A weighted three-parameter Weibull distribution

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Abstract

In this paper, extension of the weighted Weibull distribution is proposed. This class is a generalization of the two-parameter Weibull distribution as well as some other lifetime distributions. This class is found to be capable of modeling lifetime and other application data. The main properties of this class are investigated and derived. Maximum Likelihood and least squares estimation methods are used to evaluate the distribution parameters and a numerical example is also provided.

Keywords: *Weibull distribution; hazard rate function; Maximum likelihood, weighted distribution, exponential distribution*

1 Introduction

The Weibull distribution has been a powerful probability distribution in reliability analysis. It provides much wider applicability as compared with the exponential distribution. The Weibull distribution is also a good alternative to Gamma and Lognormal distribution in reliability engineering and life testing. Recently, various weighted versions of Weibull and exponential distributions have been proposed in literature. Gupta and Kundu [1] have followed a similar approach to Azzalini's novel method for introducing a skewness parameter to the normal distribution which is known in the literature as a skew-normal distribution non-symmetric distributions [2], and introduced a shape parameter to the exponential distribution to introduce the class of weighted exponential distributions. Similar approaches were followed by Shahbaz et. al., [3], and Shakhatreh [4] to introduce a weighted version of the Weibull distribution and a two parameter weighted version of the exponential distribution respectively. Ramadan [5] studied the weighted Weibull distribution and its statistical properties.

A random variable Y is said to follow the three-parameter Weibull distribution if the cumulative distribution function (cdf) and the probability density function (pdf) of Y are

$$G(x;\lambda,\beta,\theta)=1-e^{\lambda(x-\theta)^{\beta}}, \tag{1.1}$$

and

$$g(x;\lambda,\beta,\theta)=\lambda\beta(x-\theta)^{\beta-1}e^{\lambda(x-\theta)^{\beta}}, \quad x>\theta, \tag{1.2}$$

respectively, where $\lambda>0$, $\beta>0$ and $\theta\geq 0$ are the scale, shape and minimum lifespan (location) parameters respectively. In lifetime data analysis, θ is referred to as a threshold, guarantee time or a parameter, and λ is the mean time to failure [6]. This particular distribution has several advantages to give one more option for analyzing skewed lifetime data.

The main aim of this paper is to introduce a new weighted Weibull distribution based on the three-parameter Weibull distribution and discuss some of its recent developments. The rest of the paper is organized as follows. In section 2, the proposed model is introduce. Section 3, 4 and 5 are devoted for deriving some of the properties of the proposed model. In section 6, estimation of the unknown parameters of the proposed model is discussed using least squares method (LSE) and the maximum likelihood method (MLE). Simulation studies and real data example are discussed in sections 7 and 8. Finally, the paper is concluded.

2 The weighted three-parameter Weibull distribution

In this section, we introduce the definition of the probability density function and the cumulative distribution function of the weighted three parameter Weibull distribution denoted by $WTW(\lambda,\alpha,\beta,\theta)$.

Definition 2.1: Let Y and Z be two independent two parameter exponential random variables with parameters (λ,θ) and $(\lambda(1+\alpha^{\beta}),\theta)$ respectively, then the random variable $X^{\beta}=Y+Z$ will be distributed as $WTW(\lambda,\alpha,\beta,\theta)$ with pdf

$$f(x;\lambda,\alpha,\beta,\theta)=\frac{1+\alpha^{\beta}}{\alpha^{\beta}}\lambda\beta(x-\theta)^{\beta-1}e^{-\lambda(x-\theta)^{\beta}}(1-e^{-\lambda(\alpha(x-\theta))^{\beta}}),\qquad x>\theta \tag{2.1}$$

where λ, θ are scale and shift parameter respectively while α, β are shape parameters respectively. This distribution can be found using Azzalini's approach that was used to introduce the skew-normal distribution [2], and as Gupta and Kundu [1] have used it to introduce the weighted exponential distribution.

Definition 2.2: The CDF of X is given by

$$F(x;\lambda,\alpha,\beta,\theta)=\frac{1+\alpha^{\beta}}{\alpha^{\beta}}\left[\left(1-e^{-\lambda(x-\theta)^{\beta}}\right)-\frac{1}{1+\alpha^{\beta}}\left(1-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right)\right] \tag{2.2}$$

Definition 2.3: let $U=\exp[-\lambda\alpha^{2}(X-\theta)^{2}]$ be a random variable distributed as Beta($1/\alpha^{2}$, 2), (Gupta and Kundu [1], Nadarajah and Kotz, [6]), with pdf

$$f_U(u;(1/\alpha^{2}),2)=\frac{\Gamma(\frac{1}{\alpha^{2}}+2)}{\Gamma(\frac{1}{\alpha^{2}})\,\Gamma(2)}u^{\frac{1}{\alpha^{2}}-1}(1-u),\quad 0<u<1. \tag{2.3}$$

The random variable *X* will be distributed as a three parameter weighted Weibull distribution with pdf and cdf as in (2.1) and (2.2). Figures 1 and 2 show the pdfs and the cdfs of the $WTWD(\lambda,\alpha,\theta,\beta)$ for different values of λ,α and β and for $\theta=1$. From figure 1 it is immediate that the pdf's are unimodal for α, β ≥ 1 and almost inverse exponentially for α, β < 1 than can be used to represent right skewed or almost symmetric data

