

Statistical Properties of Kumaraswamy Quasi Lindley Distribution

I.Elbatat¹ and M. Elgarhy^{2*}

Institute of Statistical Studies and Research, Department of Mathematical Statistics
Cairo University.

i_elbatat @ staff.cu.edu.eg

Abstract – In this paper, we present a new class of distributions called kumaraswamy Quasi Lindley Distribution. This class of distributions contains several distributions such as kumaraswamy Lindley distribution, Quasi Lindley, and kumaraswamy gamma distribution as special cases. The hazard function, moments and moment generating function are presented. Moreover, we discuss the maximum likelihood estimation of this distribution.

Keywords– Kumaraswamy Quasi Lindley distribution, Maximum likelihood estimation, Moment generating function.

1 Introduction and Motivation

Quasi Lindley distribution with parameters α and θ is defined by its probability density function (p.d.f)

$$g(x, \theta, \alpha) = \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x}; x > 0, \theta > 0, \alpha > -1. \quad (1.1)$$

It can easily be seen that at $\alpha = \theta$, the *QLD* Equation (1.1) reduces to the Lindley distribution (1958) with probability density function

$$g(x, \theta) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x}; x > 0, \theta > 0,$$

And at $\alpha = 0$, it reduces to the gamma distribution with parameters $(2, \theta)$. The p.d.f. Equation (1.1) can be shown as a mixture of exponential (θ) and gamma $(2, \theta)$ distributions as follows

$$g(x, \theta, \alpha) = p g_1(x) + (1 - p) g_2(x)$$

Where

$$p = \frac{1}{\alpha + 1}, g_1(x) = \theta e^{-\theta x} \text{ and } g_2(x) = \theta^2 e^{-\theta x}$$

The cumulative distribution function (cdf) of *QLD* is obtained as

$$G(x, \theta) = 1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right], x > 0, \theta > 0, \alpha > -1. \quad (1.2) \text{ Where } \theta \text{ is scale parameter.}$$

Ghitany et al. (2008a) have discussed various properties of this distribution and showed that in many ways Equation (1.1) provides a better model for some applications than the exponential distribution. Ghitany et al. (2008b,c) obtained size-biased and zero-truncated version of Poisson- Lindley distribution and discussed their various properties and applications. Also Ghitany and Al-mutairi (2009) discussed as various estimation methods for the discrete Poisson- Lindley distribution. Bakouch et al. (2012) obtained an extended Lindley distribution and discussed its various properties and applications. Mazucheli and Achcar (2011) discussed the applications of Lindley distribution to competing risks lifetime data. Rama and Mishra (2013) studied quasi Lindley distribution. Ghitany et al. (2011) developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Zakerzadah and Dolati (2010) obtained a generalized Lindley distribution and discussed its various properties and applications.

The distribution introduced by Kumaraswamy (1980), also referred to as the minimax distribution, is not very common among statisticians and has been little explored in the literature, nor has its relative interchangeability with the beta distribution been widely appreciated. We use the term K_w distribution to denote the Kumaraswamy distribution. The Kumaraswamy K_w distribution is not

very common among statisticians and has been little explored in the literature. Its cdf is given by

$$F(x, a, b) = 1 - (1 - x^a)^b, 0 < x < 1, \quad (1.3)$$

Where $a > 0$ and $b > 0$ are shape parameters, and the probability density function

$$f(x, a, b) = abx^{a-1}(1 - x^a)^{b-1}, \quad (1.4)$$

This can be unimodal, increasing, decreasing or constant, depending on the parameter values. It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones (2009) explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and K_w distributions. However, the beta distribution has the following advantages over the K_w distribution: simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, we combine the works of Kumaraswamy (1980) and Cordeiro and de Castro (2011) to derive some mathematical properties of a new model, called the Kumaraswamy linear exponential ($K_w - LE$) distribution, which stems from the following

general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x) = 1 - [1 - G(x)^a]^b \quad (1.5)$$

