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The Kumaraswamy Exponentiated Pareto Distribution

Abstract: Modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. For the first time, the called Kumaraswamy Exponentiated Pareto distribution, is introduced. Some structural properties of the proposed distribution are studied including explicit expressions for the moments and generating function.An explicit expression for Rényi entropy is obtained. The method of maximum likelihood is used for estimating the model parameters.

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1 Introduction

The Pareto distribution was originally proposed to model the unequal distribution of wealth since Pareto observed the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people. Ever since, it plays an important role in analyzing a wide range of real-world situations, not only in the field of economics. Examples of approximately Pareto distributed phenomena may be found in sizes of sand particles and clusters of Bose–Einstein condensate close to absolute zero.

Another application of this distribution could be in On-Line Analytical Processing (OLAP). Nadeau and Teorey [21] used Pareto distribution for OLAP aims at gaining useful information quickly from large amounts of data residing in a data warehouse. There are several forms and extensions of the Pareto distribution in the literature. Pickands [23] was the first to propose an extension of the Pareto distribution with the generalized Pareto (*GP*) distribution when analyzing the upper tail of a distribution function. The *GP* has been used for modelling extreme value data because of its long tail feature (see Choulakian & Stephens, [2]. Of course, the Pareto distribution is a special case of the *GP*. The exponentiated Pareto (*EP*) distribution was introduced by Gupta et al. [11] in the same settings that the generalized exponential (*GE*) distribution extends the exponential distribution (see Gupta & Kundu, [8]). The *EP* distribution can be defined by raising the distribution function of a Pareto distribution to a positive power. Thus, the random variable X with *EP* distribution has a distribution function given by:

$$G_{X|(\lambda,\mu,\theta)}(x) = \left(1 - \left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta}, \quad \lambda, \mu, \theta > 0, \text{ for } x \geq \lambda. \tag{1.1}$$

The corresponding density function is given by

$$g_{(\lambda,\mu,\theta)}(x) = \frac{\theta\mu\,\lambda^{\mu}}{x^{\mu+1}}\left(1 - \left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta-1}. \tag{1.2}$$

This kind of extension has been receiving considerable attention over the last decade. See, for instance, the exponentiated Fréchet, exponentiated Weibull, exponentiated gamma and exponentiated Gumbel distributions, which extend the Fréchet, Weibull, gamma and Gumbel distributions in the same way that the GE distribution extends the exponential distribution. All these generalizations were proposed by Nadarajah and Kotz [20]. In a recent paper, Silva et al. [26] introduced the generalized exponential-geometric (*GEG*) distribution by raising the distribution function of a exponential geometric distribution to a positive power, as well as the mentioned distributions before the exponential distribution.

The Kumaraswamy (Kw) distribution is not very common among statisticians and has been little explored in the literature. Its distribution function is given by

$$F_{X|(a,b)}(x) = 1 - \left(1 - x^a\right)^b, \quad 0 < x < 1, \tag{1.3}$$

where $a > 0$ and $b > 0$ are shape parameters. The density function is given by:

$$f_{X|(a,b)}(x) = abx^{a-1}\left(1 - x^a\right)^{b-1}$$

which can be unimodal, increasing, decreasing or constant, depending on the parameter values. This distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones [13] explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and Kw distributions. However, the beta distribution has the following advantages over the Kw distribution: simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, we combine the works of Kumaraswamy [15] and Cordeiro and de Castro [3] to derive some mathematical properties of a new model, called the Kumaraswamy Exponentiated Pareto (Kw-EP) distribution, which stems from the following general construction: if G denotes the distribution function of a random variable X, then a generalized class of distributions can be defined by

$$F_{X|(a,b)}(x) = 1 - \left(1 - G_X(x)^a\right)^b, \tag{1.4}$$

where $a > 0$ and $b > 0$ are two additional shape parameters. The Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function $f_{X|(ab)}$ has a very simple form

$$f_{X|(a,b)}(x) = abg(x)G_X(x)^{a-1}\left(1 - G_X(x)^a\right)^{b-1}. \tag{1.5}$$

The density family (1.5) has many of the same properties of the class of beta-G distributions (see Eugene et al. [7], but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special Kw-G distributions can be generated as follows: the Kw-normal distribution is obtained by taking $G_X(x)$ in (1.4) to be the normal distribution function. Analogously, the Kw-Weibull (Cordeiro et al. [4]), Kw-generalized gamma (Pascoa et al. [22]), Kw-Birnbaum–Saunders (Saulo et al. [24]) and Kw-Gumbel (Cordeiro et al. [6]) distributions are obtained by taking $G_X(x)$ to be the distribution function of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new Kw-G distribution can be generated from a specified G distribution.

