8 = 0$

or $5p^2 - 10p + 4.8 = 0 \quad \Rightarrow \quad p = 1.2,\ 0.8$

$\therefore\ p = 0.8,\ q = 0.2$ ($\because$ p cannot be greater than 1)

The binomial distribution is

$P(X = x) = 5C_x\,(0.8)^x\,(0.2)^{5-x}$

Where x = 0, 1, 2, 3, 4, 5

2. In tossing 10 coin. What is the chance of having exactly 5 heads.

Solution:

Let X denote number of heads

p = Probability of getting a head = $\frac{1}{2}$

$q = 1 - p$

$= 1 - \frac{1}{2}$

$q = \frac{1}{2}$

n = number of trial is 10.

∴ The binomial distribution is

$P(X = x) \quad = nC_x\, p^x q^{n-x}$

Probability of getting exactly 5 heads

$$P(X = 5) = 10C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{10-5}$$

$$= 10C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5$$

$$= 10C_5 \left(\frac{1}{2}\right)^{10}$$

$$= \frac{63}{256}$$

PART – C

1. Ten coins are tossed simultaneously find the probability of getting atleast seven heads.

Solution:

Given $n = 10,\quad p = \frac{1}{2},\quad q = 1 - \frac{1}{2} = \frac{1}{2}$

$$P(X = x) = nC_x\, p^x q^{n-x}$$

$$= 10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$$

$$= 10C_x \left(\frac{1}{2}\right)^{x+10-x}$$

$$P(X = x) = 10C_x \left(\frac{1}{2}\right)^{10}$$

Probability of getting atleast seven heads is

$$P(x \geq 7) = P(x = 7) + P(x = 8 + P(x = 9) + P(x = 10)$$

$$= 10C_7\left(\frac{1}{2}\right)^{10} + 10C_8\left(\frac{1}{2}\right)^{10} + 10C_9\left(\frac{1}{2}\right)^{10} + 10C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10}[10C_7 + 10C_8 + 10C_9 + 10C_{10}]$$

$$= \frac{1}{2^{10}}[120 + 45 + 10 + 1]$$

$$= \frac{176}{1024}$$

$$P(x \geq 7) = \frac{11}{64}$$

A.4. Poisson Distribution

Poisson distribution is a limiting case of binomial distribution under the following conditions.

(i) n, the number of Independent trials is indefinitely large. i.e $n \to \infty$.

(ii) p, the constant probability of success in each trial is very small ie $p \to 0$.

(iii) $np = \lambda$ is finite where λ is a positive real number.

Definition:

A discrete random variable X is said to have a poisson distribution if the probability mass function of X is $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots\ldots$ for some $\lambda > 0$ and λ is the parameter of the poisson distribution.

Constants of poisson distribution:

(i) Mean = λ

(ii) Variance = λ

(iii) Standard deviation = $\sqrt{\lambda}$

Examples of poisson distribution:

1. The number of printing mistakes in each page of a book.
2. The number of suicides reported in a particular city in a particular month.
3. The number of road accidents at a particular junction per month.
4. The number of child born blind per year in a hospital.

Examples

PART – A

1. If the mean of the poisson distribution is 2. Find P (x = 0).

Solution:

We know $P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}$

Given : $\lambda = 2$

$$\therefore P(X=0)=\frac{e^{-2}2^{0}}{0!}$$
$$=e^{-2}$$

2. If the mean of the poisson distribution is 4. Find the value of the variance and standard deviation.

Solution:

Given Mean $= \lambda = 4$

Variance $= \lambda = 4$

$S.D=\sqrt{\lambda}=\sqrt{4}=2$

3. In a poisson distribution if the variance is 2. and $P=\frac{1}{100}$ what is n.

Solution:

We know $\lambda = np$

Given: $\lambda = 2,\ P=\frac{1}{100}$

$$\therefore 2 = n\frac{1}{100}$$

i.e $n = 200$

PART – B

1. In a poisson distribution if P (x = 3) = P (x = 2) find P (x = 0).

Solution:

Given: P (x = 3) = P (x = 2)

We know $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$\therefore \quad \frac{e^{-\lambda}\lambda^{3}}{3!}=\frac{e^{-\lambda}\lambda^{2}}{2!}$$

$$\frac{\lambda}{1\times 2\times 3}=\frac{1}{1\times 2}$$

$$\lambda=3$$

$$\therefore P(x=0)=\frac{e^{-3}3^{0}}{0!}$$

$$=e^{-3}$$

2. If 10% of the screws produced by an automobile machine are defective. Find the probability that out of 15 screws selected at random only 3 are defective.

Solution:

Let p-denote the probability of defective screw.

Given: $p = 10\% = \dfrac{10}{100} = \dfrac{1}{10}$, $n = 15$

$$\lambda = np$$

$$= 15 \times \frac{1}{10}$$

$$= \frac{3}{2}$$

X-denote the number of defective screws.

$$P(X=3) = \frac{e^{-1.5}(1.5)^3}{3!} = 0.1255 \qquad \Bigg| \qquad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Probability for 3 screws to be defective = 0.1255

PART – C

1. If X is a poisson variable with $P(X=2)=\frac{2}{3}P(X=1)$, find $P(X=3)$ and $P(X=0)$.

Solution:

Given: $P(X=2)=\frac{2}{3}P(X=1)$

We know $P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}$

i.e $\frac{e^{-\lambda}\lambda^{2}}{2!}=\frac{2}{3}\frac{e^{-\lambda}\lambda^{1}}{1!}$

$$\frac{\lambda}{1\times 2}=\frac{2}{3}\cdot\frac{1}{1}$$

$$\lambda=\frac{4}{3}$$

$$\therefore P(X=3)=\frac{e^{-4/3}\left(4/3\right)^{3}}{3!}$$

$$=\frac{4^{3}\,e^{-4/3}}{3^{3}\,.\,3!}$$

$$=\frac{64}{27\times 6}e^{-4/3}$$

$$\therefore P(X=3)=\frac{32}{81}e^{-4/3}$$

$$=0.0998$$

$$P(X=0)=\frac{e^{-4/3}\left(4/3\right)^{4}}{0!}$$

$$=e^{-4/3}$$

$$=0.2725$$

2. The probability that an electric bulb to be defective from a manufacturing unit is 0.02. In a box of 200 electric bulbs find the probability that

(i) Exactly 4 bulbs are defective

(ii) More than 3 bulbs are defective.

Solution:

Let p-denote the probability of one bulb to be defective.

Given: $p = 0.02$

$n = 200$

$\lambda = np$

$= 200 \times 0.02$

$= 4$

X-denote number of defective bulbs.

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\text{(i)}\ P(X = 4) = \frac{e^{-4}(4)^4}{4!} = \frac{e^{-4} \times 256}{24}$$

Probability for 4 bulbs to be defective = 0.1952

$$\text{(ii)}\ P(X > 3) = 1 - (P \le 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - \left[\frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!}\right]$$

$$= 1 - e^{-4}\left[1 + 4 + \frac{16}{2} + \frac{64}{6}\right]$$

$$= 1 - e^{-4}\left[\frac{142}{6}\right]$$

$$= 1 - 0.0183 \times 23.67$$

$$= 1 - 0.4332$$

$$= 0.5668$$

Probability for more than 3 bulbs = 0.5668

A.5. Normal Distribution

Normal distribution is a continous distribution. Like poisson distribution, the normal distribution is obtained as a limiting case of Binomial distribution. This is the most important probability model in statistical analysis.

Definition:

A random variable X is normally distributed with parameters μ (called Mean) and σ^2 (called Variance) if its p.d.f (Probability density function) is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \; -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0$$

Note:

$X \sim N(\mu, \sigma)$ denotes that the random variable X follows normal distirbution with mean μ and standard deviation σ.