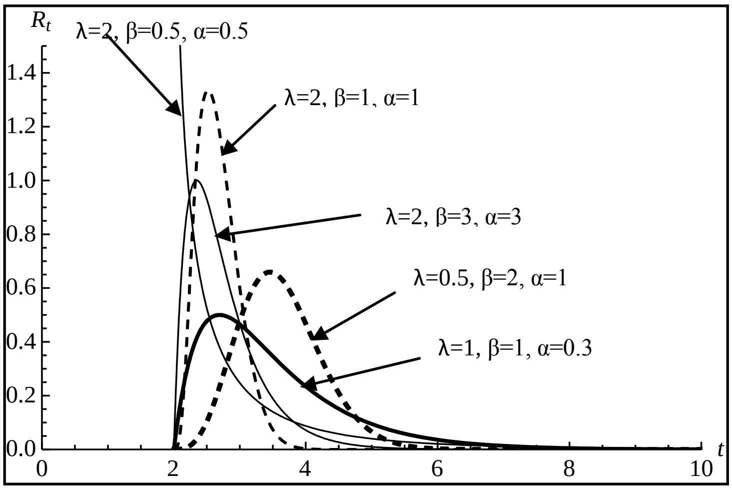

Fig. 1: Examples of the *WTW* probability density function for different couples of shape and scale parameters and for location parameter θ = 2

Definition 2.4: The mode of the weighted three-parameter Weibull distribution denoted by x_0 can be obtained directly from the derivatives of the pdf in (2.1), which at the end is the solution of the nonlinear equation

$$\frac{\beta-1}{x_0-\theta}-\frac{\lambda\beta(x_0-\theta)^{\beta-1}(1+\alpha^{\beta}-e^{-\lambda(\alpha(x_0-\theta))^{\beta}})}{1-e^{-\lambda(\alpha(x_0-\theta))^{\beta}}}=0 \tag{2.4}$$

3 Relationship to other distributions

When X is a random variable distributed as weighted three parameter Weibull distribution $WTW(\lambda,\alpha,\beta,\theta)$, with pdf as in (2.1), X can be related to some other distribution as shown in Table 1.

Table 1: related distribution To the $WTW(\lambda,\alpha,\beta,\theta)$ distribution

	Related Distribution
$\beta=1$	Weighted Two Parameter Exponential $WTE(\lambda,\alpha,\theta)$
$\theta=0$	Weighted Weibull $WW(\lambda,\alpha,\beta)$
$\beta=1,\theta=0$,	Weighted Exponential $WE(\lambda,\alpha)$
$\alpha\to\infty$	Three Parameter Weibull $W(\lambda,\beta)$ l
$\alpha\to\infty$, $\beta=1$	Two Parameter Exponential $TE(\lambda,\theta)$ l
$\alpha\to\infty$, $\beta=1,\theta=0$	Exponential $Exp(\lambda)$

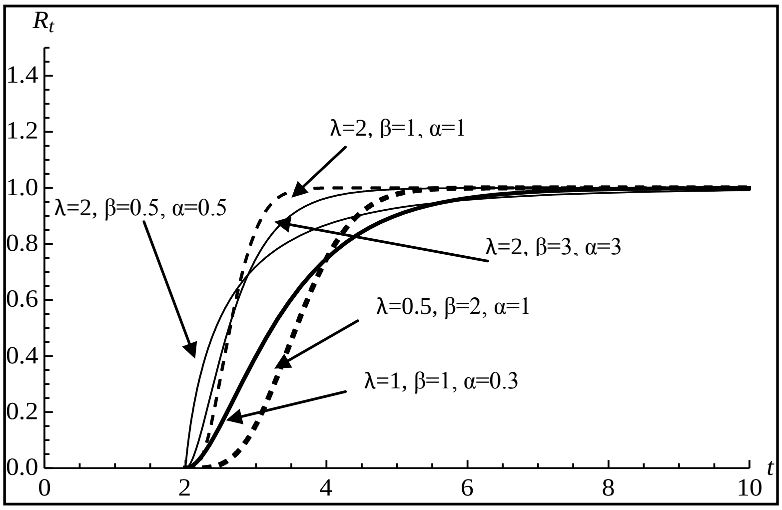

Fig. 2: Examples of the *WTW* probability distribution function for different values of shape and scale parameters and for location parameter θ = 2

4 The moment generating function

In this section, we discuss the formulation of the moment generating function. Let X denote a random variable with pdf as in (2.1). The moment generating function (mgf) of X, $M_X(t)=E(\exp(tX))$, is given by

$$M_X(t) = \frac{\lambda\beta(1+\alpha^{\beta})}{\alpha^{\beta}}\int_{\theta}^{\infty} e^{tx}(x-\theta)^{\beta-1}e^{-\lambda(x-\theta)^{\beta}}(1-e^{-\lambda(\alpha(x-\theta))^{\beta}})dx$$

$$= \frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^{\infty}(y^{1/\beta}+\theta)^j e^{-\lambda y}(1-e^{-\lambda\alpha^{\beta}y})dy \tag{4.1}$$

