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*Research Article*

# On Extended Neoteric Ranked Set Sampling Plan: Likelihood Function Derivation and Parameter Estimation

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The extended neoteric ranked set sampling (ENRSS) plan proposed by Taconeli and Cabral has proven to outperform many one stages and two stages ranked set sampling plans when estimating the mean and the variance for different populations. Therefore, in this paper, the likelihood function based on ENRSS is proposed and used for estimation of the parameters of the inverted Nadarajah–Haghighi distribution. An extensive Monte Carlo simulation study is conducted to assess the performance of the proposed likelihood function, and the efficiency of the estimated parameters based on ENRSS is compared with the well-known ranked set sampling (RSS) plan and some of its modifications. These modifications include the extended ranked set sampling (ERSS) plan and the neoteric ranked set sampling (NRSS) plan. The results as foreseeable were very satisfactory and gave similar results to Taconeli and Cabral's 2019 results.

## 1. Introduction

Ranked set sampling (RSS) plans were proposed to provide estimators that are more efficient than those derived under simple random sampling (SRS) plans. RSS plans were first proposed by McIntyre in 1952, to find efficient estimates of the mean pasture yields. These plans assume that there are no errors in ranking the units concerning the variable of interest. In most practical applications, imperfect ranking exists and there will be an efficiency loss in the estimators [1]. To reduce such losses, several modifications to the RSS procedure were proposed. The main purpose was to allow for the achievement of higher statistical efficiency and probably a lower operating effort. The first modification of RSS was the extreme ranked set sampling (ERSS) plan introduced by Samwai et al. [2]. Muttlak [3] proposed the median ranked set sampling (MRSS) plan, Al-Odat and Al-Saleh [4] proposed the moving extreme ranked set sampling (MERSS) plan, Al-Saleh and Al-Omari [5] proposed the multistage ranked set sampling (MSRSS), and others proposed the multistage ranked set sampling (MSRSS). Recently, Zamanzade E, Al-Omari [1] proposed a new ranked set sampling plan based on a dependent scheme, namely, the neoteric ranked set sampling (NRSS) plan which showed relative improvement in the efficiency of the population mean and variance estimates. Moreover, Taconeli and Cabral [6] proposed several modifications to the NRSS plan. One of these plans is the extended neoteric ranked set sampling (ENRSS) plan. They showed that the ENRSS plan is superior to NRSS and other plans. Unlike RSS and ERSS plans, NRSS and ENRSS are classified as dependent RSS plans as the resulting samples have a dependence structure.

In 2020, Sabry and Shabaan proposed the likelihood function of the NRSS plan and used it to estimate the parameters of the inverse Weibull distribution. Chen et al. [7] obtained RSS for efficient estimation of a population proportion. Terpstra and Liudahl [8] constructed concomitant-based rank set sampling proportion estimates. Mahdizadeh [9] discussed entropy-based test of exponentiality in ranked set sampling. Mahdizadeh and Strzalkowska-Kominiak [10] discussed resampling-based inference for a distribution function using censored ranked set samples. Akhter et al. [11] discussed RSS for generalized Bilal distribution. Strzalkowska-Kominiak and Mahdizadeh [12] discussed Kaplan–Meier estimator based on ranked set samples. Aljohani et al. [13] discussed ranked set sampling with an application of modified Kies exponential distribution. Sabry et al. [14] used a hybrid approach to evaluate the performance of some ranked set sampling strategies. Sabry and Almetwally [15] used under-ranked and double-ranked set sampling designs to estimate the parameters of the exponential Pareto distribution. The results showed similar results to Zamanzade and Al-Omari [16] and Taconeli and Cabral [6] results when estimating the population means and variance.

This paper aims to compute the joint order distribution of an ENRSS sample and consequently propose the associated likelihood function. Also, to use the proposed likelihood function to estimate the parameters of the inverted Nadarajah–Haghighi distribution and to conduct Monte Carlo simulations to assess the performance of the ENRSS plan and compare the results with RSS, ERSS, and NRSS plans.

The inverted Nadarajah–Haghighi [INH$(\lambda, \alpha)$] distribution was proposed by Taher and co-authors (2018). The cumulative distribution function (CDF), probability density function (PDF), and quantile function used are as follows:

$$F(x;\lambda,\alpha) = \exp\left\{1 - \left(1 + \lambda x^{-1}\right)^{\alpha}\right\} \quad x > 0, \tag{1}$$

$$f(x;\lambda,\alpha) = \lambda\alpha x^{-2}\left(1 + \lambda x^{-1}\right)^{\alpha-1} \exp\left\{1 - \left(1 + \lambda x^{-1}\right)^{\alpha}\right\}, \tag{2}$$

$$x = \lambda\left[(1 - \ln u)^{1/\alpha} - 1\right]^{-1}, \tag{3}$$

where $\lambda$, $\alpha > 0$, and $u$ have a uniform $U(0, 1)$ distribution. Figure 1 illustrates different PDF plots for the INH distribution.

The remainder of the paper is laid out as follows: the second section is devoted to a brief overview of the various RSS plans discussed in this study. In Section 3, maximum likelihood analysis for the INH distribution is considered for all plans, and in Section 4, an extensive simulation study is conducted and the different plans are compared and the results are reported. Finally, in Section 5, the paper is concluded.

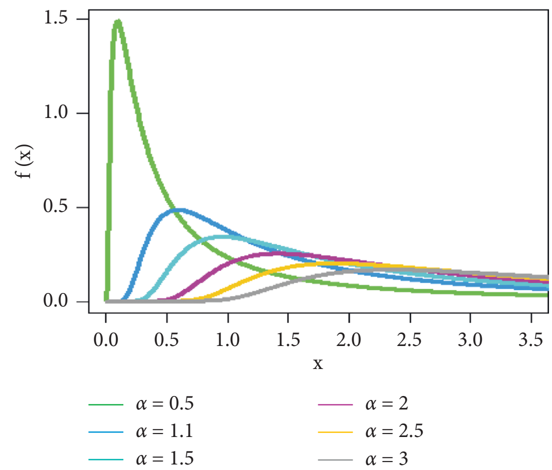


Figure 1: Different PDF plot for INH distribution when scale parameter $\lambda = 1$.

## 2. Ranked Set Sampling Designs

The ranked set sampling plans covered in this study are discussed in this section, with the number of cycles of the RSS plans assumed to be one for simplicity.

*2.1. RSS Design.* The RSS design according to Wolfe [17] can be described as follows:

Step 1: select $m^2$ units randomly from the target population with CDF $F(x;\theta)$ and PDF $f(x;\theta)$

Step 2: allocate the $m^2$ selected units as randomly as possible into $m$ sets, each of size $m$

Step 3: rank the units within each set without yet knowing any values for a variable of interest

Step 4: choose a sample for real quantification by selecting the smallest ranked unit from the first set, the second smallest ranked unit from the second set, and so on until the largest ranked unit from the last set is chosen

Step 5: to get a sample of size $n = mr$, repeat steps 1 through 4 for $r$ cycles

Figure 2 describes the process for choosing an RSS sample from one cycle where $X_{(11)}$ is the lowest observation in the first raw, $X_{(22)}$ is the second-lowest observation from the second raw, and finally the greatest observation from the previous raw is $X_{(mm)}$.