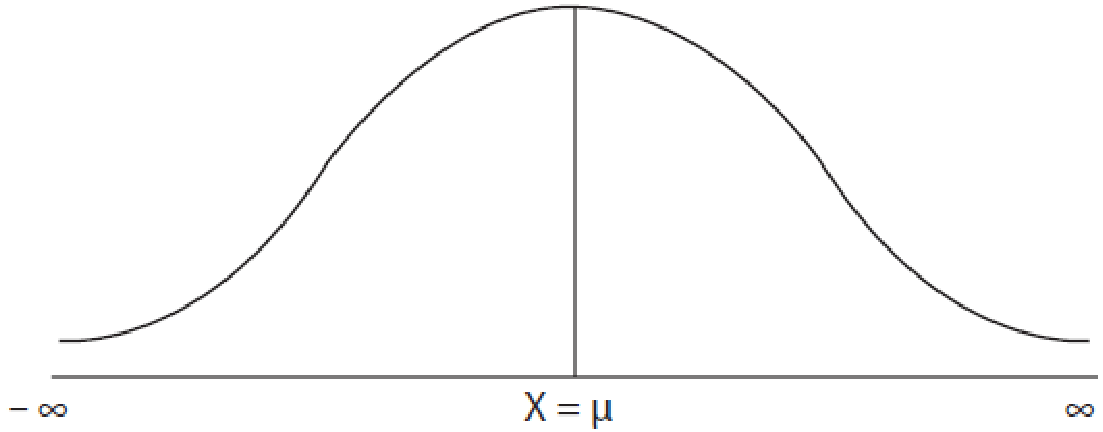
Constants of Normal distribution:

(i) Mean = μ

(ii) Variance = σ^2

(iii) Standard deviation = σ

Properties of Normal Probability curve and Normal distribution:



1. The normal curve is bell shaped.
2. The curve is symmetrical about $X = \mu$.
3. Mean = Median = Mode = μ.
4. The curve attains its maximum value at $X = \mu$ and the maximum value is $\frac{1}{\sigma\sqrt{2\pi}}$.
5. The normal curve is asymptotic to the x-axis.
6. The points of inflection are at $X = \mu \pm \sigma$.
7. Since the curve is symmetrical about $X = \mu$, the skewness is zero.
8. A normal distribution is a close approximation to the Binomial distribution when n is very large and p is close to $\frac{1}{2}$.
9. Normal distribution is also a limiting form of poisson distribution when $\lambda \rightarrow \infty$.

Standard normal distribution:

A random variable Z is called a standard normal variate if its mean is zero and standard deviation is 1 and is denoted by N (0, 1).

where $Z = \dfrac{X - \mu}{\sigma}$, X – normal variate.

Note:

1. The total area under normal probability curve is unity.

   $$\text{i.e} \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \varphi(z)\, dz = 1$$

   $$\int_{-\infty}^{0} \varphi(z)\, dx = \int_{0}^{\infty} \varphi(z)\, dz = 0.5$$

   (Since the normal curve is symmetrical about z = 0)

2. $\int_0^{z_1} \phi(x)dx$ is known as normal probability Integral and gives the area under standard normal curve between the ordinates at z = 0 and $z = z_1$.
3. In the normal probability tables, the areas are given under standard normal curve. We shall deal with the standard normal variate z rather than the variable X itself.

Examples

PART –A

1. When X = 100, μ = 80, σ = 10 what is the value of z?

Solution:

$$\text{We know } Z = \frac{X-\mu}{\sigma}$$

$$= \frac{100-80}{10}$$

$$= 2$$

2. If X is normally distributed with mean 80 and standard deviation 10. Express P (70 ≤ X ≤ 100) in terms of standard normal variate.

