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Kumaraswamy Generalized Linear Failure Rate Distribution

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Abstract

Motivated by the recent work of Cordeiro and Castro (2011), we study the Kumaraswamy Generalized linear failure rate (KGLFR).We derive some mathematical properties of the (KGLFR) including moments, moment generating function and quantile function. We provide explicit expressions for the density function of the order statistics and their moments. In addition, the method of maximum likelihood and least squares and weighted least squares estimators are discuss for estimating the model parameters.

Keywords: Kumaraswamy distribution; Hazard function;Generalized Linear failure rate distribution; Maximum likelihood estimation; Moments.

1. Introduction and Motivation

One of the important research directions for statisticians is to look for suitable distributions that have some specific properties, which can be used to describe the lifetimes of some devices. In analyzing lifetime data one often uses the exponential, Rayleigh, linear failure rate or generalized exponential distributions. It is well known that exponential can have only constant hazard function whereas Rayleigh, linear failure rate and generalized exponential distribution can have only monotone (increasing in case of Rayleigh or linear failure rate and increasing/ decreasing in case of generalized exponential distribution) hazard functions. Unfortunately, in practice often one needs to consider non-monotonic function such as bathtub shaped hazard function also, see, for example, Lai et al. (2001). The linear exponential distribution is also known as the Linear Failure Rate distribution , having exponential and Rayleigh distributions as special cases, is a very well-known distribution for modeling lifetime data in reliability and medical studies. It is also models phenomena with increasing failure rate. However, the LE distribution does not provide a reasonable parametric fit for modeling phenomenon with decreasing, non linear increasing, or non-monotone failure rates such as the bathtub shape, which are common in firm ware reliability modeling, biological studies, see Lai et al. (2001) and Zhang et al. (2005).

A random variable X is said to have the linear failure rate distribution with two parameters λ and θ , if it has the cumulative distribution function

$$F_{X|(\lambda,\theta)}(x) = 1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}, x > 0, \lambda, \theta > 0, \quad (1.1)$$

and the corresponding probability density function (pdf) is given by

$$f_{X|(\lambda,\theta)}(x) = (\lambda + \theta x)e^{-(\lambda x + \frac{\theta}{2}x^2)}, x > 0, \lambda, \theta > 0. \quad (1.2)$$

It is easily observed that the exponential distribution (ED) and the Rayleigh distribution (RD) can be obtained from LFRD (λ,θ) by putting $\theta = 0$ and $\lambda = 0$ respectively. Moreover, the probability density function (pdf) of the LFRD (λ,θ) can be decreasing or unimodal but the failure rate function is either constant or increasing only. See for example Bain (1974), Sen and Bhattacharya (1995), Lin et al. (2006), Ghitany and Kotz (2007) and the references cited their in this connection. Gupta and Kundu (1999) has been introduced and studied quite extensively the generalized exponential (GE) distribution with the parameters $\lambda > 0$ and $\alpha > 0$, where cumulative distribution function (cdf) given by

$$F_{X|(\lambda,\alpha)}(x) = \left(1 - e^{-\lambda x}\right)^{\alpha}, x \geq 0. \quad (1.3)$$

It is observed that the GE($\lambda;\alpha$) can have decreasing or unimodal PDF and monotone (increasing / decreasing) hazard functions, depending on the shape parameter α . A new generalization of the linear failure rate distribution is generalized linear failure rate $(GLFR)$ distribution. This distribution is important since it contains as special sub-models some widely well known distributions. It also provides more flexibility to analyze complex real data sets. (see Sarhan and Kundo (2009)).