Let $\{y_{is}, i = 1, 2, \ldots, m, s = 1, 2, \ldots, r$ and $-\infty < x_{(i)} < \infty\}$ denote a ranked set sample derived from a distribution with PDF $f(x;\theta)$ and CDF $F(x;\theta)$, where $m$ denotes the set size, $r$ denotes the number of cycles, and $\theta$ is the parameter space. The probability function for this design is as follows:

| | | | |
|---|---|---|---|
| *RSS* matrix | $x_{(11)}$ | $x_{(12)}$ | $x_{(13)}$ |
| | $x_{(21)}$ | $x_{(22)}$ | $x_{(23)}$ |
| | $x_{(31)}$ | $x_{(32)}$ | $x_{(33)}$ |

$\Downarrow$

| | | | |
|---|---|---|---|
| *RSS* sample | $y_1 = x_{(11)}$ | $y_2 = x_{(22)}$ | $y_3 = x_{(33)}$ |

Figure 2: Display of $m^2$ observations in one cycle and the selected RSS sample of size $m = 3$.

$$L_R(\theta; y) = \prod_{s=1}^{r}\prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!} f(y_{is};\theta)[F(y_{is};\theta)]^{i-1} [1 - F(y_{is};\theta)]^{m-i}. \quad (4)$$

*2.2. ERSS Design.* It is the first variation of RSS proposed by Samawi et al. [2] which only uses a maximum or minimum ranked unit from each set to determine the population mean. The following approach is used to estimate based on ERSS.

Step 1: repeat Steps 1 through 3 in the RSS design.

Step 2: the selection mechanism can be altered depending on whether the set size is even or odd. Select the lowest-ranked unit of each set from the first $m/2$ sets and the highest-ranked unit of each set from the other $m/2$ sets if the set size $m$ is even. If the number of units in the set is odd, choose the lowest-ranked unit from the first $m-1/2$ sets, the highest-ranked unit from the second $m-1/2$ sets, and the median from the remaining set.

Step 3: to get a sample of size $n = mr$, repeat the previous procedures $r$ times.

The process for one cycle, as well as the cases of $m = 4$ and $m = 5$, is as shown in Figure 3.

Let $\{z_i, i = 1, 2, \ldots, m\}$ be a ranked set sample (RSS) generated from a distribution with pdf $g(x;\theta)$ and cdf $G(x;\theta)$, where the set size is $m$ and the parameter space is $\theta$. Let $g = m/2$, $h = m-1/2$, and $u_i = m - i + 1$, then the likelihood function of the ERSS sample is given by

*Case 1. m* odd:

$$\begin{aligned} L_{E_o}(\boldsymbol{\theta}; z) &= \left(\prod_{i=1}^{h}\left[g_{1:m}(z_i;\boldsymbol{\theta})\, g_{m:m}\left(z_{u_i};\boldsymbol{\theta}\right)\right]\right)\left(g_{h+1:m}(z_{h+1};\boldsymbol{\theta})\right) \\ &= \prod_{i=1}^{h}\left[m\, g(z_i;\boldsymbol{\theta})[1 - G(z_i,\boldsymbol{\theta})]^{m-1} \times m g\left(z_{u_i};\boldsymbol{\theta}\right)\left[G\left(z_{u_i};\boldsymbol{\theta}\right)\right]^{m-1}\right] \times \frac{m!}{(h!)^2} g(z_{h+1};\boldsymbol{\theta})\, \left(G(z_{h+1};\boldsymbol{\theta})\left(1 - G(z_{h+1};\boldsymbol{\theta})\right)\right)^{h}. \end{aligned} \quad (5)$$

*Case 2. m* even:

$$\begin{aligned} L_{E_e}(\boldsymbol{\theta}; z) &= \prod_{i=1}^{g} g_{1:m}(z_i;\boldsymbol{\theta})\, g_{m:m}\left(z_{u_i};\boldsymbol{\theta}\right), \\ &= \prod_{i=1}^{g}\left(m g(z_i;\boldsymbol{\theta})[1 - G(z_i;\boldsymbol{\theta})]^{m-1}\right)\left(m g\left(z_{u_i};\boldsymbol{\theta}\right)\left[G\left(z_{u_i};\boldsymbol{\theta}\right)\right]^{m-1}\right). \end{aligned} \quad (6)$$

*2.3. Neoteric Ranked Set Sampling (NRSS) Design.* The NRSS sampling plan presented by Zamanzade E and Al-Omari [1] is described in the following steps:

Step 1: select $m^2$ random units from the target population.

Step 2: rank the $m^2$ sample units according to some predetermined criteria.

Step 3: for $i = 1, \ldots, m$, choose the sample unit rated $[(i-1)m + l]^{\text{th}}$ for the final sample. $l = \begin{cases} m + 1/2 & m \text{ is odd} \\ m + 1/2 & m \text{ is even} \end{cases}$ and while $l = \begin{cases} m + 2/2 & i \text{ is odd} \\ m/2 & i \text{ is even} \end{cases}$.

*Even set size*

| | | | |
|---|---|---|---|
| $x_{(11)}$ | $x_{(12)}$ | $x_{(13)}$ | $x_{(14)}$ |
| $x_{(21)}$ | $x_{(22)}$ | $x_{(23)}$ | $x_{(24)}$ |
| $x_{(31)}$ | $x_{(32)}$ | $x_{(33)}$ | $x_{(34)}$ |
| $x_{(41)}$ | $x_{(42)}$ | $x_{(43)}$ | $x_{(44)}$ |

⇓

ERSS sample

| $z_1$ | $z_2$ | $z_3$ | $z_4$ |
|---|---|---|---|
| $= x_{(11)}$ | $= x_{(21)}$ | $= x_{(34)}$ | $= x_{(44)}$ |

(a)

*Odd set size*

| | | | | |
|---|---|---|---|---|
| $x_{(11)}$ | $x_{(12)}$ | $x_{(13)}$ | $x_{(14)}$ | $x_{(15)}$ |
| $x_{(21)}$ | $x_{(22)}$ | $x_{(23)}$ | $x_{(24)}$ | $x_{(25)}$ |
| $x_{(31)}$ | $x_{(32)}$ | $x_{(33)}$ | $x_{(34)}$ | $x_{(35)}$ |
| $x_{(41)}$ | $x_{(42)}$ | $x_{(43)}$ | $x_{(44)}$ | $x_{(45)}$ |
| $x_{(51)}$ | $x_{(52)}$ | $x_{(53)}$ | $x_{(54)}$ | $x_{(55)}$ |

⇓

ERSS sample

| $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|
| $= x_{(11)}$ | $= x_{(12)}$ | $= x_{(33)}$ | $= x_{(45)}$ | $= x_{(55)}$ |

(b)

FIGURE 3: Display of $m^2$ observations in one cycle and the selected ERSS sample of size: (a)$m = 4$ and (b)$m = 5$.

Step 4: to obtain a final sample of size $n = mr$, repeat steps 1–3$r$ times.

Figure 4 displays the step for establishing an NRSS sample in one cycle when $m = 3$

**Lemma 1.** *Let $\{u_i, i = 1, 2, \ldots, n\}$ be a neoteric ranked set sample obtained from a distribution with PDF $h(u;\theta)$ and CDF $H(u;\theta)$, and let $\{u_{(k_i)s}, i = 1, 2, \ldots, m,\ s = 1, 2, \ldots, r\}$ be a random sample of size n from a continuous population, where m is the set size, r is the number of cycles, $\theta$ is the parameter space, and n = mr. Then, the likelihood function of NRSS samples is given by*

$$L(\boldsymbol{\theta}) = \frac{m^2!}{\prod_{i=1}^{m+1}(k_i - k_{i-1} - 1)!}\prod_{s=1}^{r}\prod_{i=1}^{m} h\left(u_{(k_i)s}\right)\prod_{i=1}^{m+1}\left[H\left(u_{(k_i)s}\right) - H\left(u_{(k_{i-1})s}\right)\right]^{(k_i - k_{i-1} - 1)}, \tag{7}$$

*where*

$$k_i = \begin{cases} \dfrac{m+1}{2} + (i-1)m, & m \text{ odd}, \\ \dfrac{m}{2} + (i-1)m, & m \text{ even}, i \text{ even}, \\ \dfrac{m+2}{2} + (i-1)m, & m \text{ even}, i \text{ odd}, \end{cases} \tag{8}$$

*and $k_0 = 0$, $k_{m+1} = m^2 + 1$, and $u_{(k_0)} = -\infty$, $u_{(k_{m+1})} = \infty$*

*Proof.* (see Sabry and Shabaan [18]). □

*2.4. Extended Neoteric Ranked Set Sampling (ENRSS) Plan.* In ENRSS, Taconeli and Cabral's [6] proposal is based on a single ranking stage, where $m^3$ sample units are observed instead $m^2$ sample units as in all one-stage ranked set sampling designs. These $m^3$ sample units are arranged in a single set and ordered utilizing some inexpensive ranking criteria. The units that will make up the final sample must next be chosen from (almost) evenly spaced spots. The ENRSS technique is described in the steps as follows:

Step 1: choose $m^3$ elements from the target population and combine them into a single ranking set.