Solution:

Given: Mean = μ = 80

S.D = σ = 10

$$\text{We know } Z = \frac{X-\mu}{\sigma}$$

$$\text{When } X = 70,\ Z = \frac{70-80}{10} = -1$$

$$\text{When } X = 100,\ Z = \frac{100-80}{10} = 2$$

$$\therefore P(70 \le X \le 100) = P(-1 \le Z \le 2)$$

PART – B

1. If X is a normal variate with mean 80 and S.D 10. Compute P (X ≤ 100).

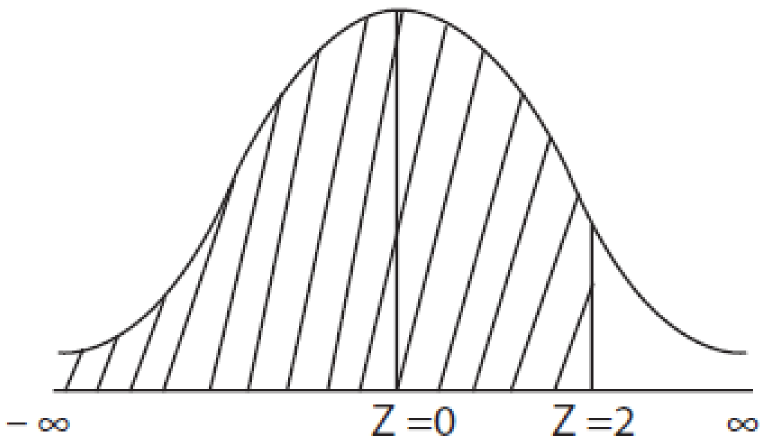
Solution:

We know $Z = \dfrac{X - \mu}{\sigma}$

When X = 100,

$$Z = \frac{100 - 80}{10}$$

$$= 2$$



$$\therefore P(X \le 100) = P(Z \le 2)$$

$$= P(-\infty < Z < 0) + P(0 < Z < z)$$

$$= 0.5 + 0.4772 \text{ (from table)}$$

$$= 0.9772$$

2. If X is normally distributed with mean 6 and standard deviation 5. Find $P(0 \leq X \leq 8)$.

Solution:

Given: Mean $= \mu = 6$

and S.D $= \sigma = 5$

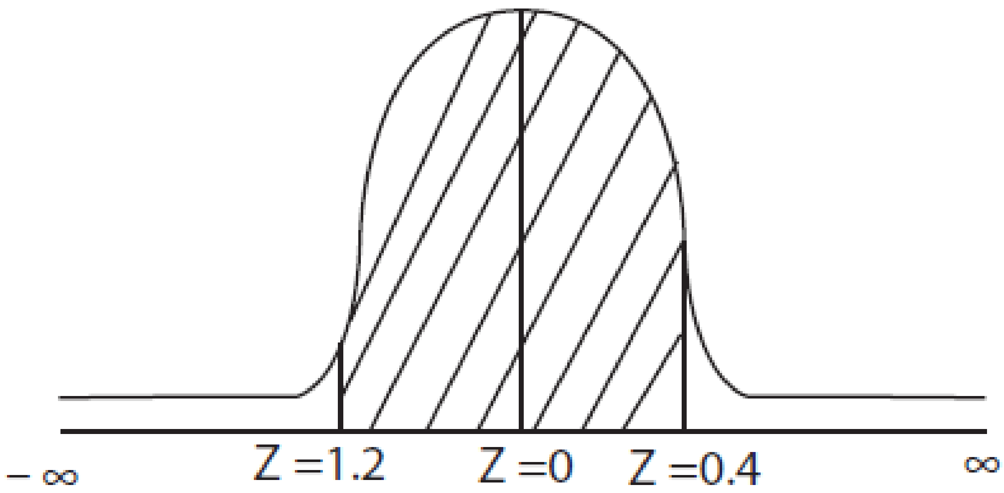
We know $Z = \dfrac{X - \mu}{\sigma}$

When $X = 0$

$$Z = \frac{0-6}{5} = -1.2$$

When $X = 8$

$$Z = \frac{8-6}{5} = \frac{2}{5} = 0.4$$



$\therefore P(0 \leq X \leq 8) = P(-1.2 \leq Z \leq 0.4)$

$= P(-1.2 \leq Z \leq 0) + P(0 \leq Z \leq 0.4)$

$= P(0 \leq Z \leq 1.2) + P(0 \leq Z \leq 0.4)$

($\because$ the curve is symmetry)

$= 0.3849 + 0.1554$ (from the table)

$= 0.5403$

PART – C

1. The mean score of 1000 students in an examination is 36 and standard deviation is 16. If the score of the students is normally distributed how many students are expected to score more than 60 marks.