A random variable X is said to have the generalized linear failure rate distribution with three parameters λ, θ and α , if it has the cumulative distribution function

$$G_{X|(\lambda,\theta,\alpha)}(x) = \left[1 - e^{-(\lambda x + \frac{\theta}{2}x^2)}\right]^{\alpha}, x > 0, \lambda, \theta, \alpha > 0, \quad (1.4)$$

where (λ,θ) denote the scale parameters and α denotes the shape parameter of the distribution. The corresponding probability density function (pdf) is given by

$$g_{X|(\lambda,\theta,\alpha)}(x)=\alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha-1}. \quad (1.5)$$

The distribution introduced by Kumaraswamy (1980), also refereed to as the minimax distribution, is not very common among statisticians and has been little explored in the literature, nor its relative interchangeability with the beta distribution has been widely appreciated. We use the term "K_w" distribution to denote the Kumaraswamy distribution. The Kumaraswamy (K_w) distribution is not very common among statisticians and has been little explored in the literature. Its cdf is given by

$$F_{X|(a,b)}(x)=1-\left(1-x^a\right)^b, 0<x<1, \quad (1.6)$$

where $a>0$ and $b>0$ are shape parameters, and the probability density function

$$F_{X|(a,b)}(x)=abx^{a-1}\left(1-x^a\right)^{b-1}, \quad (1.7)$$

which can be unimodal, increasing, decreasing or constant, depending on the parameter values. It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones (2009) explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and K_w distributions. However, the beta distribution has the following advantages over the K_w distribution: simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, we combine the works of Kumaraswamy (1980) and Cordeiro and Castro (2011) to derive some mathematical properties of a new model, called the Kumaraswamy generalized linear failure rate (K_w-GLFR) distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F_{X|(a,b)}(x)=1-\left[1-G(x)^a\right]^b \quad (1.8)$$

where $a>0$ and $b>0$ are two additional shape parameters .The $Kw-G$ distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f_{X|(a,b)}(x) = abg(x)G(x)^{a-1}\left[1-G(x)^{a}\right]^{b-1} \qquad (1.9)$$

The density family (1.9) has many of the same properties of the class of beta- G distributions (see Eugene et al. (2002)), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta $-G$ family of distributions, special $Kw-G$ distributions can be generated as follows: the K_w- normal distribution is obtained by taking $G(x)$ in (1.4) to be the normal cumulative function. Analogously, the K_w- Weibull (Cordeiro et al. (2010)), General results for the Kumaraswamy-G distribution (Nadarajah et al. (2012)). K_w - generalized gamma (Pascoa et al. (2011)), $Kw-$Birnbaum-Saunders (Saulo et al. (2012)), and K_w- Gumbel (Cordeiro et al. (2012)) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new $Kw-G$ distribution can be generated from a specified G distribution.

A physical interpretation of the $K_w - G$ distribution given by (1.5) and (1.6) (for a and b positive integers) is as follows. Suppose a system is made of b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, X_{j2}, ..., X_{ja}$ denote the life times of the subcomponents with in the j_{th} component, $j=1,...,b$ with common (cdf) G . Let X_j denote the lifetime of the j_{th} component, $j=1,...,b$ and let X denote the lifetime of the entire system. Then the (cdf) of X is given by

$$P(X \le x) = 1 - P(X_1 > x, X_2 > x, ..., X_b > x)$$

$$= 1 - \left[P(X_1 > x)\right]^b = 1 - \left\{1 - P(X_1 \le x\right\}^b$$

$$= 1 - \left\{1 - P(X_{11} \le x, X_{12} \le x, ..., X_{1a} \le x\right\}^b$$

$$= 1 - \left\{1 - P\left[X_{11} \le x\right]^a\right\}^b = 1 - \left\{1 - G^a(x)\right\}^b .$$

So, it follows that the $K_w - G$ distribution given by (1.8) and (1.9) is precisely the time to failure distribution of the entire system.

The rest of the article is organized as follows. In Section 2, we define the Kumaraswamy generalized linear failure rate distribution , the expansion for the cumulative and density functions of the *KGLFR* distribution and some special cases. Quantile function , median, moments, moment generating function are discussed in Section 3. In Section 4 included the distribution of the order statistics . Least squares and weighted least squares estimators introduced in Section 5. Finally, maximum likelihood estimation is performed in Section 6.