Step 2: for $i = 1, 2, \ldots, m$, choose the $[m^2 + 1/2 + (i-1)m^2]^{\text{th}}$ ranked unit to compose the final sample if m is odd. If $m$ is even, choose the $[l + (i-1)m^2]^{\text{th}}$ ranked unit, where $l = m^2/2$ represents even $i$ and $l = m^2 + 2/2$ represents odd $i$.

Step 3: to obtain a sample of size $n = mr$, repeat steps 1–2$r$ times.

To pick an ENRSS sample with a set size of $m = 3$, a single set of $m^3 = 27$ sample units should be taken, and the sample units ranked in positions 5, 14, and 23 should be chosen to create the final sample. If you want a sample of set size $m = 4$, use $m^3 = 64$, and the desired ENRSS sample will constitute the ranked units in positions 9, 24, 41, and 56 (see [6]).

**Lemma 2.** *Let $\{v_i, i = 1, 2, \ldots, n\}$ be a random sample of size n from a continuous population and let $\{v_{(i_t)s}, t = 1, 2, \ldots, m, s = 1, 2, \ldots, r\}$ be an extended neoteric ranked set samples drawn from a model with PDF $w(v;\theta)$ and CDF $W(v;\theta)$, where m is the set size, r is the number of cycles, $\theta$ is the parameter space, and n = mr. Then, the likelihood function of an ENRSS sample is given by*

| Random sample | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ | $u_9$ |
|---|---|---|---|---|---|---|---|---|---|

⇓

| Ordered sample | $u_{(1)}$ | $u_{(2)}$ | $u_{(3)}$ | $u_{(4)}$ | $u_{(5)}$ | $u_{(6)}$ | $u_{(7)}$ | $u_{(8)}$ | $u_{(9)}$ |
|---|---|---|---|---|---|---|---|---|---|

⇓

| NRSS sample | $u_{(2)}$ | $u_{(5)}$ | $u_{(8)}$ |
|---|---|---|---|

FIGURE 4: Display of $m^2$ observations in one cycle and the selected NRSS sample of size $m = 3$.

$$L_{EN}(\boldsymbol{\theta}) = \frac{m^3!}{\prod_{i=1}^{m+1}(i_t - i_{t-1} - 1)!}\prod_{s=1}^{r}\prod_{t=1}^{m} w\left(v_{(i_t)s}\right)\prod_{i=1}^{m+1}\left[W\left(v_{(i_t)s}\right) - W\left(v_{(i_{t-1})s}\right)\right]^{(t_i - i_{t-1} - 1)}, \tag{9}$$

*where*

$$i_t = \begin{cases} \dfrac{m^2+1}{2} + (i-1)m^2, & m \text{ odd}, \\ \dfrac{m^2}{2} + (i-1)m^2, & m \text{ even, } i \text{ even}, \\ \dfrac{m^2+2}{2} + (i-1)m^2, & m \text{ even, } i \text{ odd}, \end{cases} \tag{10}$$

*and* $k_0 = 0$, $k_{m+1} = m^3 + 1$, *and* $v_{(k_0)} = -\infty$, $v_{(k_{m+1})} = \infty$. *See Figure 5 which shows* $m^3$ *observations in one cycle and the selected ENRSS sample of size* $m = 3$.

*Proof.* Let $\{x_i, i = 1, 2, \ldots, n\}$ be a random sample from a continuous distribution with order statistics $\{x_{(i)}, i = 1, 2, \ldots, n\}$. **Let** $\left\{x_{(i_j)}, i = 1, 2, \ldots, n\right\}$ be a subset of the order statistics corresponding to $\{x_i\}$. Then, the joint PDF of $\left\{X_{(i_j)}\right\}$ according to [8] and [9] is given by

$$f_{(k)}\left(x_{(i_1)}, x_{(i_2)}, \ldots, x_{(i_k)}\right) = \frac{n!}{\prod_{j=1}^{k+1}\left(i_j - i_{j-1} - 1\right)!}\prod_{j=1}^{k} f\left(x_{(i_j)}\right)\prod_{j=1}^{k+1}\left[F\left(x_{(i_j)}\right) - F\left(x_{(i_{j-1})}\right)\right]^{\left(i_j - i_{j-1} - 1\right)}, \tag{11}$$

where $i_0 = 0$ and $i_{k+1} = n + 1$ (see Figure 6).

Since to get an ENRSS sample $\left\{X_{(i_j)}\right\}$ of size $m$, $m^3$ units are selected and ranked (ordered). Therefore and according to [9], the joint PDF of an ENRSS sample of size $m$ is given by

$$f_{(k)}\left(v_{(i_1)}, v_{(i_2)}, \ldots, v_{(i_m)}\right) = \frac{m^3!}{\prod_{t=1}^{m+1}(i_t - i_{t-1} - 1)!}\prod_{t=1}^{m} w\left(v_{(i_t)}\right)\prod_{t=1}^{m+1}\left[W\left(v_{(i_t)}\right) - W\left(v_{(i_{t-1})}\right)\right]^{\left|i_t - i_{t-1} - 1\right|}. \tag{12}$$

If the ENRSS design is repeated $r$ time to get a sample of size mr, the likelihood function of an ENRSS sample with set size $m$ and number of cycles $r$ will be given by

$$L_{EN}(\boldsymbol{\theta}) = \frac{m^3!}{\prod_{t=1}^{m+1}(i_t - i_{t-1} - 1)!}\prod_{s=1}^{r}\prod_{t=1}^{m} w\left(v_{(i_t)s}\right)\prod_{i=1}^{m+1}\left[W\left(v_{(i_t)s}\right) - W\left(v_{(i_{t-1})s}\right)\right]^{(i_t - i_{t-1} - 1)}, \tag{13}$$

| Ordered sample | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $z_{(1)}$ | $v_{(2)}$ | $v_{(3)}$ | $v_{(4)}$ | $v_{(5)}$ | $v_{(6)}$ | $v_{(7)}$ | $v_{(8)}$ | $v_{(9)}$ |
| $v_{(10)}$ | $v_{(11)}$ | $v_{(12)}$ | $v_{(13)}$ | $v_{(14)}$ | $v_{(15)}$ | $v_{(16)}$ | $v_{(17)}$ | $v_{(18)}$ |
| $v_{(19)}$ | $v_{(20)}$ | $v_{(21)}$ | $v_{(22)}$ | $v_{(23)}$ | $v_{(24)}$ | $v_{(25)}$ | $v_{(26)}$ | $v_{(27)}$ |

⇓

| *ENRSS* sample | | | |
|---|---|---|---|
| | $v_{(5)}$ | $v_{(14)}$ | $v_{(23)}$ |

Figure 5: Display of $m^3$ observations in one cycle and the selected ENRSS sample of size $m = 3$.

where $i_0 = 0, i_{m+1} = m + 1$, $v_{(i_0)} = -\infty, v_{(i_{m+1})} = \infty$ and $i_t$ is defined in (8). This completes the proof. For more elaborations, we can see in Figure 6. □

## 3. Estimation of the Inverted Nadarajah–Haghighi Distribution Parameters

The parameters of INH $(\lambda, \alpha)$ distribution are estimated using the maximum likelihood estimation (MLE) method. The estimation process is taking place when samples are drawn according to SRS, RSS, ERSS, NRSS, and ERSS sampling plans as illustrated in section 2.