When $\theta=0$, the mgf of *X* is the mgf of weighted two-parameter Weibull distribution given by

$$M_X(t) = \frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\sum_{j=0}^{\infty}\frac{t^j}{j!}\lambda^{-j/\beta}\left(1-(1+\alpha^{\beta})^{-j/\beta}\right)\Gamma\left(1+\frac{j}{\beta}\right)$$

4.1. The survival rate, hazard rate and reversed hazard rate functions MRL functions

The survival rate function $s_X(x)$, hazard rate function $h_X(x)$ and reversed hazard function $r_X(x)$ of the random variable X having the pdf in (3.1) can be written respectively as

$$s_X(x)=1-F(x;\lambda,\alpha,\beta,\theta)$$

$$=\frac{1}{\alpha^{\beta}}\left((1+\alpha^{\beta})e^{-\lambda(x-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right), \tag{4.2}$$

$$h_X(x)=\frac{\lambda\beta(1+\alpha^{\beta})\,(x-\theta)^{\beta-1}\left(1-e^{-\lambda(\alpha(x-\theta))^{\beta}}\right)}{(1+\alpha^{\beta})-e^{-\lambda(\alpha(x-\theta))^{\beta}}}, \tag{4.3}$$

and

$$r_X(x)=\frac{\lambda\beta\,(1+\alpha^{\beta})(x-\theta)^{\beta-1}\left(1-e^{-\lambda(\alpha(x-\theta))^{\beta}}\right)}{\alpha^{\beta}e^{-\lambda(x-\theta)^{\beta}}+(1+\alpha^{\beta})e^{-\lambda(\alpha(x-\theta))^{\beta}}-(1+\alpha^{\beta})} \tag{4.4}$$

Figure 3 shows examples of different hazard rate function for different values of the distribution parameter. It is immediate that the hazard functions can be increasing, decreasing, or bathtub shaped.

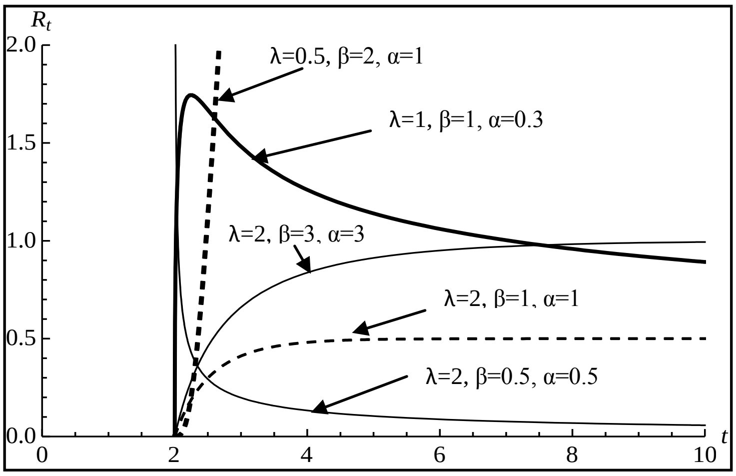

Fig. 3: Examples of the hazard rate function for different values of shape and scale parameters and for location parameter θ = 2

4.2. Mean residual life and mean waiting time functions

The mean residual life (MRL) waiting and the mean waiting time are very important in reliability and survival analysis. The MRL function, also called expected remaining life function or mean excess function, has been widely studied in lifetime random variables context. It plays an important role in, many fields such as industrial reliability, biomedical science, life insurance and demography among others and that is because it describes the aging process. It is well known that the MRL uniquely determine the distribution function, i.e. it has all the information about the model; see e.g. Barlow and Proschan [7] and Guillamon et al. [8]. The (MRL) of WTW random variable X is given by $m(t)$ and is defined by (see, Belkacem and Alexandre [9])

$$m(t)=E(T-t\,|\,T>t)=\frac{\int_t^{\infty}\overline{F}(y)\,dy}{\overline{F}(t)}=\frac{\int_t^{\infty}\frac{1}{\alpha^{\beta}}\left((1+\alpha^{\beta})e^{-\lambda(y-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(y-\theta)^{\beta}}\right)dy}{\frac{1}{\alpha^{\beta}}\left((1+\alpha^{\beta})e^{-\lambda(t-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(t-\theta)^{\beta}}\right)}$$

$$=\frac{(1+\alpha^{\beta})\,\Gamma\!\left(\frac{1}{\beta},\lambda(t-\theta)^{\beta}\right)-\frac{1}{(1+\alpha^{\beta})^{1/\beta}}\,\Gamma\!\left(\frac{1}{\beta},\lambda(1+\alpha^{\beta})(t-\theta)^{\beta}\right)}{\lambda^{1/\beta}\,\beta e^{-\lambda(t-\theta)^{\beta}}\left((1+\alpha^{\beta})-e^{-\lambda(\alpha^{\beta})(t-\theta)^{\beta}}\right)} \tag{4.5}$$