Where $a > 0$ and $b > 0$ are two additional shape parameters. The $Kw - G$ distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1} \quad (1.6)$$

The density family (1.6) has many of the same properties of the class of beta- G distributions (see Eugene et al. (2002)), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta- G family of distributions, special $Kw - G$ distributions can be generated as follows: the $K_w -$ normal distribution is obtained by taking $G(x)$ in (1.4) to be the normal cumulative function. Analogously, the $K_w -$ Weibull (Cordeiro et al. (2010)), General results for the Kumaraswamy- G distribution (Nadarajah et al.(2011)). $K_w -$ generalized gamma (Pascoa et al.(2011)), $Kw -$ Birnbaum-Saunders (Saulo et al. (2012)) and $K_w -$ Gumbel (Cordeiro et al. (2011)) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, Elbatal (2013) introduced the kumaraswamy generalized linear failure rate distribution, among several others. Hence, each new $Kw - G$ distribution can be generated from a specified G distribution.

A physical interpretation of the $K_w - G$ distribution given by (1.5) and (1.6) (for a and b positive integers) is as follows.

Suppose a system is made of b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, X_{j2}, \dots, X_{ja}$ denote the life times of the subcomponents within the j_{th} component, $j = 1, \dots, b$ with common (cdf) G . Let X_j denote the lifetime of the j_{th} component, $j = 1, \dots, b$ and let X denote the lifetime of the entire system. Then the (cdf) of X is given by

$$\begin{aligned}
 P(X \leq x) &= 1 - P(X_1 > x, X_2 > x, \dots, X_b > x) \\
 &= 1 - [P(X_1 > x)]^b = 1 - \{1 - P(X_1 \leq x)\}^b \\
 &= 1 - \{1 - P(X_{11} \leq x, X_{12} \leq x, \dots, X_{1a} \leq x)\}^b \\
 &= 1 - \{1 - P[X_{11} \leq x]^a\}^b = 1 - \{1 - G^a(x)\}^b.
 \end{aligned}$$

So, it follows that the $K_w - G$ distribution given by (1.5) and (1.6) is precisely the time to failure distribution of the entire system. The rest of the article is organized as follows. In Section 2, we define the Kumaraswamy Quasi Lindley distribution, the expansion for the cumulative and density functions of the KQL distribution and some special cases. Quantile function, moments, moment generating function are discussed in Section 3. In Section 4 included the distribution of the order statistics. Finally, maximum likelihood estimation is performed in Section 5.

2. Kumaraswamy Quasi Lindley Distribution

In this section, we introduce the four – parameter Kumaraswamy Quasi Lindley (KQL) distribution. By taking $G(x)$ in Equation (1.2) to be the cdf of quasi lindley (QL) distribution. Using (1.2) in (1.5), the cdf of the (KQL) distribution can be written as

$$F_{KQL}(x, \theta, \alpha, a, b) = 1 - \left\{ 1 - \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right]^a \right\}^b \quad (2.1)$$

The corresponding pdf and hazard rate function respectively,

$$\begin{aligned}
 f_{KQL}(x, \theta, \alpha, a, b) &= ab \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \\
 &\quad \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right]^{a-1} \\
 &\quad \times \left\{ 1 - \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right]^a \right\}^{b-1}.
 \end{aligned} \quad (2.2)$$

And

$$\begin{aligned}
 h(x, \theta, \alpha, a, b) &= \frac{f(x, \theta, \alpha, a, b)}{F(x, \theta, \alpha, a, b)} \\
 &= \frac{ab \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right]^{a-1}}{1 - \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right]^a}.
 \end{aligned} \quad (2.3)$$

Figure 1 and Figure 2 provide the pdf , cdf and the failure rate functions of KQLD (θ, α, a, b) for different parameter values.