*3.1. Estimation Based on SRS.* Let $(x_i, i = 1, 2, \ldots, m)$ be a SRS with CDF and PDF given in (1) and (2), respectively, the likelihood function of the SRS samples from INH $(\lambda, \alpha)$ distribution is

$$\begin{aligned} L(\lambda, \alpha; x) &= \prod_{i=1}^{m} f(x_i; \lambda, \alpha) \\ &= \prod_{i=1}^{m} \lambda \alpha x_i^{-2} \left(1 + \lambda x_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda x_i^{-1}\right)^{\alpha}} \\ &= \lambda^m \alpha^m e^{m-\sum_{i=1}^{m}\left(1+\lambda x_i^{-1}\right)^{\alpha}} \prod_{i=1}^{m} x_i^{-2} \left(1 + \lambda x_i^{-1}\right)^{\alpha-1}. \end{aligned} \tag{14}$$

The log-likelihood function is thus given by

$$\begin{aligned} \ell(\lambda, \alpha) &= m \log \lambda + m \log \alpha + m - \sum_{i=1}^{m} \left(1 + \lambda x_i^{-1}\right)^{\alpha} \\ &\quad + \sum_{i=1}^{m} \log x_i^{-2}, \end{aligned} \tag{15}$$

and the associated likelihood equations are therefore identified as

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} &= \frac{m}{\lambda} - \alpha \sum_{i=1}^{m} x_i^{-1} \left(1 + \lambda x_i^{-1}\right)^{\alpha-1}, \\ \frac{\partial \ell}{\partial \alpha} &= \frac{m}{\alpha} - \sum_{i=1}^{m} \left(1 + \lambda x_i^{-1}\right)^{\alpha} \log\left(1 + \lambda x_i^{-1}\right)^{\alpha} = 0. \end{aligned} \tag{16}$$

*3.2. Estimation Based on RSS.* Let $\{y_{is}, i = 1, 2, \ldots, m; s = 1, 2, \ldots, r\}$ be a ranked set sample with the CDF and PDF provided in (1) and (2), respectively, where $m$ is the set size, $r$ is the number of cycles, and $n = mr$. Assuming $r = 1$ and according to (4), the likelihood function of the RSS samples from INH $(\lambda, \alpha)$ distribution is given by

$$\begin{aligned} L_R(\lambda, \alpha; y) &= \prod_{i=1}^{m} C_i \lambda \alpha y_i^{-2} \left(1 + \lambda y_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}} \left[e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}\right]^{i-1} \left[1 - e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}\right]^{m-i}, \\ &\propto \lambda^m \alpha^m \prod_{i=1}^{m} \left(1 + \lambda y_i^{-1}\right)^{\alpha-1} e^{m(m+1)/2-\sum_{i=1}^{m} i\left(1+\lambda y_i^{-1}\right)^{\alpha}} \prod_{i=1}^{m} \left[1 - e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}\right]^{m-i}, \end{aligned} \tag{17}$$

where $C_i = n!/(i-1)!(n-i)!$. The following is a direct derivation of the associated log-likelihood function:

$$\ell_R(\lambda, \alpha) \propto m \log \lambda + m \log \alpha + (\alpha - 1) \sum_{i=1}^{m} \log\left(1 + \lambda y_i^{-1}\right) + \alpha \sum_{i=1}^{m} i\left(1 + \lambda y_i^{-1}\right)) + (m - i) \sum_{i=1}^{m} \log\left(1 - e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}\right), \tag{18}$$

and the normal likelihood equations is as follows:

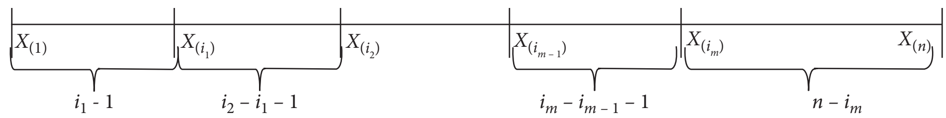


Figure 6: Order statistics of a subset of complete order statistics corresponding to the random sample $\{x_i, i = 1, 2, \ldots, n\}$.

$$\begin{aligned}\frac{\partial \ell_R}{\partial \lambda} &= \frac{m}{\lambda} + (\alpha - 1)\sum_{i=1}^{m}\frac{y_i^{-1}}{1+\lambda y_i^{-1}} + \alpha\sum_{i=1}^{m} i y_i^{-1} - (m-i)\sum_{i=1}^{m}\frac{\alpha y_{(i)}^{-1}\left(1+\lambda y_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}},\\ \frac{\partial \ell_R}{\partial \alpha} &= \frac{m}{\lambda\alpha} + \sum_{i=1}^{m}\log\left(1+\lambda y_i^{-1}\right) + \sum_{i=1}^{m} i\left(1+\lambda y_i^{-1}\right) - (m-i)\sum_{i=1}^{m}\frac{\left(1+\lambda y_i^{-1}\right)^{\alpha}\log\left(1+\lambda y_i^{-1}\right)^{\alpha} e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda y_i^{-1}\right)^{\alpha}}}.\end{aligned} \tag{19}$$

*3.3. Estimation Based on ERSS.* Let $\{z_i, i = 1, 2, \ldots, m\}$ be a ERSS drawn from a distribution with CDF and PDF as in (1) and (2), respectively, where $m$ is the set size. Let $g = m/2$, $h = m - 1/2$, and $u_i = m - i + 1$, then the likelihood function of the ERSS representative sample from INH $(\lambda, \alpha)$ according to (3) and (4) is given by

*Case 1.* $m$ odd:

$$\begin{aligned} L_{E_o}(\lambda,\alpha;z) &= \prod_{i=1}^{h}\Bigg[\left(m\lambda\alpha z_i^{-2}\left(1+\lambda z_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]^{m-1}\right)\\ &\quad\times\left(m\lambda\alpha z_{u_i}^{-2}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}}\left[e^{1-\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}}\right]^{m-1}\right)\Bigg]\\ &\quad\times\frac{m!}{(h!)^2}\lambda\alpha z_{h+1}^{-2}\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\left(e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\left(1-e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\right)\right)^{h}\\ &\propto \lambda^{2h+1}\alpha^{2h+1}\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha-1} e^{-\sum_{i=1}^{h}\left[\left(1+\lambda z_i^{-1}\right)^{\alpha}+\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\right]-(m-1)\sum_{i=1}^{h}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}} e^{-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\\ &\quad\times\left(e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\left(1-e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\right)\right)^{h}\prod_{i=1}^{h}\left(1+\lambda z_i^{-1}\right)^{\alpha-1}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1}\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]^{m-1}. \end{aligned} \tag{20}$$