The mean waiting time (MWT) describes the time, which had elapsed since the failure. For the $WTW(\lambda,\alpha,\beta,\theta)$ distribution the MWT is given by

$$\mu(t)=\frac{\alpha^{\beta}\left[(t+\theta)+\frac{1}{\lambda^{1/\beta}}\,\Gamma\left(\frac{1}{\beta},\lambda(t-\theta)^{\beta}\right)-\frac{1}{\lambda^{1/\beta}\,\beta(1+\alpha^{\beta})^{1/\beta}}\,\Gamma\left(\frac{1}{\beta},\lambda(1+\alpha^{\beta})(t-\theta)^{\beta}\right)\right]}{\alpha^{\beta}-e^{-\lambda(t-\theta)^{\beta}}\left((1+\alpha^{\beta})-e^{-\lambda(\alpha^{\beta})(t-\theta)^{\beta}}\right)} \tag{4.6}$$

5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let the random variable $X_{r:n}$ be the rth order statistic ($X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$) in a sample of size n from $WTW(\lambda,\alpha,\beta,\theta)$ distribution, for $r=1,\ldots,n$. The pdf of $X_{r:n}$ is given

$$f_{r:n}(x)=C\,f(x)\,F(x)^{r-1}[1-F(x)]^{n-r}, \qquad x>0\,,$$

where $f(x)$ comes from the pdf in (2.1), $F(x)$ comes from (2.2), and $C=n!\,\left[(r-1)!\,(n-r)!\right]^{-1}$. After some algebra we can obtain the rth order statistic density function, the pdf of the minimum value, and the pdf of the Maximum value respectively as

$$f_{r:n}(x)=C\,\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\,\lambda\beta\,x^{\beta-1}\,e^{-\lambda(x-\theta)^{\beta}}\left(1-e^{-\lambda(\alpha(x-\theta))^{\beta}}\right)\times\left(\frac{1}{\alpha^{\beta}}\left((1+\alpha^{\beta})e^{-\lambda(x-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right)\right)^{n-r}$$

$$\times\left(\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\left[\left(1-e^{-\lambda(x-\theta)^{\beta}}\right)-\frac{1}{1+\alpha^{\beta}}\left(1-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right)\right]\right)^{r-1}, \tag{5.1}$$

$$f_{1:n}(x)=n\left(\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\right)^{n}\lambda\beta\,x^{\beta-1}\,e^{-\lambda(x-\theta)^{\beta}}\left(1-e^{-\lambda\alpha(x-\theta)^{\beta}}\right)\times\left[\left((1+\alpha^{\beta})e^{-\lambda(x-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right)\right]^{n-1}, \tag{5.2}$$

and

$$f_{n:n}(x)=n\left(\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\right)^{n}\lambda\beta\,x^{\beta-1}\,e^{-\lambda(x-\theta)^{\beta}}\left(1-e^{-\lambda\alpha(x-\theta)^{\beta}}\right)\times\left[\left(1-e^{-\lambda(x-\theta)^{\beta}}\right)-\frac{1}{1+\alpha^{\beta}}\left(1-e^{-\lambda(1+\alpha^{\beta})(x-\theta)^{\beta}}\right)\right]. \tag{5.3}$$

6 Statistical inferences

In this section, we consider estimation by method of least squares estimation (LSE) and the maximum likelihood estimation (MLE) methods for the unknown parameters of the weighted three- parameter Weibull distribution.

6.1. The method of least squares

Let $X_1,\ldots,X_n$ be a random sample drawn from $WTW(\lambda,\alpha,\beta,\theta)$. Assume that $X_{(1)},\ldots,X_{(n)}$ are the ordered sample based on the random sample drawn. It is known that $E\left(F(x_{(i)})\right)=i/(n+1),\ i=1,2,\ldots,n$. The least squares estimators (LSEs) are obtained by minimizing the loss function

$$Q(\Omega)=\sum_{i=1}^{n}\left(F(x_{(i)})-E\left(F(x_{(i)})\right)\right)^{2}=\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{i}{(n+1)}\right)^{2}. \tag{6.1}$$

In the case of our proposed model ($WTW(\lambda,\alpha,\beta,\theta)$) Equation (4.1) becomes

$$Q(\Omega)=\sum_{i=1}^{n}\left(\frac{1+\alpha^{\beta}}{\alpha^{\beta}}\left[\left(1-e^{-\lambda\left(\left(x_{(i)}-\theta\right)^{\beta}\right)}\right)-\frac{1}{1+\alpha^{\beta}}\left(1-e^{-\lambda(1+\alpha^{\beta})\left(x_{(i)}-\theta\right)^{\beta}}\right)\right]-\frac{i}{(n+1)}\right)^{2}. \tag{6.2}$$