2. Kumaraswamy Generalized Linear Failure Rate Distribution

In this section ,we propose the Kumaraswamy generalized linear failure rate $(KGLFR)$ distribution and provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research. The linear exponential distribution represents only a special case of the Kumaraswamy generalized linear failure rate distribution. By taking the cdf

$$G_{X|(\lambda,\theta,\alpha)}(x)=\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha},x>0,\lambda,\theta,\alpha>0,$$

of generalized linear failure rate, the cdf and pdf of the $(KGLFR)$ distribution are obtained from Eqs.(1.8) and (1.9) as

$$F_{KGLFR}(x,a,b,\lambda,\theta,\alpha)=1-\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a}\right]^{b}, \qquad (2.1)$$

and

$$f_{KGLFR}(x,a,b,\lambda,\theta,\alpha)=\left\{ab\alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{\alpha a-1}\right.$$

$$\left.\times\left[1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{\alpha a}\right]^{b-1}\right\}. \qquad (2.2)$$

The associated hazard (failure) rate function (hrf) is

$$h_{KGLFR}(x,a,b,\lambda,\theta,\alpha)=\frac{f_{KGLFR}(x,a,b,\lambda,\theta,\alpha)}{1-F_{KGLFR}(x,a,b,\lambda,\theta,\alpha)}$$

$$=\frac{ab\alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{\alpha a-1}}{1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{\alpha a}}.(2.3)$$

Figure 1 and Figure 2 provide the pdf , cdf and the failure rate functions of KGLFRD $(a, b, \lambda, \theta, \alpha)$ for different parameter values.

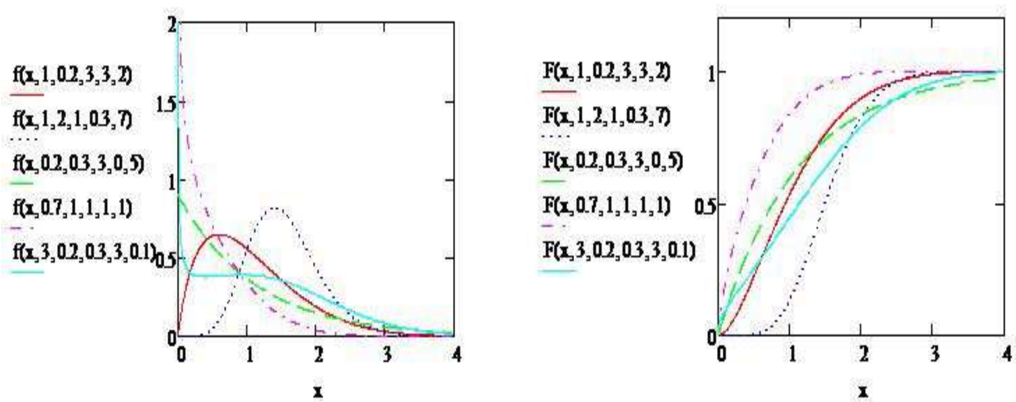

Figure1.The pdf, and cdf of KGLFRD $(a, b, \lambda, \theta, \alpha)$

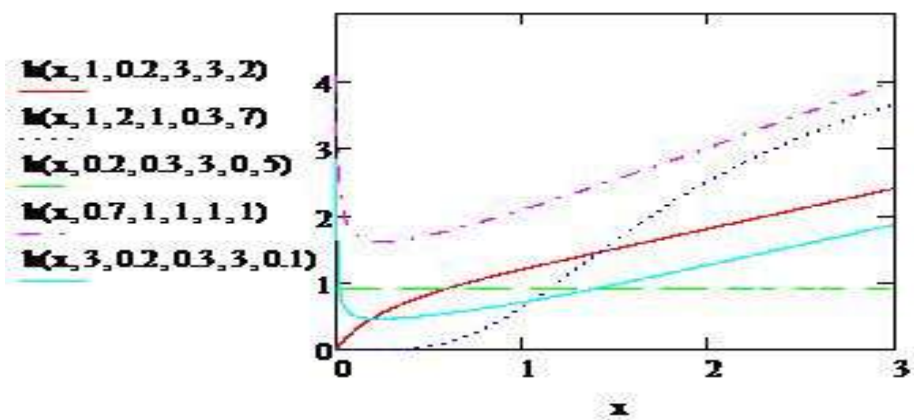

Figure2.The hazard rate of KGLFRD $(a, b, \lambda, \theta, \alpha)$

The Kumaraswamy generalized linear failure rate is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the Kumaraswamy generalized linear failure rate is explained in Table (1). The subject distribution includes as special cases the generalized linear failure rate (GLFR), the generalized exponential (GE), Linear failure rate (LFR), generalized Rayleigh (GR) and kumaraswamy generalized exponential distribution (KGE) and kumaraswamy generalized Rayleigh (KGR) distribution.