The rest of the article is organized as follows. In Section 2, we define the distribution, density and hazard functions of the Kw-EP distribution and some special cases. In addition, we study the limit behavior of its density and hazard rate functions. A range of mathematical properties of this distribution is considered in Section 3. This section includes moment, moment generating function, characteristic function, incomplete moments and mean deviations. The Rényi entropy is calculated in Section 4. Finally, the Maximum Likelihood estimator is given and the observed information matrix is determined in Section 5.

2 The Kumaraswamy Exponentiated Pareto Distribution

Let $G_{X|(\lambda,\mu,\theta)}(x)$ be the Exponentiated Pareto distribution function with parameters λ, μ, θ, then (1.4) yields the Kumaraswamy Exponentiated Pareto (Kw-EP) distribution function (for $x \geq \lambda$):

$$F_{X|(\lambda,\mu,\theta,a,b)}(x) = 1 - \left(1 - \left(1 - \left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a}\right)^{b}, \tag{2.1}$$

where $\lambda > 0$ is a scale parameter and the other positive parameters μ, θ, a and b are shape parameters. The corresponding density and hazard rate functions are:

$$f_{X|(\lambda,\mu,\theta,a,b)}(x) = \frac{ab\theta\mu\lambda^{\mu}}{x^{\mu+1}}\left[1-\left(\frac{\lambda}{x}\right)^{\mu}\right]^{\theta a-1}\left\{1-\left[1-\left(\frac{\lambda}{x}\right)^{\mu}\right]^{\theta a}\right\}^{b-1}, \quad x \geq \lambda \tag{2.2}$$

and

$$\begin{aligned} h_{X|(\lambda,\mu,\theta,a,b)}(x) &= \frac{f_{X|(\lambda,\mu,\theta,a,b)}}{\overline{F}_{X|(\lambda,\mu,\theta,a,b)}} \\ &= \frac{ab\theta\mu\lambda^{\mu}(1-(\frac{\lambda}{x})^{\mu})^{\theta a-1}}{x^{\mu+1}(1-(1-(\frac{\lambda}{x})^{\mu})^{\theta a})}, \end{aligned} \tag{2.3}$$

respectively.

We notice that the following distributions are special cases of the Kumaraswamy Exponentiated Pareto distribution function $F_{X|\lambda,\mu,\theta,a,b)}(x)$.

1. If $\theta = 1$, we get the Kumaraswamy Pareto distribution function.
2. If $a = b = 1$, we get the Exponentiated Pareto distribution function.

2.1 Expansion for the Distribution and Density Functions

In this subsection we present some representations of the distribution and density functions of Kumaraswamy Exponentiated Pareto. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is positive and $|z| < 1$, then

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}z^{i}. \tag{2.4}$$

Using the power series of (2.4) equation (2.1) becomes with $\beta = \theta a + 1$:

$$F_{X|(\lambda,\mu,\theta,a,b)}(x) = 1-\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a i},$$

and equation (2.2) becomes:

$$f_{X|(\lambda,\mu,\theta,a,b)}(x) = \frac{ab\theta\mu\lambda^{\mu}}{x^{\mu+1}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a(i+1)-1} \tag{2.5}$$

Again, by using (2.4) in the last factor of each summand in (2.5), we obtain

$$f_{X|(\lambda,\mu,\theta,a,b)}(x) = \frac{ab\theta\mu}{(j+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{\beta-1}{i}\binom{\theta a(i+1)-1}{j}\left(\frac{\lambda^{\mu(j+1)}}{x^{\mu(j+1)+1}}\right)$$

and

$$f_{X|(\lambda,\mu,\theta,a,b)}(x) = \sum_{j=0}^{\infty}w_{j}g(x,\lambda,\mu(j+1)), \tag{2.6}$$

where

$$w_{j} = \frac{ab\theta}{(j+1)}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{\beta-1}{i}\binom{\theta a(i+1)-1}{j} \tag{2.7}$$

and $g_{X|(\lambda,\mu(j+1))}(x)$ denotes the Pareto density function with parameters λ and $\mu(j+1)$. The above result ensures that some mathematical properties such as ordinary and incomplete moments, generating function and mean deviations can be derived from those quantities of the reparametrized (with parameters λ and $\mu(j+1)$) Pareto distribution.

Limiting Behaviour of *Kw-EP* Density Function

In this subsection we state the limiting behavior of Kumaraswamy Exponentiated Pareto.