The normal equations for the LSE methods are obtained by minimizing Equation (6.2) with respect to λ, α and θ , that is differentiating with respect to λ, α , β and θ respectively and equating to zero. This will lead to

$$\sum_{i=1}^{n}\Big\{\big(x_{(i)}-\theta\big)^{\beta}\Big[e^{-\lambda(x_{(i)}-\theta)^{\beta}}-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}\Big]\times\Big[\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Big[1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big]-\frac{\alpha^{\beta}}{(1+\alpha^{\beta})}\frac{1}{n+i}\Big]\Big\}=0 \tag{6.3}$$

$$\sum_{i=1}^{n}\Bigg\{\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Bigg(\frac{\Big(1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}\Big)}{\alpha^{\beta-1}(1+\alpha^{\beta})^{2}}-\frac{\lambda\beta\big(x_{(i)}-\theta\big)^{\beta}e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{\alpha^{\beta-1}(1+\alpha^{\beta})^{2}}\Bigg)\Bigg)\Bigg[\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Big[1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big]-\frac{\alpha^{\beta}}{(1+\alpha^{\beta})}\frac{1}{n+i}\Bigg]$$

$$+\frac{\beta\Big(1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big)}{\alpha}-\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\beta\Big(1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big) \tag{6.4}$$

$$\sum_{i=1}^{n}\Big\{\lambda\beta\big(x_{(i)}-\theta\big)^{\beta-1}\Big[-\exp\Big(-\lambda\big(x_{(i)}-\theta\big)^{\beta}\Big)+\exp\Big(-\lambda(1+\alpha^{\beta})\big(x_{(i)}-\theta\big)^{\beta}\Big)\Big]$$

$$\times\Big[\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Big[1-\exp\Big(-\lambda\big(x_{(i)}-\theta\big)^{\beta}\Big)-\frac{1-\exp\Big(-\lambda(1+\alpha^{\beta})\big(x_{(i)}-\theta\big)^{\beta}\Big)}{(1+\alpha^{\beta})}\Big]-\frac{\alpha^{\beta}}{(1+\alpha^{\beta})}\frac{1}{n+i}\Big]\Big\}=0 \tag{6.5}$$

$$\sum_{i=1}^{n}\Bigg\{\frac{\beta}{\alpha}\Big(1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big)-\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\beta\Big(1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big)$$

$$+\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Bigg(\frac{\Big(1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}\Big)}{\alpha^{\beta-1}(1+\alpha^{\beta})^{2}}-\frac{\lambda\beta\big(x_{(i)}-\theta\big)^{\beta}e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{\alpha^{\beta-1}(1+\alpha^{\beta})^{2}}\Bigg)\Bigg)\Bigg[\frac{(1+\alpha^{\beta})}{\alpha^{\beta}}\Big[1-e^{-\lambda(x_{(i)}-\theta)^{\beta}}-\frac{1-e^{-\lambda(1+\alpha^{\beta})(x_{(i)}-\theta)^{\beta}}}{(1+\alpha^{\beta})}\Big]-\frac{1}{n+i}\Bigg]=0\Bigg\} \tag{6.6}$$

Equations (6.3)-(6.6) are a system of nonlinear equations that needs to be solved numerically.

6.2. The maximum likelihood method

Let $X_1,\ldots,X_n$ be a random sample drawn from $WTW(\lambda,\alpha,\beta,\theta)$. The likelihood function based on $X_1,\ldots,X_n$ is given by

$$L(\lambda,\alpha,\beta,\theta)=\alpha^{-n\beta}\ \lambda^{n}\beta^{n}\,(1+\alpha^{\beta})^{n}\ \prod_{i=1}^{n}(x-\theta)_{i}^{\beta-1}\ \ e^{-\lambda\sum_{1}^{n}(x_i-\theta)^{\beta}}\ \prod_{1}^{n}(1-e^{-\lambda(\alpha(x_i-\theta))^{\beta}})$$

and the log-likelihood function is then given by

$$\ln L(\lambda,\alpha,\beta,\theta)=n\ln\lambda-n\beta\ln\alpha+n\ln\beta+n\ln(1+\alpha^{\beta})+(\beta-1)\sum_{i=1}^{n}\ln(x_i-\theta)$$

$$-\lambda\sum_{1}^{n}(x_i-\theta)^{\beta}+\sum_{1}^{n}\ln(1-e^{-\lambda(\alpha(x_i-\theta))^{\beta}}) \tag{6.7}$$

Our goal is to maximize the log-likelihood function with respect to λ, α and θ. This is done by partially differentiate Equation (6.7) with respect to λ, α and θ and equating to zero. The following normal equations are obtained

$$\frac{\partial\ln L}{\partial\lambda}=\frac{n}{\lambda}-\sum_{1}^{n}(x_i-\theta)^{\beta}+\alpha^{\beta}\sum_{1}^{n}\frac{(x_i-\theta)^{\beta}e^{-\lambda(\alpha(x_i-\theta))^{\beta}}}{(1-e^{-\lambda(\alpha(x_i-\theta))^{\beta}})}=0 \tag{6.8}$$