Table (1): Some recent sub-models from the KGLFR $(a,b,\lambda,\theta,\alpha)$ distribution.

Model	a	b	λ	θ	α	*CDF*
KLFR	–	–	–	–	1	$1-\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^a\right]^b$
KGE	–	–	–	0	–	$1-\left[1-\left[1-e^{-\lambda x}\right]^{\alpha a}\right]^b$
KGR	–	–	0	–	–	$1-\left[1-\left[1-e^{-\frac{\theta}{2}x^2)}\right]^{\alpha a}\right]^b$
KE	–	–	–	0	1	$1-\left[1-\left[1-e^{-\lambda x}\right]^a\right]^b$
KR	–	–	0	–	1	$1-\left[1-\left[1-e^{-\frac{\theta}{2}x^2)}\right]^a\right]^b$
GLFR	1	1	–	–	–	$\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^\alpha$
LFR	1	1	–	–	1	$\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]$
GE	1	1	–	–	–	$\left[1-e^{-\lambda x}\right]^\alpha$
GR	1	1	0	–	–	$\left[1-e^{-\frac{\theta}{2}x^2)}\right]^\alpha$
E	1	1	–	0	1	$\left[1-e^{-\lambda x}\right]$
R	1	1	0	–	1	$\left[1-e^{-\frac{\theta}{2}x^2)}\right]$

2.1 Expansion for the cumulative and density functions

In this subsection we present some representations of cdf, pdf of Kumaraswamy generalized linear failure rate. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z|<1$, then

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}z^i. \qquad (2.4)$$

Equation (2.1) becomes

$$F(x,\lambda,\alpha,\theta,a,b)=1-\sum_{j=0}^{\infty}(-1)^j\binom{b}{j}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a j},$$

also using the power series of (2.4) the equation (2.2) becomes

$$f_{KGLFR}(x,a,b,\lambda,\theta,\alpha)=\sum_{j=0}^{\infty}(-1)^{j}\binom{b-1}{j}ab\alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^2)}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right)^{\alpha a(j+1)-1}, \qquad (2.5)$$

Now using (2.4) in the last term of (2.5), we obtain

$$f_{KGLFR}(x,a,b,\lambda,\theta,\alpha)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{b}{j}\binom{\alpha a(j+1)-1}{k}ab\alpha(\lambda+\theta x)e^{-k(\lambda x+\frac{\theta}{2}x^2)}e^{-(\lambda x+\frac{\theta}{2}x^2)}$$

$$=w_{j,k}ab\alpha(\lambda+\theta x)e^{-\lambda(k+1)x}\ \ e^{-\frac{\theta}{2}(k+1)x^2} \qquad (2.6)$$

where

$$w_{j,k}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{b}{j}\binom{\alpha a(j+1)-1}{k} \qquad (2.7)$$

3. Statistical Properties

In this section we studied the statistical properties of the $(KGLFR)$ distribution, specifically quantile function , moments and moment generating function.

3.1 Quantile and Median

Starting with the well known definition of the $100q-th$ quantile, which is simply the solution of the following equation, with respect to x_q , $0<q<1$,

$$q=P(X\le x_q)=F(x_q)$$

$$=1-\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a}\right]^{b}$$

$$-\lambda x-\frac{\theta}{2}x^2=\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}^{\frac{1}{\alpha a}},$$

which finally produces the following equation

$$\frac{\theta}{2}x_q^2+\lambda x_q+\frac{1}{\alpha a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}=0. \qquad (3.1)$$

Solving equation (3.1) with respect to x_q , we get

$$x_q=\frac{-\lambda\pm\sqrt{\lambda^2-\frac{2\theta}{\alpha a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}}}{\theta}$$

Since x_q is positive, then

$$x_q=\frac{-\lambda+\sqrt{\lambda^2-\frac{2\theta}{\alpha a}\ln\left\{1-\left[1-(1-q)^{\frac{1}{b}}\right]\right\}}}{\theta}, \qquad (3.2)$$

which completes the proof .