Lemma. *The limit of the Kw-EP density function as $x \to \infty$ is 0, and $x \to \lambda$ are:*

$$\lim_{x \to \lambda} f_{X|(\lambda,\mu,\theta,a,b)}(x) = \begin{cases} \infty & \text{for } 0 < a < 1, \\ \frac{ab\mu}{\lambda} & \text{for } 1 < a. \end{cases} \tag{2.8}$$

3 Moments and Moments Generating Function

In this section we derive the moment and moment generating function of the *Kw-EP* distribution.

3.1 Moments

Here after, let $X|(\lambda, \mu, \theta, a, b)$ be a random variable having the *Kw-EP* distribution (2.2). Using (2.6), it is easy to obtain the rth moment of X. Let Y be a Pareto distributed random variable, then the rth moment of Y is given by

$$E[Y^r|\lambda] = \frac{\mu\lambda^r}{\mu - r}, \quad r < \mu. \tag{3.1}$$

From (2.6), we obtain

$$\begin{aligned} E[X^r|(\lambda, \mu, \theta, a, b)] &= \sum_{j=0}^{\infty} w_j \int_{\lambda}^{\infty} x^r g_{X|(\lambda,\mu(j+1))}(x)dx \\ &= \mu\lambda^r \sum_{j=0}^{\infty} \frac{w_j(j+1)}{\mu(j+1) - r}, \quad r < b\lambda. \end{aligned} \tag{3.2}$$

By setting $r = 1$ in (3.2), gives the first moment of $X|(\lambda, \mu, \theta, a, b)$

$$E[X||(\lambda, \mu, \theta, a, b)] = \mu\lambda \sum_{j=0}^{\infty} \frac{w_j(j+1)}{[\mu(j+1) - 1]} \quad \text{for } b\lambda > 1. \tag{3.3}$$

Setting $a = b = \theta = 1$, we obtain:

$$w_j = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \geq 1 \end{cases} \tag{3.4}$$

and the first moment $E[X|(\lambda, \mu, 1, 1, 1)]$ reduces to:

$$E[X|(\lambda, \mu, 1, 1, 1)] = \frac{\mu\lambda}{\mu - 1} \tag{3.5}$$

which is identical to the first moment of the Pareto distribution.

3.2 Moments Generating Function

The moment generating function (mgf) $M_{Y|(\lambda,\mu)}$ corresponding to a random variable Y with Pareto distribution with parameters λ and μ is only defined for negative values of its argument t. It is given by:

$$M_{Y|(\lambda,\mu)}(t) = \mu(-\lambda t)^{\mu}\Gamma(-\mu, -\lambda t), \quad t < 0, \tag{3.6}$$

where Γ denotes the incomplete gamma function:

$$\Gamma(x,n) = \int_x^{\infty} t^{n-1}e^{-t}dt. \tag{3.7}$$

Thus, using $M_{Y(\lambda,\mu)}(t)$ and (2.6), we can write for $t < 0$

$$\begin{aligned} M_{X|(\lambda,\mu,\theta,a,b)}(t) &= \sum_{j=0}^{\infty} w_j \int_{\lambda}^{\infty} e^{tx} g_{X|(\lambda,\mu(j+1),\theta,a,b)}((x)dx \\ &= \mu \sum_{j=0}^{\infty} (j+1) w_j (-\lambda t)^{\mu(j+1)} \Gamma(-\mu(j+1), -\lambda t). \end{aligned} \tag{3.8}$$

In the same way, the characteristic function of the *Kw-EP* distribution becomes

$$\varphi_{X|(\lambda,\mu,\theta,a,b)}(t) = M_{X|(\lambda,\mu,\theta,a,b)}(it)$$

where $i = \sqrt{-1}$ is the unit imaginary number.

3.3 Incomplete Moments

Let $Y|(\lambda,\mu)$ be a random variable with a Pareto distribution with parameters λ and μ, then the r_{th} incomplete moment of $Y|(\lambda,\mu)$ is given by

$$\begin{aligned} M_{Y|(\lambda,\mu);r}(z) &= \int_{\lambda}^{z} y^r g(x,\lambda,\mu)dy \\ &= \frac{\mu\lambda^r}{\mu - r}\left((1 - \left(\frac{\lambda}{z}\right)^{\mu - r}\right). \end{aligned} \tag{3.9}$$