Solution:

Given: Mean = μ = 36

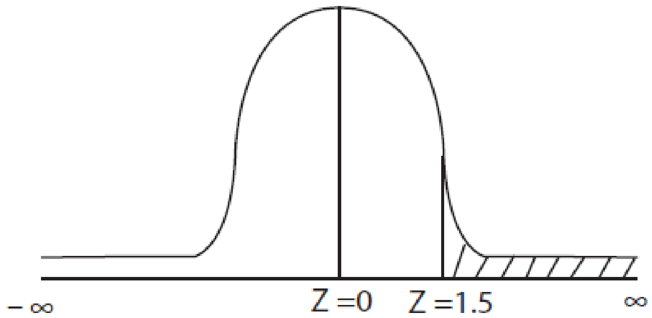
S.D = σ = 16

Number of students = 1000

We know $Z = \dfrac{X - \mu}{\sigma}$

When X = 60

$$Z = \frac{60 - 36}{16} = \frac{24}{16} = \frac{3}{2} = 1.5$$



$\therefore$ P (X > 60) = P (Z > 1.5)

P (X > 60) $= P(0 < Z < \infty) - P(0 < Z < 1.5)$

$= 0.5 - 0.4332 = 0.0668$

Probability for one student to score more than 60 = 0.0668

The number of students expected to score more than 60 marks out of 1000 students

$= 0.0668 \times 1000$

$= 66.8$

$\cong 67$

2. When X is normally distributed with mean 12, standard deviation is 4. Find (i) $P(X \geq 20)$, (ii) $P(0 < X < 12)$ (iii) $P(X \leq 20)$.

Solution:

Given: Mean $= \mu = 12$

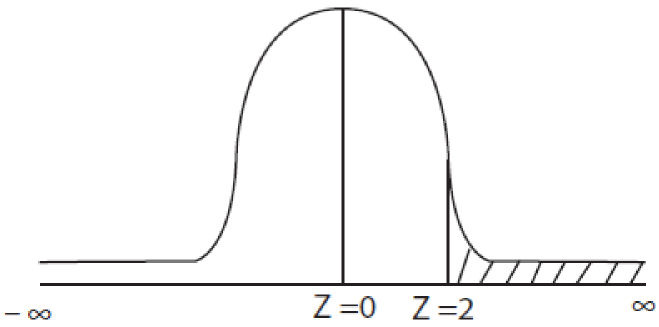
$S.D = \sigma = 4$

We know $Z = \dfrac{X - \mu}{\sigma}$

When $X = 20$, $Z = \dfrac{20 - 12}{4} = 2$

When $X = 0$, $Z = \dfrac{0 - 12}{4} = -3$

When $X = 12$, $Z = \dfrac{12 - 12}{4} = 0$



(i) $P(X \geq 20) = P(Z \geq 2)$

$= P(0 \leq Z \leq \infty) - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228$