The log-likelihood function follows directly as

$$\begin{aligned} \ell_{E_o}(\lambda,\alpha) &\propto (2h+1)\log\lambda + (2h+1)\log\alpha + (\alpha-1)\log\left(1+\lambda z_{h+1}^{-1}\right) - \sum_{i=1}^{h}\left[\left(1+\lambda z_i^{-1}\right)^{\alpha}+\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\right]\\ &\quad-(m-1)\sum_{i=1}^{h}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha} - (1+h)\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}\\ &\quad+h\log\left(1-e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}\right) + (\alpha-1)\sum_{i=1}^{h}\log\left(1+\lambda z_i^{-1}\right)\\ &\quad+(\alpha-1)\sum_{i=1}^{h}\log\left(1+\lambda z_{u_i}^{-1}\right) + (m-1)\sum_{i=1}^{h}\log\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]. \end{aligned} \tag{21}$$

Therefore, the associated likelihood normal equations will be

$$\frac{\partial \ell_{E_o}}{\partial \lambda} = \frac{2h+1}{\lambda} + \frac{(\alpha-1)z_{h+1}^{-1}}{1+\lambda z_{h+1}^{-1}} - \alpha\sum_{i=1}^{h}\left[z_i^{-1}\left(1+\lambda z_i^{-1}\right)^{\alpha-1} + z_{u_i}^{-1}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1}\right] - \alpha(1+h)z_{h+1}^{-1}\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha-1}$$
$$- \alpha(m-1)\sum_{i=1}^{h} z_{u_i}^{-1}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1} - \frac{\alpha h z_{h+1}^{-1}\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha-1} h e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}} + (\alpha-1)\sum_{i=1}^{h}\frac{z_i^{-1}}{1+\lambda z_i^{-1}}$$
$$+ (\alpha-1)\sum_{i=1}^{h}\frac{z_{u_i}^{-1}}{1+\lambda z_{u_i}^{-1}} - (m-1)\sum_{i=1}^{h}\frac{\alpha z_i^{-1}\left(1+\lambda z_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}},$$

$$\frac{\partial \ell_{E_o}}{\partial \alpha} = \frac{2h+1}{\alpha} + \log\left(1+\lambda z_{h+1}^{-1}\right) - \sum_{i=1}^{h}\left[\left(1+\lambda z_i^{-1}\right)^{\alpha}\log\left(1+\lambda z_i^{-1}\right) + \left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\log\left(1+\lambda z_{u_i}^{-1}\right)\right] \tag{22}$$
$$- (m-1)\sum_{i=1}^{h}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\log\left(1+\lambda z_{u_i}^{-1}\right) - (1+h)\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}\log\left(1+\lambda z_{h+1}^{-1}\right)$$
$$+ \sum_{i=1}^{h}\log\left(1+\lambda z_i^{-1}\right) + \sum_{i=1}^{h}\log\left(1+\lambda z_{u_i}^{-1}\right) - \frac{h\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}\log\left(1+\lambda z_{h+1}^{-1}\right)e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda z_{h+1}^{-1}\right)^{\alpha}}}$$
$$- (m-1)\sum_{i=1}^{h}\frac{\left(1+\lambda z_i^{-1}\right)^{\alpha}\log\left(1+\lambda z_i^{-1}\right)e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}{1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}.$$

*Case 2. m* even:

$$L_{E_e}(\lambda,\alpha;z) = \prod_{i=1}^{g}\left[\left(m\lambda\alpha z_i^{-2}\left(1+\lambda z_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]^{m-1}\right)\right.$$
$$\left.\times\left(m\lambda\alpha z_{u_i}^{-2}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}}\left[e^{1-\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}}\right]^{m-1}\right)\right]$$
$$\propto \lambda^{2g}\alpha^{2g}e^{-\sum_{i=1}^{h}\left[\left(1+\lambda z_i^{-1}\right)^{\alpha}+\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\right]-(m-1)\sum_{i=1}^{h}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}} \times \prod_{i=1}^{g}\left(1+\lambda z_i^{-1}\right)^{\alpha-1}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha-1}\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]^{m-1}. \tag{23}$$

Similarly, the log-likelihood function is given directly as

$$\ell_{E_e}(\lambda,\alpha) \propto 2g\log\lambda + 2g\log\alpha - \sum_{i=1}^{g}\left[\left(1+\lambda z_i^{-1}\right)^{\alpha} + \left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}\right] - (m-1)\sum_{i=1}^{g}\left(1+\lambda z_{u_i}^{-1}\right)^{\alpha}$$
$$+ (\alpha-1)\sum_{i=1}^{g}\left[\log\left(1+\lambda z_i^{-1}\right) + \log\left(1+\lambda z_{u_i}^{-1}\right)\right] + (m-1)\sum_{i=1}^{g}\log\left[1-e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}\right]. \tag{24}$$

Thus, the associated likelihood normal equations will be given as

$$\frac{\partial \ell_{E_e}}{\partial \lambda} = \frac{2g}{\lambda} - \alpha \sum_{i=1}^{h} \left[ z_i^{-1} \left(1 + \lambda z_i^{-1}\right)^{\alpha-1} + z_{u_i}^{-1} \left(1 + \lambda z_{u_i}^{-1}\right)^{\alpha-1} \right] - \alpha (m-1) \sum_{i=1}^{h} z_{u_i}^{-1} \left(1 + \lambda z_{u_i}^{-1}\right)^{\alpha-1} + (\alpha - 1) \sum_{i=1}^{h} \left[ \frac{z_i^{-1}}{1 + \lambda z_i^{-1}} + \frac{z_{u_i}^{-1}}{1 + \lambda z_{u_i}^{-1}} \right]$$

$$- \alpha (m-1) \sum_{i=1}^{h} \frac{z_i^{-1} \left(1 + \lambda z_i^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}{1 - e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}},$$

$$\frac{\partial \ell_{E_e}}{\partial \alpha} = \frac{2g}{\alpha} - \sum_{i=1}^{h} \left[ \left(1 + \lambda z_i^{-1}\right)^{\alpha} \log\left(1 + \lambda z_i^{-1}\right) + \left(1 + \lambda z_{u_i}^{-1}\right)^{\alpha} \log\left(1 + \lambda z_{u_i}^{-1}\right) \right] - (m-1) \sum_{i=1}^{h} \left(1 + \lambda z_{u_i}^{-1}\right)^{\alpha} \log\left(1 + \lambda z_{u_i}^{-1}\right)$$

$$+ \sum_{i=1}^{h} \left[ \log\left(1 + \lambda z_i^{-1}\right) + \log\left(1 + \lambda z_{u_i}^{-1}\right) \right] - (m-1) \sum_{i=1}^{g} \frac{\left(1 + \lambda z_i^{-1}\right)^{\alpha} \log\left(1 + \lambda z_i^{-1}\right)^{\alpha} e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}{1 - e^{1-\left(1+\lambda z_i^{-1}\right)^{\alpha}}}. \quad (25)$$

*3.4. Estimation Based on NRSS.* Assume $\{u_{(k_i)}, i = 1, 2, \ldots, m$ and $k_i$ is defined as in (6)$\}$ be a neoteric ranked set sample, where $m$ is the set size. According to (5), the likelihood function of NRSS samples drawn from INH $(\lambda, \alpha)$ for one cycle is given by

$$L_N(\lambda, \alpha; u) \propto \prod_{i=1}^{m} \lambda\alpha \left(1 + \lambda u_{(k_i)}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda u\left(-1/\left(k_i\right)\right)\right)^{\alpha}} \prod_{i=1}^{m+1} \left[ e^{1-\left(1+\lambda u\left(-1/\left(k_i\right)\right)\right)^{\alpha}} - e^{1-\left(1+\lambda u\left(-1/\left(k_i\right)\right)\right)^{\alpha}} \right]^{\left(k_i - k_{i-1} - 1\right)},$$

$$\propto \lambda^m \alpha^m \prod_{i=1}^{m} \left(1 + \lambda u_{(k_i)}^{-1}\right)^{\alpha-1} \prod_{i=1}^{m+1} \left[ e^{1-\left(1+\lambda u\left(-1/\left(k_i\right)\right)\right)^{\alpha}} - e^{1-\left(1+\lambda u\left(-1/\left(k_i-1\right)\right)\right)^{\alpha}} \right]^{\left(k_i - k_{i-1} - 1\right)}, \quad (26)$$

$$\propto \lambda^m \alpha^m \prod_{i=1}^{m} \left(1 + \lambda u_{(k_i)}^{-1}\right)^{\alpha-1} \left[ e^{-\left(1+\lambda u\left(-1/\left(k_i\right)\right)\right)^{\alpha}} - e^{-\left(1+\lambda u\left(-1/\left(k_i-1\right)\right)\right)^{\alpha}} \right]^{\left(k_i - k_{i-1} - 1\right)}.$$