The median can be derived from (3.2) be setting $q=\frac{1}{2}$. That is, the median is given by the following relation

$$M(X)=\frac{-\lambda+\sqrt{\lambda^2-\frac{2\theta}{\alpha a}\ln\left\{1-\left[1-(\frac{1}{2})^{\frac{1}{b}}\right]\right\}}}{\theta}$$

3.2 The Moments

In this subsection, we derive the r_{th} moments and moment generating function $(M_x(t))$ of the $KGLFR(\Phi)$ where $\Phi=(a,b,\lambda,\theta,\alpha)$.

Lemma (3.1): If X has $KGLFR(\Phi)$, then the r_{th} moment of X, $r=1,2,...$. has the following form:

$$\mu_r^{'}=\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{j+k+m}\frac{\theta^m(k+1)^m}{2^m\ m!}\binom{b}{j}\binom{\alpha a(j+1)-1}{k}\right.$$

$$\left.\times\left[\frac{\lambda\Gamma(r+2m+1)}{\left(\lambda(k+1)\right)^{r+2m+1}}+\frac{\theta\Gamma(r+2m+2)}{\left(\left(\lambda(k+1)\right)^{r+2m+2}\right)}\right]\right\}. \qquad (3.3)$$

Proof:

We start with the well known definition of the r_{th} moment of the random variable X with pdf $f(x)$ given by

$$\mu_r^{'} = E(X^r) = \int_0^\infty x^r f_{KGLFR}(x,\Phi)dx$$

$$= w_{j,k} ab\alpha \int_0^\infty x^r (\lambda + \theta x) e^{-\lambda(k+1)x}\ e^{-\frac{\theta}{2}(k+1)x^2} dx$$

Since $0 < e^{-\frac{\theta}{2}(k+1)x^2} < 1$ for $x > 0$, then by using the binomial series expansion of $e^{-\frac{\theta}{2}(k+1)x^2}$ given by

$$e^{-\frac{\theta}{2}(k+1)x^2} = \sum_{m=0}^{\infty} \frac{\left(-\frac{\theta}{2}(k+1)x^2\right)^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\ \theta^m (k+1)^m}{2^m\ m!} x^{2m} \qquad (3.4)$$

Thus

$$\mu_r^{/} = w_{j,k} ab\alpha \sum_{m=0}^{\infty} \frac{(-1)^m\ \theta^m (k+1)^m}{2^m\ m!} \int_0^\infty x^{r+2m} (\lambda + \theta x) e^{-\lambda(k+1)x} dx$$

$$= w_{j,k,m} ab\alpha \left[\int_0^\infty \lambda x^{r+2m} e^{-\lambda(k+1)x} dx + \theta \int_0^\infty x^{r+2m+1} e^{-\lambda(k+1)x} dx \right]$$

$$= w_{j,k,m} ab\alpha \left[\frac{\lambda \Gamma(r+2m+1)}{\left(\lambda(k+1)\right)^{r+2m+1}} + \frac{\theta \Gamma(r+2m+2)}{\left(\left(\lambda(k+1)\right)^{r+2m+2}\right)} \right] \qquad (3.5)$$

where

$$w_{j,k,m} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{j+k+m} \frac{\theta^m (k+1)^m}{2^m\ m!} \binom{b}{j} \binom{\alpha a(j+1)-1}{k} \qquad (3.6)$$

which completes the proof .