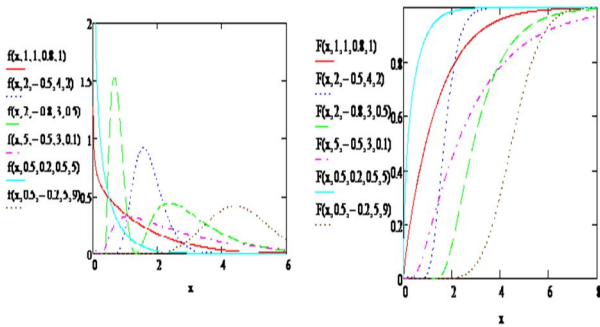


Figure1. pdf, and cdf of KGLFRD(a, b, θ, α)

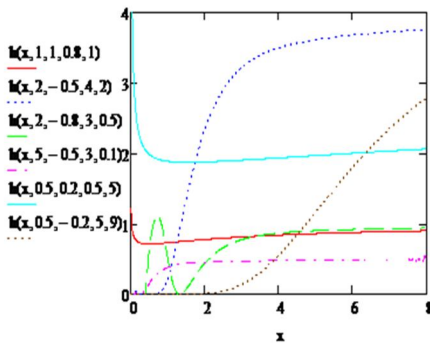


Figure2. Failure rate functions of KQLD for different parameter values.

Special Cases of the (KQL) Distribution

The Kumaraswamy quasi Lindley is very flexible model that approaches to different distributions when its parameters are changed. The *KQL* distribution contains as special- models the following well known distributions. If X is a random variable with cdf (2.1), then we have the following cases.

- 1- If $a = b = 1$, then Equation (2.1) gives Quasi Lindley distribution which is introduced by Rama. S and A. Mishra (2013).
- 2- If $\alpha = \theta$ we get the kumaraswamy Lindley distribution.
- 3- If $\alpha = 0$ we get the kumaraswamy gamma distribution with parameters $(2, \theta)$.
- 4- If $a = b = 1$, and $\alpha = \theta$ we get the Lindley distribution.
- 5- If $a = b = 1$, and $\alpha = \theta$ we get the gamma distribution with parameters $(2, \theta)$.

2.1. Expansion for the cumulative and density functions.

In this subsection we present some representations of cdf, pdf of Kumaraswamy quasi Lindley distribution. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z| < 1$, then

$$(1 - z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i, \quad (2.4)$$

The equation (2.2) becomes

$$f_{KQL}(x, \theta, \alpha, a, b) = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} ab \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta x} \times \left\{ 1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] \right\}^{a(j+1)-1} \quad (2.5)$$

Now using (2.4) in the last term of (2.5), we obtain

$$f_{KQL}(x, \theta, \alpha, a, b) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{b-1}{j} \binom{a(j+1)-1}{k} ab \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta(k+1)x} \left[1 + \frac{\theta x}{\alpha + 1} \right]^k = W_{jk} ab \frac{\theta}{\alpha + 1} (\alpha + \theta x) e^{-\theta(k+1)x} \left[1 + \frac{\theta x}{\alpha + 1} \right]^k \quad (2.6)$$

where

$$w_{j,k} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{b-1}{j} \binom{a(j+1)-1}{k}$$

3. Statistical Properties

This section is devoted to studying statistical properties of the *KQL* distribution, specifically quantile function, moments, and moment generating function.

3.1. Quantile Function

The *KQL* quantile function, say $Q(u) = F^{-1}(u)$, is straightforward to be computed by inverting (2.1), we have

$$e^{-\theta x} \left[1 + \frac{\theta x}{\alpha + 1} \right] = 1 - \left\{ 1 - (1 - q)^{\frac{1}{b}} \right\}^{\frac{1}{a}} \quad (3.1)$$

we can easily generate X by taking u as a uniform random variable in $(0,1)$.