The log-likelihood associated with this design is then given by

$$\ell_N(\lambda, \alpha) \propto m \log \lambda + m \log \alpha + (\alpha - 1) \sum_{i=1}^{m} \log\left(1 + \lambda u\left(\frac{-1}{\left(i_t\right)}\right)\right)$$

$$+ \sum_{i=1}^{m+1} \left(k_i - k_{i-1} - 1\right) \log\left[ e^{1-\left(1+\lambda u\left(-1/\left(i_t\right)\right)\right)^{\alpha}} - e^{1-\left(1+\lambda u\left(-1/\left(i_t-1\right)\right)\right)^{\alpha}} \right], \quad (27)$$

where $k_0 = 0$, $k_{m+1} = m^2 + 1$, and $u_{(k_0)} = -\infty$, $u_{(k_{m+1})} = \infty$. The associated normal equations are directly derived as

$$\frac{\partial \ell_N}{\partial \lambda} = \frac{m}{\lambda} + (\alpha - 1) \sum_{i=1}^{m} \frac{u_{(k_i)}^{-1}}{1 + \lambda u_{(k_i)}^{-1}} + \alpha \sum_{i=1}^{m+1} \left(k_i - k_{i-1} - 1\right) \frac{u_{(k_{i-1})}^{-1} \left(1 + \lambda u_{(k_{i-1})}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda u\left(-1/\left(i_t-1\right)\right)\right)^{\alpha}}}{e^{1-\left(1+\lambda u\left(-1/\left(i_t\right)\right)\right)^{\alpha}} - e^{1-\left(1+\lambda u\left(-1/\left(i_t-1\right)\right)\right)^{\alpha}}}$$

$$- \alpha \sum_{i=1}^{m+1} \left(k_i - k_{i-1} - 1\right) \frac{u_{(k_i)}^{-1} \left(1 + \lambda u_{(k_i)}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda u\left(-1/\left(i_t\right)\right)\right)^{\alpha}}}{e^{1-\left(1+\lambda u\left(-1/\left(i_t\right)\right)\right)^{\alpha}} - e^{1-\left(1+\lambda u\left(-1/\left(i_t-1\right)\right)\right)^{\alpha}}},$$

$$\frac{\partial \ell_N}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} \log\left(1 + \lambda u_{(k_i)}^{-1}\right) + \sum_{i=1}^{m+1} \frac{(k_i - k_{i-1} - 1)\left(1 + \lambda u_{(k_{i-1})}^{-1}\right)^{\alpha} \log\left(1 + \lambda u_{(k_{i-1})}^{-1}\right) e^{1-(1+\lambda u(-1/(i_t-1)))^{\alpha}}}{e^{1-(1+\lambda u(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda u(-1/(i_t-1)))^{\alpha}}}$$
$$- \sum_{i=1}^{m+1} \frac{(k_i - k_{i-1} - 1)\left(1 + \lambda u_{(k_i)}^{-1}\right)^{\alpha} \log\left(1 + \lambda u_{(k_i)}^{-1}\right) e^{1-(1+\lambda u(-1/(i_t)))^{\alpha}}}{e^{1-(1+\lambda u(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda u(-1/(i_t-1)))^{\alpha}}}. \tag{28}$$

*3.5. Estimation Based on ENRSS.* Let $\{v_{(i_t)}, t = 1, 2, \ldots, m$ and $i_t$ is defined as in (10)$\}$ be a random sample of size $n$ from a continuous population and let $\{z_{(i_t)}, t = 1, 2, \ldots, m\}$ be an extended neoteric ranked set sample. Therefore, the likelihood function of the INH $(\lambda, \alpha)$ distribution for one cycle based on (7) is computed as

$$L_{\mathrm{EN}}(\lambda, \alpha; v) \propto \prod_{t=1}^{m} \lambda\beta\left(1 + \lambda v_{(i_t)}^{-1}\right)^{\alpha-1} e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} \prod_{t=1}^{m+1} \left[e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}}\right]^{(i_t - i_{t-1} - 1)},$$
$$\propto \lambda^m \alpha^m \prod_{t=1}^{m} \left(1 + \lambda v_{(i_t)}^{-1}\right)^{\alpha-1} e^{m - \sum_{t=1}^{m}(1+\lambda v(-1/(i_t)))^{\alpha}} \prod_{t=1}^{m+1} \left[e^{-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{-(1+\lambda v(-1/(i_t)))^{\alpha}}\right]^{(i_t - i_{t-1} - 1)}, \tag{29}$$

where $i_0 = 0, i_{m+1} = m^3 + 1$, and $v_{(i_0)} = -\infty, v_{(i_{m+1})} = \infty$. The log-likelihood associated with this design is then given by

$$\ell_{\mathrm{EN}}(\lambda, \alpha) \propto m \log \lambda + m \log \alpha + (\alpha - 1) \sum_{i=1}^{m} \log\left(1 + \lambda v_{(i_t)}^{-1}\right),$$
$$+ \sum_{i=1}^{m+1} (i_t - i_{t-1} - 1) \log\left[e^{-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{-((-1/(i_t)))^{\alpha}}\right]. \tag{30}$$

The associated normal equations are directly derived as

$$\frac{\partial \ell_{\mathrm{EN}}}{\partial \lambda} = \frac{m}{\lambda} + (\alpha - 1) \sum_{i=1}^{m} \frac{v_{(i_t)}^{-1}}{1 + \lambda v_{(i_t)}^{-1}} + \alpha \sum_{t=1}^{m+1} (i_t - i_{t-1} - 1) \frac{v_{(i_{t-1})}^{-1}\left(1 + \lambda v_{(i_{t-1})}^{-1}\right)^{\alpha-1} e^{1-(1+\lambda v(-1/i_{(t-1)}))^{\alpha}}}{e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda v(-1/i_{(t-1)}))^{\alpha}}}$$
$$- \alpha \sum_{t=1}^{m+1} (i_t - i_{t-1} - 1) \frac{v_{(k_i)}^{-1}\left(1 + \lambda v_{(i_t)}^{-1}\right)^{\alpha-1} e^{1-\left(1+\lambda v_{(i_t)}^{-1}\right)^{\alpha}}}{e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}}},$$
$$\frac{\partial \ell_{\mathrm{EN}}}{\partial} = \frac{m}{\alpha} + \sum_{i=1}^{m} \log\left(1 + \lambda v_{(i_t)}^{-1}\right) + \sum_{t=1}^{m+1} \frac{(i_t - i_{t-1} - 1)\left(1 + \lambda v_{(i_{t-1})}^{-1}\right)^{\alpha} \log\left(1 + \lambda v_{(i_{t-1})}^{-1}\right) e^{1-(1+\lambda v(-1/(i_t-1)))^{\alpha}}}{e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{1-((-1/(i_t-1)))^{\alpha}}}$$
$$- \sum_{t=1}^{m+1} \frac{(i_t - i_{t-1} - 1)\left(1 + \lambda v_{(i_t)}^{-1}\right)^{\alpha} \log\left(1 + \lambda v_{(i_t)}^{-1}\right) e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}}}{e^{1-(1+\lambda v(-1/(i_t)))^{\alpha}} - e^{1-(1+\lambda v(-1/(i_t+1)))^{\alpha}}}. \tag{31}$$

The equations presented in this section are nonlinear and complex to be solved analytically; therefore, a numerical solution will be addressed in the next section by the use of simulation algorism.