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n\beta\alpha^{\beta-1}}{1+\alpha^{\beta}}-\frac{n}{\alpha}+\lambda\beta\alpha^{\beta-1}\sum_{1}^{n}\frac{(x_i-\theta)e^{-\lambda(\alpha(x_i-\theta))^{\beta}}}{(1-e^{-\lambda\alpha(x_i-\theta)})}=0 \tag{6.9}$$

$$\frac{\partial\ln L}{\partial\beta}=\frac{n(\ln\alpha)\alpha^{\beta}}{1+\alpha^{\beta}}-n\ln\alpha+\frac{n}{\beta}+\sum_{i=1}^{n}\ln(x_i-\theta)$$

$$-\lambda\sum_{1}^{n}(x_i-\theta)^{\beta}\ln(x_i-\theta)-\lambda\alpha^{\beta}\sum_{1}^{n}\frac{(x_i-\theta)^{\beta}\ln(\alpha(x_i-\theta))e^{-\lambda(\alpha(x_i-\theta))^{\beta}}}{(1-e^{-\lambda(\alpha(x_i-\theta))^{\beta}})}=0 \qquad (6.10)$$

$$\frac{\partial \ln L}{\partial\theta}=-(\beta-1)\sum_{1}^{n}(x_i-\theta)^{-1}+\lambda\beta\sum_{1}^{n}(x_i-\theta)^{\beta-1}-\lambda\beta\alpha^{\beta-1}\sum_{1}^{n}\frac{(x_i-\theta)^{\beta-1}e^{-\lambda(\alpha(x_i-\theta))^{\beta}}}{(1-e^{-\lambda(\alpha(x_i-\theta))^{\beta}})}=0 \qquad (6.11)$$

On solving Eqs. (6.8)-(6.11) we get the maximum likelihood estimators of the parameters of the $WTW(\lambda,\alpha,\beta,\theta)$.

7 Numerical Example

In this section, we study the performance of the ML estimators of the unknown parameters of the weighted 3p Weibull distribution. This is done using MATHEMATICA v.8.1 and is made by generating samples from the WTW distribution with sample sizes 20, 35, 50 and 100. The simulation is made for different values of the parameters α, θ and β when the parameter value of the scale parameter λ is assumed known and fixed. Without loss of generality we assume that $\lambda=1$. The results of this simulation study are in table 2 - 6. We comparison of different methods are made using the estimated values of parameters, the bias and the mean squared error MSE for 1000 replicates.

From the results, it is observed that, the performance of the MLE method is better than the LSE method. When estimating α and θ, the bias MSE and decreases as the sample size increases and as the values of α and θ increases the bias and MSE deceases as well. It is also noted that, when both θ and β are unknown, the performance of the shape parameter α is less efficient. Furthermore, when both α and β are unknown the performance of the estimated values of θ is also less efficient which indicates that shape parameter β has an influence in the process of estimating both α or θ. Finally, when both α or θ. are unknown the performance of the estimators of β using both LSE or the MLE methods is quite satisfactory where the MLEs are performing better than the LSEs, while as the value of β increases the bias and MSEs increases.

Table 2: MLE of α when $\lambda=1$ and $\beta=3$ and θ is assumed unknown with population parameters ($\theta = 2$)

		LSE				MLE			
		n = 20	n = 35	n = 50	n = 100	n = 20	n = 35	n = 50	n = 100
α =0.1	$\hat{\alpha}$	0.2976	0.2259	0.2036	0.1649	0.2097	0.1592	0.1435	0.1162
	$\hat{\alpha}/\alpha$	2.9758	2.2592	2.0364	1.6490	2.0970	1.5920	1.4350	1.1620
	Bias	0.1976	0.1259	0.1036	0.0649	0.1097	0.0592	0.0435	0.0162
	MSE	2.4707	1.8473	1.3904	1.0612	1.7411	1.3018	0.9798	0.7478
α=0.5	$\hat{\alpha}$	0.6801	0.6635	0.6462	0.5981	0.5854	0.5711	0.5562	0.5148
	$\hat{\alpha}/\alpha$	1.3603	1.3270	1.2924	1.1962	1.1708	1.1422	1.1124	1.0296
	Bias	0.1801	0.1635	0.1462	0.0981	0.0854	0.0711	0.0562	0.0148
	MSE	1.3405	1.1713	0.9637	0.8233	0.9446	0.8254	0.6791	0.5802
α=1	$\hat{\alpha}$	1.1672	1.2092	1.1960	1.1831	1.0561	1.0408	1.0294	1.0183
	$\hat{\alpha}/\alpha$	1.1672	1.2092	1.1960	1.1831	1.0561	1.0408	1.0294	1.0183
	Bias	0.1672	0.2092	0.1960	0.1831	0.0561	0.0408	0.0294	0.0183
	MSE	1.2329	1.1073	0.8304	0.7646	0.8688	0.7803	0.5852	0.5388
α=3	$\hat{\alpha}$	3.1441	3.1374	3.1319	3.1298	3.0208	3.0144	3.0091	3.0071
	$\hat{\alpha}/\alpha$	1.0480	1.0458	1.0440	1.0433	1.0069	1.0048	1.0030	1.0024
	Bias	0.1441	0.1374	0.1319	0.1298	0.0208	0.0144	0.0091	0.0071
	MSE	0.9602	0.8624	0.6467	0.5955	0.6766	0.6077	0.4558	0.4196

Table 3: MLE of α when λ=1 and β and θ are assumed unknown with population parameters ($\beta = 3$, $\theta = 2$)