Lemma(3.2): If X has $KGLFR(\Phi)$, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = w_{j,k,m} ab\alpha \left[\frac{\lambda\Gamma(2m+1)}{\left[(\lambda(k+1)-t)\right]^{2m+1}} + \frac{\theta\Gamma(2m+2)}{\left[(\lambda(k+1)-t)\right]^{2(m+1)}} \right] \qquad (3.7)$$

Proof.
We start with the well known definition of the moment generating function given by

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f_{KLE}(x,\Phi)dx$$

$$= w_{j,k} ab\alpha \int_0^{\infty} e^{tx} (\lambda + \theta x) e^{-\lambda(k+1)x} \; e^{-\frac{\theta}{2}(k+1)x^2} dx \qquad (3.8)$$

Substituting (3.4) and (3.6) into (3.8), we get

$$M_X(t) = w_{j,k,m} ab\alpha \left[\lambda \int_0^{\infty} x^{2m} e^{-(\lambda(k+1)-t)x} dx + \theta \int_0^{\infty} x^{2m+1} e^{-(\lambda(k+1)-t)x} dx \right]$$

$$= w_{j,k,m} ab\alpha \left[\frac{\lambda\Gamma(2m+1)}{\left[(\lambda(k+1)-t)\right]^{2m+1}} + \frac{\theta\Gamma(2m+2)}{\left[(\lambda(k+1)-t)\right]^{2(m+1)}} \right].$$

which completes the proof .

In the same way, the characteristic function of the $KGLFR$ distribution becomes $\phi_x(t) = M_X(it)$ where $i = \sqrt{-1}$ is the unit imaginary number.

4. Distribution of the Order Statistics

Moments of order statistics play an important role in quality control testing and reliability,where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. In this section, we derive closed form expressions for the pdfs of the r_{th} order statistic of the $(KGLFR)$ distribution, also, the measures of skewness and kurtosis of the distribution of the r_{th} order statistic in a sample of size n for different choices of $n;r$ are presented in this section. Let $X_1, X_2, ..., X_n$ be a simple random sample from $(KGLFR)$ distribution with cdf and pdf given by (2.1) and (2.2), respectively. We now derive an explicit expression for the density function of the r_{th} order statistic $X_{r:n}$, say $f_{r:n}(x)$, in a random sample of size n from the $KGLFR$ distribution. To prove the r_{th} order statistic $X_{r:n}$ we need the following Lemma.

Lemma (4.1):
The probability density function of $X_{r:n}$, $r = 1, 2, ..., n$ of $KGLFR$ distribution is

$$f_{r:n}(x) = \sum_{j=0}^{n-r} d_j(n,r) f_{KLE}(x, a_{r+j}, b_{r+j}, \lambda, \theta) \qquad (4.1)$$

where

$$a_i = ai \text{ and } d_j(n,r) = \frac{n(-1)^j \binom{n-1}{r-1}\binom{n-r}{j}}{r+j}. \qquad (4.2)$$

Proof:
The pdf of $X_{r:n}$, $r = 1, 2, ..., n$ is given by, David (1981)

$$f_{r:n}(x) = \frac{1}{\beta(r, n-r+1)} \left[F(x,\Phi)\right]^{r-1} \left[1 - F(x,\Phi)\right]^{n-r} f(x,\Phi)$$

where $F(x,\Phi)$ and $f(x,\Phi)$ are cdf and pdf given by (2.1) and (2.2), respectively. Since $0 < F(x,\Phi) < 1$ for $x > 0$, by using the binomial series expansion of $\left[1 - F(x,\Phi)\right]^{n-r}$, given by

$$\left[1 - F(x,\Phi)\right]^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[F(x,\Phi\right]^j$$

we have

$$f_{r:n}(x) = \frac{1}{\beta(r, n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \left[F(x)\right]^{r+j-1} f(x) \qquad (4.3)$$

Substituting (2.1) and (2.2) into (4.3), we get

$$f_{r\,:\,n}(x)=\sum_{j=0}^{n-r}d_j(n,r)\,f_{KLE}(x,a_{r+j},b_{r+j},\lambda,\theta,\alpha). \qquad (4.4)$$

The coefficients $d_j(n,r),\ j=1,2,...,n-r$ do not depend on $a,b,\lambda,\theta,\alpha$. Thus $f_{r\,:\,n}(x)$ is the weighted average of the *KGLFR* distribution with different shape parameters.