(ii) $P(0 < X < 12) = P(-3 < Z < 0)$

$= P(0 < Z < 3)$

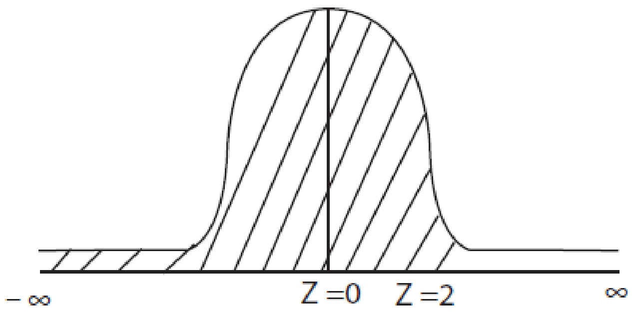
$= 0.4987$

(iii) $P(X \leq 20) = P(Z \leq 2)$

$= P(-\infty < Z < 0) + P(0 < Z < 2)$

$= 0.5 + 0.4772$

$= 0.9772$



3. In an aptitude test administered to 900 students, the scores obtained by the students are distributed normally with mean 50 and standard deviation 20. Find the number of students whose score is (i) between 30 and 70.

Solution:

Given: Mean $= \mu = 50$

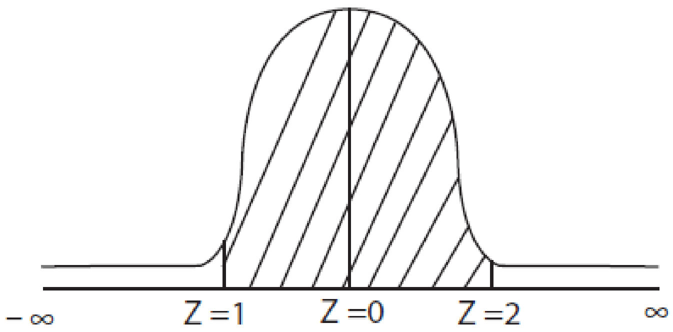
$S.D = \sigma = 20$

Number of students = 900

We know $Z = \dfrac{X - \mu}{\sigma}$

When $X = 30$; $Z = \dfrac{30 - 50}{20} = -1$

When $X = 70$; $Z = \dfrac{70 - 50}{20} = 1$



$$\begin{aligned}\therefore P(30 < X < 70) &= P(-1 < Z < 1) \\ &= P(-1 < Z < 0) + P(0 < Z < 1) \\ &= P(0 < Z < 1) + P(0 < Z < 1) \\ &= 2P(0 < Z < 1) \\ &= 2 \times 0.3413 \\ &= 0.6826\end{aligned}$$

Probability for a students score is between 30 and 70 = 0.6826

The number of students whose score is between 30 and 70 out of 900 students

$$\begin{aligned}&= 0.6826 \times 900 \\ &= 614.34 \\ &\cong 614\end{aligned}$$

4. Obtain K, μ, σ^2 of the normal distribution whose probability distribution function is $f(x) = Ke^{-2x^2+4x}, -\infty < x < \infty$.

Solution:

The normal distribution is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$.

First consider, the power of the exponential (e)

i.e

$$\begin{aligned} -2x^2 + 4x &= -2(x^2 - 2x) \\ &= -2(x^2 - 2x + 1 - 1) \\ &= -2[(x-1)^2 - 1] \\ &= -2(x-1)^2 + 2 \end{aligned}$$

$$\begin{aligned} e^{-2x^2+4x} &= e^{-2(x-1)^2+2} \\ &= e^2 . e^{\frac{-(x-1)^2}{\frac{1}{2}}} \\ &= e^2 . e^{-\frac{1}{2}\frac{(x-1)^2}{(\frac{1}{2})^2}} \\ &= e^2 . e^{-\frac{1}{2}\left[\frac{x-1}{\frac{1}{2}}\right]^2} \end{aligned}$$

Comparing with f(x)