## 4. Simulation Study and Results

We use a Monte Carlo simulation to compare the performance of the various ranked set sample designs in this

Table 1: Relative efficiency for RSS-based estimators for different sampling plans and INH distribution with scale parameter $\lambda = 1$.

| $\alpha$ | $m$ | eff ($\alpha_{(\text{RSS-based})}, \alpha_{\text{SRS}}$) | | | | eff ($\lambda_{(\text{RSS-based})}, \lambda_{\text{SRS}}$) | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | RSS | ERSS | NRSS | ENRSS | RSS | ERSS | NRSS | ENRSS |
| 0.4 | 3 | 1.50 | 1.80 | 2.40 | 6.30 | 1.28 | 1.59 | 2.07 | 5.59 |
| | 6 | 1.95 | 2.34 | 3.12 | 8.58 | 1.66 | 2.07 | 2.69 | 7.27 |
| | 9 | 2.80 | 3.36 | 4.48 | 12.04 | 2.38 | 2.98 | 3.87 | 10.44 |
| | 12 | 3.20 | 3.84 | 5.12 | 14.40 | 2.72 | 3.40 | 4.42 | 11.93 |
| | 15 | 3.91 | 4.69 | 6.26 | 19.55 | 3.32 | 4.15 | 5.40 | 14.58 |
| | 20 | 4.32 | 5.18 | 6.91 | 22.03 | 3.67 | 4.59 | 5.97 | 16.11 |
| 1.1 | 3 | 1.65 | 1.98 | 2.64 | 6.93 | 1.40 | 1.75 | 2.28 | 6.15 |
| | 6 | 2.02 | 2.42 | 3.23 | 8.89 | 1.72 | 2.15 | 2.79 | 7.53 |
| | 9 | 2.57 | 3.08 | 4.11 | 11.05 | 2.18 | 2.73 | 3.55 | 9.58 |
| | 12 | 3.18 | 3.82 | 5.09 | 14.31 | 2.70 | 3.38 | 4.39 | 11.86 |
| | 15 | 4.05 | 4.86 | 6.48 | 20.25 | 3.44 | 4.30 | 5.59 | 15.10 |
| | 20 | 4.61 | 5.53 | 7.38 | 23.51 | 3.92 | 4.90 | 6.37 | 17.19 |
| 1.5 | 3 | 1.58 | 1.90 | 2.53 | 6.64 | 1.34 | 1.68 | 2.18 | 5.89 |
| | 6 | 1.97 | 2.36 | 3.15 | 8.67 | 1.67 | 2.09 | 2.72 | 7.35 |
| | 9 | 2.31 | 2.77 | 3.70 | 9.93 | 1.96 | 2.45 | 3.19 | 8.61 |
| | 12 | 3.01 | 3.61 | 4.82 | 13.55 | 2.56 | 3.20 | 4.16 | 11.23 |
| | 15 | 3.53 | 4.24 | 5.65 | 17.65 | 3.00 | 3.75 | 4.88 | 13.16 |
| | 20 | 4.19 | 5.03 | 6.70 | 21.37 | 3.56 | 4.45 | 5.79 | 15.63 |
| 2.0 | 3 | 1.63 | 1.96 | 2.61 | 6.85 | 1.39 | 1.73 | 2.25 | 6.08 |
| | 6 | 2.15 | 2.58 | 3.44 | 9.46 | 1.83 | 2.28 | 2.97 | 8.02 |
| | 9 | 2.58 | 3.10 | 4.13 | 11.09 | 2.19 | 2.74 | 3.56 | 9.62 |
| | 12 | 2.91 | 3.49 | 4.66 | 13.10 | 2.47 | 3.09 | 4.02 | 10.85 |
| | 15 | 3.35 | 4.02 | 5.36 | 16.75 | 2.85 | 3.56 | 4.63 | 12.49 |
| | 20 | 4.11 | 4.93 | 6.58 | 20.96 | 3.49 | 4.37 | 5.68 | 15.33 |
| 2.5 | 3 | 1.79 | 2.15 | 2.87 | 7.53 | 1.52 | 1.90 | 2.47 | 6.68 |
| | 6 | 2.37 | 2.84 | 3.78 | 10.41 | 2.01 | 2.52 | 3.27 | 8.84 |
| | 9 | 2.84 | 3.41 | 4.54 | 12.20 | 2.41 | 3.02 | 3.92 | 10.59 |
| | 12 | 3.20 | 3.84 | 5.12 | 14.40 | 2.72 | 3.40 | 4.42 | 11.93 |
| | 15 | 3.69 | 4.42 | 5.90 | 18.43 | 3.14 | 3.92 | 5.10 | 13.76 |
| | 20 | 4.52 | 5.43 | 7.23 | 23.06 | 3.84 | 4.80 | 6.24 | 16.86 |
| 3.0 | 3 | 1.88 | 2.26 | 3.01 | 7.91 | 1.60 | 2.00 | 2.60 | 7.01 |
| | 6 | 2.48 | 2.98 | 3.97 | 10.93 | 2.11 | 2.64 | 3.43 | 9.25 |
| | 9 | 2.98 | 3.58 | 4.77 | 12.81 | 2.53 | 3.17 | 4.12 | 11.11 |
| | 12 | 3.36 | 4.03 | 5.38 | 15.12 | 2.86 | 3.57 | 4.64 | 12.53 |
| | 15 | 3.87 | 4.64 | 6.19 | 19.35 | 3.29 | 4.11 | 5.35 | 14.43 |
| | 20 | 4.75 | 5.70 | 7.60 | 24.21 | 4.04 | 5.05 | 6.56 | 17.71 |

section. The information was derived and generated from INH distribution with scale parameter $\lambda = 1$ and shape parameter $\alpha = 0.4$, 1.1, 1.5, 2, 2.5, and 3 for different sample sizes ($m = 3, 6, 9,\ 15$ and 20). The simulation algorithm is as follows.

For SRS,

(1) With 50,000 replicates, generate $m$ random samples from the INH distribution using the quantile function specified in (3)

(2) Obtain the MLEs for both scale and shape parameters

For different ranked set samples,

(1) Use the different RSS designs discussed in Section 2 to simulate different RSS designs samples and generate sample using the quantile function defined in (3) with 50,000 replicates

(2) Obtain the MLEs for RSS, NRSS, and ENRSS plans

(3) Calculate the relative bias (RB), mean squared error (MSE), and relative efficiency (RE) of the different RSS estimators compared with the SRS estimators where the relative bias and MSE of an estimator $\widehat{\theta}$ of a parameter $\theta$ are defined, respectively, as

$$\text{RB} = \sum_{l=1}^{50,000} \frac{\widehat{\lambda}_l - \Phi_l}{\Phi_l},$$
$$\text{MSE} = \frac{1}{50,000} \sum_{l=1}^{50,000} \left(\widehat{\Phi}_l - \Phi_l\right)^2, \tag{32}$$

and the RE of the estimator $\widehat{\theta}_2$ compared with the estimator $\widehat{\theta}_1$ is defined as

$$\text{eff}\,(\text{SRS, Ranked}) = \frac{\text{MSE}\left(\widehat{\Phi}_l\right)_{\text{SRS}}}{\text{MSE}\,(\widehat{\Phi})_{l\text{Ranked}}}. \tag{33}$$