From (3.9), we note that whenever $r < \mu$ the incomplete moments converge for $z \to \infty$ to the rth moment $M_{Y|(\lambda,\mu);r}(z) \to E[Y^r|(\lambda,\mu)]$. Let $X|\lambda,\mu,a,b)$ has *Kw-EP* distribution. For $r < b\mu$ the rth incomplete moment of $X|\lambda,\mu,a,b)$ is given by:

$$\begin{aligned} M_{X|(\lambda,\mu,a,b);r}(z) &= \int_{\lambda}^{z} x^r g_{X|(\lambda,\mu(j+1))}(x)dx \\ &= \mu\lambda^r \sum_{j=0}^{\infty} \frac{w_j(j+1)}{\mu(j+1) - r}\left(1 - \left(\frac{\lambda}{z}\right)^{\mu(j+1)-r}\right). \end{aligned} \tag{3.10}$$

3.4 First Moment of Deviations

The first moments of the deviations from the first moment and from the median are usually used as a measure of spread in a population. Let $\psi = E[X|(\lambda,\mu,\theta,a,b)]$ and $\phi = \phi_{X|(\lambda,\mu,\theta,a,b)}$ be the first moment and the median of the *Kw-EP* distribution, respectively. The first moment of deviations about the ψ and about ϕ can be calculated as:

$$D[\psi] = E\left[|X|(\lambda,\mu,\theta,a,b) - \psi|\right] = \int_{\lambda}^{\infty} (|x-\psi|) f_{X|(\lambda,\mu,\theta,a,b)}(x)dx, \tag{3.11}$$

and

$$D[\phi] = E[|X|(\lambda,\mu,\theta,a,b) - \phi|] = \int_{\lambda}^{\infty} (|x-\phi|) f_{X|(\lambda,\mu,\theta,a,b)}(x)dx, \tag{3.12}$$

respectively. We obtain:

$$D[\psi] = \int_{\lambda}^{\infty} (|x-\psi|) f_{X|(\lambda,\mu,\theta,a,b)}(x)dx = 2\psi F_{X|(\lambda,\mu,\theta,a,b)}(\psi) - 2M_{X|(\lambda,\mu,\theta,a,b);1}(\psi), \tag{3.13}$$

where $M_{X|(\lambda,\mu,\theta,a,b);1}(\psi)$ denotes the first incomplete moment calculated from (3.10) for $r = 1$. Similarly,the first moment of deviation about the median follows as:

$$D[\phi] = \int_{\lambda}^{\infty} (|X|(\lambda,\mu,\theta,a,b) - \phi|) f_{X|(\lambda,\mu,\theta,a,b)}(x)dx = \psi - 2M_{X|(\lambda,\mu,\theta,a,b);1}(\phi). \tag{3.14}$$

4 Rényi Entropy

Next, we provide the Rényi entropy, which is a measure of variation of uncertainty. The theory of entropy has been successfully used in a wide diversity of applications and has also been used for the characterization of numerous standard probability distributions. The entropy of a random variable X is a measure of uncertainty variation. The Rényi entropy is defined as

$$I_{X;R}(\delta) = \frac{1}{1-\delta}\log\left[I_X(\delta)\right],$$

where

$$I_X(\delta) = \int_R f_X^{\delta}(x)dx \quad \text{for } \delta > 0 \text{ and } \delta \neq 1. \tag{4.1}$$

For a *Kw-EP* random variable $X|\lambda,\mu,\theta,a,b)$, we obtain:

$$I_{X|\lambda,\mu,\theta,a,b)}(\delta) = (ab\theta\mu)^{\delta}\lambda^{\mu\delta}\int_{\lambda}^{\infty}\frac{1}{x^{(\mu+1)\delta}}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\delta(\theta a-1)}\left(1-\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a}\right)^{(b-1)\delta}dx. \tag{4.2}$$

Applying the binomial expansion to the last factor in the above integrand yields:

$$I_{X|\lambda,\mu,\theta,a,b)}(\delta) = (ab\theta\mu)^{\delta}\lambda^{\mu\delta}\int_{\lambda}^{\infty}\frac{1}{x^{(\mu+1)\delta}}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\delta(\theta a-1)}\sum_{j=0}^{\infty}(-1)^j\binom{(b-1)\delta}{j}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a\, j}dx \tag{4.3}$$

$$= (ab\theta\mu)^{\delta}\lambda^{\mu\delta}\sum_{j=0}^{\infty}(-1)^j\binom{(b-1)\delta}{j}\int_{\lambda}^{\infty}\frac{1}{x^{(\mu+1)\delta}}\left(1-\left(\frac{\lambda}{x}\right)^{\mu}\right)^{\theta a(\delta+j)-\delta}dx. \tag{4.4}$$