		LSE				MLE			
		n = 20	n = 35	n = 50	n = 100	n = 20	n = 35	n = 50	n = 100
α =0.1	$\hat{\alpha}$	0.3184	0.2391	0.2145	0.1717	0.2212	0.1654	0.1481	0.1179
	$\hat{\alpha}/\alpha$	3.1838	2.3914	2.1450	1.7173	2.2124	1.6543	1.4807	1.1790
	Bias	0.2184	0.1391	0.1145	0.0717	0.1212	0.0654	0.0481	0.0179
	MSE	0.2413	0.1538	0.1265	0.0793	0.1340	0.0723	0.0531	0.0198
α=0.5	$\hat{\alpha}$	0.6990	0.6807	0.6616	0.6084	0.5944	0.5786	0.5621	0.5164
	$\hat{\alpha}/\alpha$	1.3981	1.3614	1.3232	1.2168	1.1888	1.1572	1.1242	1.0327
	Bias	0.1990	0.1807	0.1616	0.1084	0.0944	0.0786	0.0621	0.0164
	MSE	0.2200	0.1997	0.1786	0.1198	0.1043	0.0868	0.0686	0.0181
α=1	$\hat{\alpha}$	1.1848	1.2312	1.2166	1.2024	1.0620	1.0451	1.0325	1.0202
	$\hat{\alpha}/\alpha$	1.1848	1.2312	1.2166	1.2024	1.0620	1.0451	1.0325	1.0202
	Bias	0.1848	0.2312	0.2166	0.2024	0.0620	0.0451	0.0325	0.0202
	MSE	0.2042	0.2555	0.2394	0.2236	0.0685	0.0498	0.0359	0.0224
α=3	$\hat{\alpha}$	3.1593	3.1519	3.1458	3.1435	3.0230	3.0159	3.0101	3.0078
	$\hat{\alpha}/\alpha$	1.0531	1.0506	1.0486	1.0478	1.0077	1.0053	1.0034	1.0026
	Bias	0.1593	0.1519	0.1458	0.1435	0.0230	0.0159	0.0101	0.0078
	MSE	0.3184	0.2391	0.2145	0.1717	0.2212	0.1654	0.1481	0.1179

Table 4: MLE of θ (θ =1) when λ=1 and β=3 and α is assumed unknown with population parameters (α = 1)

		LSE				*MLE*			
		n = 20	n = 35	n = 50	n = 100	n = 20	n = 35	n = 50	n = 100
α =0.1	$\hat{\theta}$	1.7809	1.6180	1.5018	1.4926	1.4581	1.3247	1.2296	1.2220
	$\hat{\theta}/\theta$	1.7809	1.6180	1.5018	1.4926	1.4581	1.3247	1.2296	1.2220
	Bias	0.7809	0.6180	0.5018	0.4926	0.4581	0.3247	0.2296	0.2220
	MSE	1.6516	1.6179	1.5643	1.4712	1.3522	1.3246	1.2807	1.2045
α =0.5	$\hat{\theta}$	1.6941	1.5391	1.4286	1.4198	1.3194	1.1987	1.1126	1.1057
	$\hat{\theta}/\theta$	1.6941	1.5391	1.4286	1.4198	1.3194	1.1987	1.1126	1.1057
	Bias	0.6941	0.5391	0.4286	0.4198	0.3194	0.1987	0.1126	0.1057
	MSE	1.5247	1.4958	1.4097	1.3381	1.1874	1.1649	1.0979	1.0421
α =2	$\hat{\theta}$	1.5329	1.3926	1.2926	1.2847	1.1938	1.0846	1.0583	1.0518
	$\hat{\theta}/\theta$	1.5329	1.3926	1.2926	1.2847	1.1938	1.0846	1.0583	1.0518
	Bias	0.5329	0.3926	0.2926	0.2847	0.1938	0.0846	0.0583	0.0518
	MSE	1.3076	1.1825	1.1269	1.1094	1.0184	0.9209	0.8776	0.8640
α =3	$\hat{\theta}$	1.2550	1.1402	1.0583	1.0518	1.0275	1.0109	1.0098	1.0051
	$\hat{\theta}/\theta$	1.2550	1.1402	1.0583	1.0518	1.0275	1.0109	1.0098	1.0051
	Bias	0.2550	0.1402	0.0583	0.0518	0.0275	0.0109	0.0098	0.0051
	θ=0.1	1.2286	1.1275	1.0958	1.0519	0.7834	0.7189	0.6987	0.6707

8 Data analysis

In this section, we illustrate performance of the proposed model with data obtained from Singfat [10] which represents the weight of the diamond stones in carat. The data set is

1.4575	0.3092	0.3642	0.0119	0.0664	2.6125	0.6027	0.1693	0.5894	0.1558
0.7701	0.0626	0.5350	0.1352	0.4024	0.2872	1.2177	2.6257	0.3954	0.4107

To examine the proposed model, we fit the Weibull distribution, the weighted exponential distribution, the Shahbaz's weighted Weibull distribution [3], and the proposed distribution for these data. We use the Kolmogorov-Smirnov (K-S) distances between the empirical distribution function and the fitted distribution function and the chi-squared goodness of fit test to determine the appropriateness of the models. The results are shown in Table 6.

From table 7 one can note that the proposed model is very close to Shahbaz's two parameter Weibull distribution and better than both Weibull and Weighted one parameter exponential distribution.