Theorem (4.2):
The k_{th} moment of order statistic $X_{r\,:\,n}$ is

$$\mu_{r\,:\,n}^{(k)}=\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n-r}d_j(n,r)(-1)^{i+j+k}\binom{b_{r+j}-1}{l}\binom{a_{r+j}(l+1)-1}{k}$$

$$\times\left[\frac{\lambda\Gamma(2i+k+1)}{\left[\lambda(j+1)\right]^{2i+k+1}}+\frac{\theta\Gamma(2i+k+2)}{\left[\lambda(k+1)\right]^{2i+k+2}}\right]. \qquad (4.5)$$

Proof: The general definition of the k_{th} moment of order statistic $X_{r\,:\,n}$ is

$$\mu_{r\,:\,n}^{(k)}=\int_0^{\infty}x^k f_{r\,:\,n}(x,a,b,\lambda,\theta,\alpha)dx. \qquad (4.6)$$

Substituting from (4.4) into (4.6), one gets

$$\mu_{r\,:\,n}^{(k)}=\sum_{j=0}^{n-r}d_j(n,r)\int_0^{\infty}x^k f(x,a_{r+j},b_{r+j},\lambda,\theta,\alpha)dx. \qquad (4.7)$$

Since the integral in (4.7) is the k_{th} moment of $KGLFR(\lambda,\theta,\alpha,a_{r+j},b_{r+j})$, then from (4.7) with the Lemma (4.1) we get (4.5) which completes the proof.

Based on the results given in theorem (4.1), the measures of skewness and kurtosis of the distribution of the r_{th} order statistic can be evaluated from the following expressions

$$A(\Phi)=\frac{\mu_3(\theta)-3\mu_1(\theta)\mu_2(\theta)+2\mu_1^3(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^{\frac{3}{2}}},$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}.$$

5. Least Squares and Weighted Least Squares Estimators

In this section we provide the regression based method estimators of the unknown parameters of the Kumaraswamy generalized linear failure rate, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1,...,Y_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $Y_{(i)}$; $i=1,2,...,n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n , we have

$$E\left(G(Y_{(j)})\right)=\frac{j}{n+1},V\left(G(Y_{(j)})\right)=\frac{j(n-j+1)}{(n+1)^2(n+2)}$$

$$\text{and } Cov\left(G(Y_{(j)}),G(Y_{(k)})\right)=\frac{j(n-k+1)}{(n+1)^2(n+2)};\text{for } j<k,$$

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators) . Obtain the estimators by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2, \qquad (5.1)$$

with respect to the unknown parameters. Therefore in case of $KGLFR$ distribution the least squares estimators of $\lambda,\theta,\alpha,$ a and b , say λ_{LSE}, $\widehat{\theta}_{LSE},\alpha_{LSE},\widehat{a}_{LSE}$ and $\widehat{b}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left[1-\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a}\right]^{b}-\frac{j}{n+1}\right]^2$$

with respect to $\lambda, \theta, \alpha,$ a and b .

Method 2 (Weighted Least Squares Estimators). The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left(G(Y_{(j)} - \frac{j}{n+1} \right)^2, \qquad (5.2)$$

with respect to the unknown parameters, where

$$w_j = \frac{1}{V\left(G(Y_{(j)})\right)} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in case of $KGLFR$ distribution the weighted least squares estimators of $\lambda, \theta, \alpha,$ a and b , say λ_{WLSE}, $\widehat{\theta}_{WLSE}, \alpha_{WLSE}, \widehat{a}_{WLSE}$ and $\widehat{b}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left[1-\left[1-\left[1-e^{-(\lambda x+\frac{\theta}{2}x^2)}\right]^{\alpha a}\right]^{b}-\frac{j}{n+1}\right]^2$$

with respect to the unknown parameters only.

6. Estimation and Inference

In this section, we derive the maximum likelihood estimates of the unknown parameters $\Phi = (a, b, \lambda, \theta, \alpha)$ of $KGLFR$ distribution based on a complete sample. Let us assume that we have a simple random sample $X_1, X_2, ..., X_n$ from $KGLFR(a, b, \lambda, \theta, \alpha)$. The likelihood function of this sample is