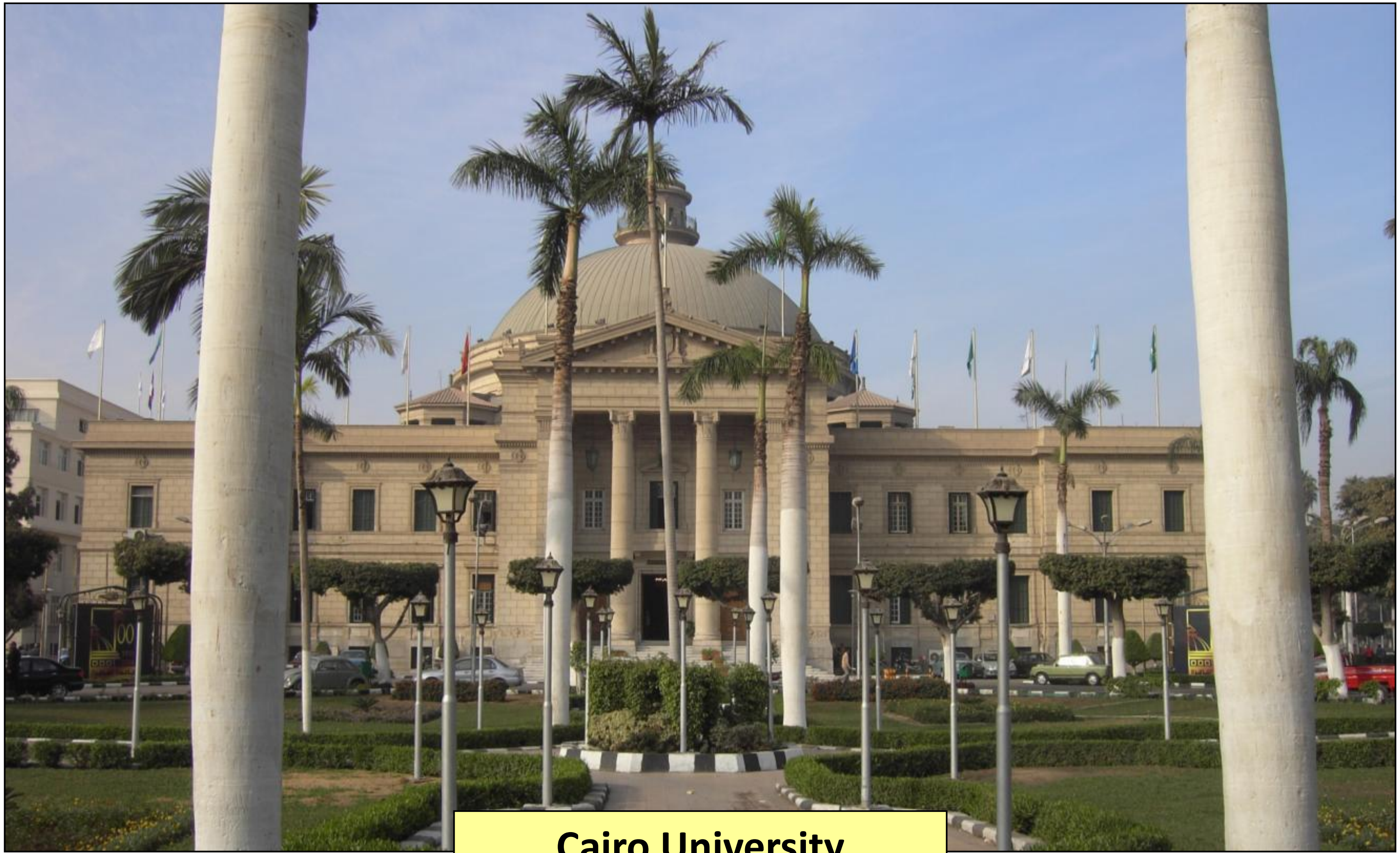
$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = Ke^2 . e^{-\frac{1}{2}\left[\frac{x-1}{\frac{1}{2}}\right]^2}$$

We get $\sigma = \dfrac{1}{2}, \mu = 1, Ke^2 = \dfrac{1}{\sigma\sqrt{2\pi}}$

$$\begin{aligned} \therefore K &= \frac{1}{\frac{1}{2}\sqrt{2\pi}} . e^{-2} \\ &= \frac{2e^{-2}}{\sqrt{2\pi}} \\ &= \sqrt{\frac{2}{\pi}} . e^{-2} \end{aligned}$$

Exercises and Solutions

Thank You!



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