Table 2: Relative bias for RSS-based estimators for different sampling plans and INH distribution with scale parameter $\lambda = 1$.

| $\alpha$ | m | RB $(\hat{\alpha}_{\text{RSS-based}})$ | | | | RB $(\hat{\lambda}_{\text{RSS-based}})$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | RSS | ERSS | NRSS | ENRSS | RSS | ERSS | NRSS | ENRSS |
| 0.4 | 3 | 0.00115 | 0.00092 | 0.00064 | 0.00032 | 0.00173 | 0.00138 | 0.00097 | 0.00048 |
| | 6 | 0.00213 | 0.0017 | 0.00119 | 0.0006 | 0.0032 | 0.00256 | 0.00179 | 0.00089 |
| | 9 | 0.00107 | 0.00086 | 0.0006 | 0.0003 | 0.00161 | 0.00128 | 0.0009 | 0.00045 |
| | 12 | 0.00112 | 0.0009 | 0.00063 | 0.00031 | 0.00168 | 0.00134 | 0.00094 | 0.00047 |
| | 15 | 0.00101 | 0.00081 | 0.00057 | 0.00028 | 0.00152 | 0.00121 | 0.00085 | 0.00042 |
| | 20 | 0.00099 | 0.00079 | 0.00055 | 0.00028 | 0.00149 | 0.00119 | 0.00083 | 0.00042 |
| 1.1 | 3 | 0.00098 | 0.00078 | 0.00055 | 0.00027 | 0.00113 | 0.0009 | 0.00063 | 0.00032 |
| | 6 | 0.00181 | 0.00145 | 0.00101 | 0.00051 | 0.00209 | 0.00167 | 0.00117 | 0.00059 |
| | 9 | 0.00091 | 0.00073 | 0.00051 | 0.00025 | 0.00105 | 0.00084 | 0.00059 | 0.00029 |
| | 12 | 0.00095 | 0.00076 | 0.00053 | 0.00027 | 0.0011 | 0.00088 | 0.00062 | 0.00031 |
| | 15 | 0.00086 | 0.00069 | 0.00048 | 0.00024 | 0.00099 | 0.00079 | 0.00056 | 0.00028 |
| | 20 | 0.00084 | 0.00067 | 0.00047 | 0.00024 | 0.00097 | 0.00078 | 0.00054 | 0.00027 |
| 1.5 | 3 | 0.00083 | 0.00066 | 0.00047 | 0.00023 | 0.00096 | 0.00077 | 0.00054 | 0.00027 |
| | 6 | 0.00154 | 0.00123 | 0.00086 | 0.00043 | 0.00178 | 0.00142 | 0.001 | 0.0005 |
| | 9 | 0.00077 | 0.00062 | 0.00043 | 0.00022 | 0.00089 | 0.00071 | 0.0005 | 0.00025 |
| | 12 | 0.00081 | 0.00065 | 0.00045 | 0.00023 | 0.00094 | 0.00075 | 0.00052 | 0.00026 |
| | 15 | 0.00073 | 0.00058 | 0.00041 | 0.0002 | 0.00084 | 0.00067 | 0.00047 | 0.00024 |
| | 20 | 0.00072 | 0.00057 | 0.0004 | 0.0002 | 0.00083 | 0.00066 | 0.00046 | 0.00023 |
| 2 | 3 | 0.00056 | 0.00045 | 0.00031 | 0.00016 | 0.00064 | 0.00051 | 0.00036 | 0.00018 |
| | 6 | 0.00103 | 0.00082 | 0.00058 | 0.00029 | 0.00119 | 0.00095 | 0.00067 | 0.00033 |
| | 9 | 0.00052 | 0.00041 | 0.00029 | 0.00015 | 0.0006 | 0.00048 | 0.00034 | 0.00017 |
| | 12 | 0.00054 | 0.00043 | 0.0003 | 0.00015 | 0.00063 | 0.0005 | 0.00035 | 0.00018 |
| | 15 | 0.00049 | 0.00039 | 0.00027 | 0.00014 | 0.00057 | 0.00045 | 0.00032 | 0.00016 |
| | 20 | 0.00048 | 0.00038 | 0.00027 | 0.00013 | 0.00055 | 0.00044 | 0.00031 | 0.00016 |
| 2.5 | 3 | 0.00031 | 0.00024 | 0.00017 | 0.00009 | 0.00035 | 0.00028 | 0.0002 | 0.0001 |
| | 6 | 0.00057 | 0.00045 | 0.00032 | 0.00016 | 0.00065 | 0.00052 | 0.00037 | 0.00018 |
| | 9 | 0.00028 | 0.00023 | 0.00016 | 0.00008 | 0.00033 | 0.00026 | 0.00018 | 0.00009 |
| | 12 | 0.0003 | 0.00024 | 0.00017 | 0.00008 | 0.00034 | 0.00027 | 0.00019 | 0.0001 |
| | 15 | 0.00027 | 0.00021 | 0.00015 | 0.00008 | 0.00031 | 0.00025 | 0.00017 | 0.00009 |
| | 20 | 0.00026 | 0.00021 | 0.00015 | 0.00007 | 0.0003 | 0.00024 | 0.00017 | 0.00009 |
| 3 | 3 | 0.00012 | 0.0001 | 0.00007 | 0.00003 | 0.00014 | 0.00011 | 0.00008 | 0.00004 |
| | 6 | 0.00023 | 0.00018 | 0.00013 | 0.00006 | 0.00026 | 0.00021 | 0.00015 | 0.00007 |
| | 9 | 0.00011 | 0.00009 | 0.00006 | 0.00003 | 0.00013 | 0.0001 | 0.00007 | 0.00004 |
| | 12 | 0.00012 | 0.0001 | 0.00007 | 0.00003 | 0.00014 | 0.00011 | 0.00008 | 0.00004 |
| | 15 | 0.00011 | 0.00009 | 0.00006 | 0.00003 | 0.00012 | 0.0001 | 0.00007 | 0.00003 |
| | 20 | 0.00011 | 0.00008 | 0.00006 | 0.00003 | 0.00012 | 0.0001 | 0.00007 | 0.00003 |

The results of the simulation study are reported in Tables 1 and 2. From the table, some important conclusions can be observed from the results.

As expected, the efficiency of all RSS-based designs increases as the sample size increases and as the shape of the distribution is near symmetry.

(i) It is clear that the proposed likelihood function for the ENRSS plan is working effectively and provides efficient estimators similar to the results reported by Taconeli and Cabral [6]

(ii) The ENRSS plan estimators outperform the one-stage RSS plans when the process does not include ranking errors

(iii) The NRSS plan estimators outperform the one-stage ERSS and RSS and SRS plans

(iv) The RSS plan estimators outperform the one-stage SRS plans

(v) Biases are almost negligible when the shape of the distribution is near symmetry

## 5. Conclusion

Different sampling designs have been discussed. The likelihood function for the INH distribution was studied based on SRS, RSS, ERSS, NRSS, and ENRSS. By Monte Carlo simulation, the different sampling designs have been compared. It is clear from the simulation results that the likelihood function used to estimate the parameters of the INH distribution based on ENRSS showed relatively efficient estimates compared with SRS, RSS, ERSS, and NRSS estimators. During parameter estimation of an INH distribution based on different sampling designs, the authors advocate employing ENRSS and other two-stage RSS designs as ranked sampling methods. For future research on this topic, the author plans to start a more general study regarding most of the important one-stage RSS plans in both perfect and imperfect ranking cases and study their performance when making parameter estimations for several symmetric and asymmetric distributions [19–21].

## Data Availability

All data are included in the paper.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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