$$L = \Pi_{i=1}^{n} f(x_i, a, b, \lambda, \theta, \alpha). \qquad (6.1)$$

Substituting from (2.2) into (6.1), we get

$$L=\Pi_{i=1}^{n}\left\{ab\alpha(\lambda+\theta x)e^{-(\lambda x+\frac{\theta}{2}x^{2})}\left(1-e^{-(\lambda x+\frac{\theta}{2}x^{2})}\right)^{\alpha a-1}\right.$$

$$\left.\times\left[1-\left(1-e^{-(\lambda x+\frac{\theta}{2}x^{2})}\right)^{\alpha a}\right]^{b-1}\right\}$$

$$=(ab\alpha)^{n}\Pi_{i=1}^{n}(\lambda+\theta x_i)e^{-\sum_{i=1}^{n}(\lambda x_i+\frac{\theta}{2}x_i^{2})}\Pi_{i=1}^{n}\left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^{2})}\right)^{\alpha a-1}$$

$$\times\Pi_{i=1}^{n}\left[1-\left(1-e^{-(\lambda x_i+\frac{\theta}{2}x_i^{2})}\right)^{\alpha a}\right]^{b-1}. \qquad (6.2)$$

The log-likelihood function for the vector of parameters $\Phi=(a,b,\lambda,\theta,\alpha)$ can be written as

$$\log\ L=n\log a+n\log b+n\log\alpha+\sum_{i=1}^{n}\log(\lambda+\theta x_i)+\sum_{i=1}^{n}z_{(i)}$$

$$+(\alpha a-1)\sum_{i=1}^{n}\log\left(1-z_{(i)}\right)$$

$$+(b-1)\sum_{i=1}^{n}\log\left[1-\left(1-z_{(i)}\right)^{\alpha a}\right] \qquad (6.3)$$

where

$$z_{(i)}=e^{-(\lambda x_i+\frac{\theta}{2}x_i^{2})}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.3). The components of the score vector W (Φ) are given by

$$W_a(\Phi)=\frac{\partial \log\ L}{\partial a}=\frac{n}{a}+\alpha\sum_{i=1}^{n}\log\left(1-z_{(i)}\right)$$

$$-\alpha(b-1)\sum_{i=1}^{n}\frac{(1-z_{(i)})^{\alpha a}\log(1-z_{(i)})}{1-(1-z_{(i)})^{\alpha a}}, \qquad (6.4)$$

$$W_b(\Phi)=\frac{\partial \log\ L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-\left(1-z_{(i)}\right)^{\alpha a}\right], \qquad (6.5)$$

$$W_\lambda(\Phi)=\frac{\partial \log\ L}{\partial \lambda}=\sum_{i=1}^{n}\frac{1}{(\lambda+\theta x_i)}-\sum_{i=1}^{n}x_i+(\alpha a-1)\sum_{i=1}^{n}\frac{x_i z_{(i)}}{1-z_{(i)}}$$

$$+(b-1)\sum_{i=1}^{n}\frac{\alpha a x_i z_{(i)}(1-z_{(i)})^{\alpha a-1}}{1-\left(1-z_{(i)}\right)^{\alpha a}}, \qquad (6.6)$$

$$W_\theta(\Phi)=\frac{\partial \log\ L}{\partial \theta}=\sum_{i=1}^{n}\frac{x_i}{(\lambda+\theta x_i)}-\sum_{i=1}^{n}z_{(i)}\frac{x_i^2}{2}$$

$$+(\alpha a-1)\sum_{i=1}^{n}\frac{x_i^2 z_{(i)}}{2\left(1-z_{(i)}\right)}+\frac{\alpha a(b-1)}{2}\sum_{i=1}^{n}\frac{x_i^2 z_{(i)}(1-z_{(i)})^{\alpha a-1}}{1-(1-z_{(i)})^{\alpha a}}. \qquad (6.7)$$

and

$$W_\alpha(\Phi)=\frac{\partial \log\ L}{\partial \alpha}=\frac{n}{\alpha}+a\sum_{i=1}^{n}\log\left(1-z_{(i)}\right)+a(b-1)\sum_{i=1}^{n}\frac{(1-z_{(i)})^{\alpha a}\log(1-z_{(i)})}{1-(1-z_{(i)})^{\alpha a}} \qquad (6.8)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (5.4)- (5.8) to zero and solve them simultaneously.

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