Changing variables and simplifying, $I_{X|\lambda,\mu,\theta,a,b)}(\delta)$ reduces to

$$I_{X|\lambda,\mu,\theta,a,b)}(\delta) = (ab\theta)^{\delta}\mu^{\delta-1}\lambda^{\delta+1}\sum_{j=0}^{\infty}(-1)^j\binom{(b-1)\delta}{j}\beta\left(\theta a(\delta+j)-\delta+1,\frac{\delta(\mu+1)}{\mu}\right), \tag{4.5}$$

where

$$\beta = \int_0^1 t^{a-1}(1-t)dt. \tag{4.6}$$

Hence, the formula for the Rényi entropy becomes:

$$I_{X|\lambda,\mu,\theta,a,b);R}(\delta) = \frac{\delta}{1-\delta}\log(ab\theta) - \log(\mu) + \frac{1+\delta}{1-\delta}\log(\lambda)$$
$$+\frac{1}{1-\delta}\log\sum_{j=0}^{\infty}(-1)^j\binom{(b-1)\delta}{j}\beta\left(\theta a(\delta+j)-\delta+1,\frac{\delta(\mu+1)}{\mu}\right). \tag{4.7}$$

5 Maximum Likelihood Estimation

In this section we derive the non-linear equations for finding the Maximum Likelihood Estimation (MLE) and inference of the parameters for the *Kw-EP* distribution. The Maximum Likelihood Estimation is one of the most widely used estimation method for finding the unknown parameters. Here we find the estimators for the *Kw-EP*. Let

$$\left(X|(\lambda,\mu,\theta,a,b)_1, X|(\lambda,\mu,\theta,a,b)_2,\dots,X|(\lambda,\mu,\theta,a,b)_n\right)$$

be a random sample from $X|(\lambda,\mu,\theta,a,b) \sim$ *Kw-EP* with observed values $x_1, x_2, \dots, x_n$ and let $\Psi = (\lambda,\mu,\theta,a,b)^T$ be the vector of the model parameters. The log likelihood function of (2.2) is defined as

$$\begin{aligned}\text{Log}(L) &= n\log a + n\log b + n\log\theta + n\log\mu + n\mu\log\lambda - (\mu+1)\sum_{i=1}^{n}\log(x_i)\\ &+ (\theta a - 1)\sum_{i=1}^{n}\log\left(1-\left(\frac{\lambda}{x_i}\right)^{\mu}\right) + (b-1)\sum_{i=1}^{n}\log\left(1-\left(1-\left(\frac{\lambda}{x_i}\right)^{\mu}\right)^{\theta a}\right).\end{aligned} \tag{5.1}$$

Firstly, since $x \geq \lambda$, the MLE of λ is the first order statistic $x_{(1)}$. The score vector is

$$U(\Psi) = (\partial\ell/\partial\mu, \partial\ell/\partial\theta, \partial\ell/\partial a, \partial\ell/\partial b)^T,$$

where the components corresponding to the model parameters are calculated by differentiating (5.1). By setting $z_i = 1 - \left(\frac{\lambda}{x}\right)^{\mu}$ we obtain

$$\frac{\partial\,\text{Log}(L)}{\partial a} = \frac{n}{a} + \theta\sum_{i=1}^{n}\log z_i - \theta(b-1)\sum_{i=1}^{n}\frac{z_i^{\theta a}\log z_i}{1-z_i^{\theta a}}, \tag{5.2}$$

$$\frac{\partial\,\text{Log}(L)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-z_i^{\theta a}), \tag{5.3}$$

$$\begin{aligned}\frac{\partial\,\text{Log}(L)}{\partial \mu} &= \frac{n}{\mu} + \frac{1}{\mu}\sum_{i=1}^{n}\log(1-z_i) - \frac{(\theta a-1)}{\mu}\sum_{i=1}^{n}\frac{(1-z_i)\log(1-z_i)}{z_i}\\ &+ \frac{\theta a(b-1)}{\mu}\sum_{i=1}^{n}\frac{(1-z_i)\log(1-z_i)z_i^{\theta a-1}}{(1-z_i^{\theta a})},\end{aligned} \tag{5.4}$$

$$\frac{\partial\,\text{Log}(L)}{\partial \theta} = \frac{n}{\theta} + a\sum_{i=1}^{n}\log z_i + (b-1)\sum_{i=1}^{n}\frac{a z_i^{\theta a}\log z_i}{(1-z_i^{\theta a})}. \tag{5.5}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (5.2)–(5.5) to zero and solve them simultaneously.

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