3.2. Moments

In this subsection we discuss the r th non-central moment for *KQL* distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has *KQL*(x, ϕ), $\phi = (\alpha, \theta, a, b)$ then the r th non-central moment of X is given by the following

$$\mu_r'(x) = E(X^r) = w_{i,j,k} \left[\frac{\alpha \Gamma(r+i+1)}{(\theta(k+1))^{r+i+1}} + \frac{\theta \Gamma(r+i+2)}{(\theta(k+1))^{r+i+2}} \right] \quad (3.2)$$

Where

$$w_{i,j,k} = \sum_{i,j,k} \left(\frac{\theta}{\alpha+1}\right)^i (-1)^{j+k} \binom{b-1}{j} \binom{c(a+j)-1}{k} \binom{k}{i}$$

Proof:

Let X be a random variable with density function (2.2). The r_{th} non-central moment of the KQL distribution is given by

$$\begin{aligned} \mu_r'(x) &= E(X^r) = \int_0^\infty x^r f(x, \phi) dx \\ &= \frac{ab\theta}{\alpha+1} \int_0^\infty x^r (\alpha + \theta x) e^{-\theta x} \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right]^{a-1} \\ &\quad \times \left[1 - \left\{ 1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right\}^a \right]^{b-1} dx \quad (3.3) \end{aligned}$$

using the fact that

$$\begin{aligned} \left[1 - \left\{ 1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right\}^a \right]^{b-1} &= \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \\ &\quad \times \left\{ 1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right\}^{aj} \quad (3.4) \end{aligned}$$

then

$$\begin{aligned} \mu_r'(x) &= \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \frac{ab\theta}{\alpha+1} \int_0^\infty x^r (\alpha + \theta x) e^{-\theta x} \\ &\quad \times \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right]^{a(j+1)-1} dx. \end{aligned}$$

a gain

$$\begin{aligned} \left[1 - e^{-\theta x} \left[1 + \frac{\theta x}{\alpha+1} \right] \right]^{a(j+1)-1} &= \sum_{k=0}^{\infty} (-1)^k \binom{a(j+1)-1}{k} e^{-\theta k x} \\ &\quad \times \left[1 + \frac{\theta x}{\alpha+1} \right]^k \quad (3.5) \end{aligned}$$

therefore

$$\begin{aligned} \mu_r'(x) &= \sum_{j,k} (-1)^{j+k} \binom{b-1}{j} \binom{a(j+1)-1}{k} \frac{ab\theta}{\alpha+1} \int_0^\infty x^r (\alpha + \theta x) \\ &\quad \times e^{-\theta(k+1)x} \left[1 + \frac{\theta x}{\alpha+1} \right]^k dx \quad (3.6) \end{aligned}$$

also by using binomial expansion, we get

$$\left[1 + \frac{\theta x}{\alpha + 1}\right]^k = \sum_{i=0}^k \binom{k}{i} \left(\frac{\theta}{\alpha + 1}\right)^i x^i$$

Now

$$\begin{aligned} \mu_r(x) &= w_{i,j,k} \int_0^\infty x^{r+i} (\alpha + \theta x) e^{-\theta(k+1)x} dx \\ &= w_{i,j,k} \left[\alpha \int_0^\infty x^{r+i} e^{-\theta(k+1)x} dx + \theta \int_0^\infty x^{r+i+1} e^{-\theta(k+1)x} dx \right] \\ &= w_{i,j,k} \left[\frac{\alpha \Gamma(r+i+1)}{(\theta(k+1))^{r+i+1}} + \frac{\theta \Gamma(r+i+2)}{(\theta(k+1))^{r+i+2}} \right] \end{aligned} \quad (3.7)$$

Where

$$w_{i,j,k} = \sum_{j,k} \sum_{i=0}^k ab \left(\frac{\theta}{\alpha + 1}\right)^{i+1} (-1)^{j+k} \binom{b-1}{j} \binom{a(j+1)-1}{k} \binom{k}{i}$$

Which completes the proof .

Based on the first four moments of the *KQL* distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the *KQL* distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{\frac{3}{2}}},$$

and

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}.$$

Moment Generating function

In this subsection we derived the moment generating function of *KQL* distribution.

Theorem (3.2): If X has *KQL* distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = w_{i,j,k} \left[\frac{\alpha \Gamma(i+1)}{(\theta(k+1) - t)^{i+1}} + \frac{\theta \Gamma(i+2)}{(\theta(k+1) - t)^{i+2}} \right]. \quad (3.8)$$

Proof.

We start with the well known definition of the moment generating function given by

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} f_{KQL}(x, \phi) dx \\ &= w_{i,j,k} \int_0^\infty x^i (\alpha + \theta x) e^{-x(\theta(k+1)-t)} dx \\ &= w_{i,j,k} \left[\frac{\alpha \Gamma(i+1)}{(\theta(k+1) - t)^{i+1}} + \frac{\theta \Gamma(i+2)}{(\theta(k+1) - t)^{i+2}} \right]. \end{aligned} \quad (3.9)$$

This completes the proof.

4. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the KQL distribution from complete samples only. Let X_1, X_2, \dots, X_n be a random sample of size n from $KQL(x, \phi)$. The likelihood function for the vector of parameters $\phi = (\alpha, \theta, a, b)$ can be written as

$$L(x_{(i)}, \Phi) = \prod_{i=1}^n f(x_{(i)}, \phi)$$

$$= \left(\frac{ab\theta}{\alpha + 1} \right)^n \prod_{i=1}^n (\alpha + \theta x_{(i)}) e^{-\theta \sum_{i=1}^n x_{(i)}} \prod_{i=1}^n \left[1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right]^{a-1} \times \prod_{i=1}^n \left[1 - \left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\}^a \right]^{b-1}. \quad (4.1)$$

Taking the log-likelihood function for the vector of parameters $\phi = (\alpha, \theta, a, b)$ we get $\log L = n \log a + n \log b + n \log \theta - n \log(1 + \alpha)$

$$+ \sum_{i=1}^n \log(\alpha + \theta x_{(i)}) - \theta \sum_{i=1}^n x_{(i)} + (a-1) \sum_{i=1}^n \log \left[1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right] + (b-1) \sum_{i=1}^n \log \left[1 - \left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\}^a \right]. \quad (4.2)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.2). The components of the score vector are given by

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^n \log \left[1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right] - (b-1) \sum_{i=1}^n \frac{\left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\}^a}{\left[1 - \left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\}^a \right]} \times \log \left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\} \quad (4.3)$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^n \log \left[1 - \left\{ 1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha + 1} \right] \right\}^a \right], \quad (4.4)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^n \frac{x_{(i)}}{(\alpha + \theta x_{(i)})} - \sum_{i=1}^n x_{(i)} + \\ & (a-1) \sum_{i=1}^n \frac{x_{(i)} (\alpha + \theta x_{(i)}) (e^{-\theta x_{(i)}})}{\left[1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]\right]} \\ & + a(b-1) \sum_{i=1}^n \frac{x_{(i)} (\alpha + \theta x_{(i)}) (e^{-\theta x_{(i)}})}{\left[1 - \left\{1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^a\right]} \\ & \times \left\{1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{a-1}, \end{aligned} \quad (4.5)$$

And

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \frac{-n}{\alpha+1} - \sum_{i=1}^n \frac{1}{(\alpha + \theta x_{(i)})} + (a-1) \\ & \sum_{i=1}^n \frac{\theta}{(\alpha+1)^2} x_{(i)} e^{-\theta x_{(i)}} \\ & \left[\frac{1}{1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]} - a(b-1) \right. \\ & \left. \sum_{i=1}^n \frac{\theta x_{(i)} e^{-\theta x_{(i)}} \left\{1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{a-1}}{(\alpha+1)^2 \left[1 - \left\{1 - e^{-\theta x_{(i)}} \left[1 + \frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^a\right]}, \end{aligned} \quad (4.6)$$

Where $\psi(\cdot)$ is the digamma function. We can find the estimates of

the unknown parameters by maximum likelihood method by setting these above non-linear equations (4.3) - (4.6) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

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