Table 4: MLE of θ (θ =1) when both α and β are assumed unknown with population parameters (α = 1, β = 3)

		LSE				MLE			
		n = 20	*n* = 35	*n* = 50	*n* = 100	*n* = 20	*n* = 35	*n* = 50	*n* = 100
α =0.1	$\hat{\theta}$	1.8630	1.6830	1.5546	1.5444	1.5063	1.3588	1.2537	1.2453
	$\hat{\theta}/\theta$	1.8630	1.6830	1.5546	1.5444	1.5063	1.3588	1.2537	1.2453
	Bias	0.8630	0.6830	0.5546	0.5444	0.5063	0.3588	0.2537	0.2453
	MSE	2.0173	1.9761	1.9106	1.7969	1.6516	1.6179	1.5643	1.4712
α =0.5	$\hat{\theta}$	1.7671	1.5958	1.4737	1.4640	1.3530	1.2196	1.1244	1.1168
	$\hat{\theta}/\theta$	1.7671	1.5958	1.4737	1.4640	1.3530	1.2196	1.1244	1.1168
	Bias	0.7671	0.5958	0.4737	0.4640	0.3530	0.2196	0.1244	0.1168
	MSE	1.8623	1.8270	1.7218	1.6344	1.4503	1.4228	1.3410	1.2728
α =2	$\hat{\theta}$	1.5889	1.4339	1.3234	1.3146	1.2142	1.0935	1.0644	1.0572
	$\hat{\theta}/\theta$	1.5889	1.4339	1.3234	1.3146	1.2142	1.0935	1.0644	1.0572
	Bias	0.5889	0.4339	0.3234	0.3146	0.2142	0.0935	0.0644	0.0572
	MSE	1.5971	1.4443	1.3764	1.3550	1.2439	1.1248	1.0719	1.0553
α =3	$\hat{\theta}$	1.2818	1.1549	1.0644	1.0572	1.0304	1.0120	1.0108	1.0056
	$\hat{\theta}/\theta$	1.2818	1.1549	1.0644	1.0572	1.0304	1.0120	1.0108	1.0056
	Bias	0.2818	0.1549	0.0644	0.0572	0.0304	0.0120	0.0108	0.0056
	θ=0.1	1.8630	1.6830	1.5546	1.5444	1.5063	1.3588	1.2537	1.2453

Table 6: MLE of β when λ=1 and both α and θ are assumed unknown with population parameters (α = 1, θ = 2)

		LSE				MLE			
		n = 20	*n* = 35	*n* = 50	*n* = 100	*n* = 20	*n* = 35	*n* = 50	*n* = 100
β=0.1	$\hat{\beta}$	0.4276	0.3192	0.2479	0.2075	0.3196	0.2469	0.1991	0.1721
	$\hat{\beta}/\beta$	4.2760	3.1920	2.4790	2.0750	3.1960	2.4693	1.9914	1.7206
	Bias	0.3276	0.2192	0.1479	0.1075	0.2196	0.1469	0.0991	0.0721
	MSE	0.4196	0.3287	0.2918	0.2019	0.2813	0.2203	0.1956	0.1353
β=0.5	$\hat{\beta}$	1.0896	1.0527	0.8971	0.7056	0.8236	0.8033	0.7179	0.6128
	$\hat{\beta}/\beta$	2.1792	2.1054	1.7942	1.4112	1.6472	1.6067	1.4359	1.2257
	Bias	0.5896	0.5527	0.3971	0.2056	0.3236	0.3033	0.2179	0.1128
	MSE	0.2032	0.2007	0.1612	0.1225	0.1232	0.1345	0.1081	0.0821
β=1	$\hat{\beta}$	1.6535	1.5849	1.4273	1.2256	1.3586	1.3210	1.2345	1.1238
	$\hat{\beta}/\beta$	1.6535	1.5849	1.4273	1.2256	1.3586	1.3210	1.2345	1.1238
	Bias	0.6535	0.5849	0.4273	0.2256	0.3586	0.3210	0.2345	0.1238
	MSE	0.3451	0.2712	0.1859	0.1347	0.2825	0.2220	0.1522	0.1103
β=3	$\hat{\beta}$	3.7102	3.6828	3.6549	3.5943	3.3898	3.3747	3.3594	3.3262
	$\hat{\beta}/\beta$	1.2367	1.2276	1.2183	1.1981	1.1299	1.1249	1.1198	1.1087
	Bias	0.7102	0.6828	0.6549	0.5943	0.3898	0.3747	0.3594	0.3262
	MSE	0.2914	0.2319	0.1793	0.1059	0.2386	0.1899	0.1468	0.0867

Table 7: Comparison between the different distributions in relation to the proposed model and in trems of the K-S values, and *p* value.

Distribution	**K-S**	**Anderson Darling**	**Chi-Squared**	***p* value**
Weibull	0.298795	2.8114	9.0521	0.11573
Weighted 1p Exponential	0.28756	2.4855	2.8529	0.11984
Saman's Weighted 2p Weibull	0.169785	0.4014	1.5866	0.28628
Weighted 3pWeibull	0.167652	0.39149	1.5074	0.30176

9 Conclusion

In this paper, we have introduced an extension to the weighted Weibull distribution. The proposed model is another generalization to the Weibull distribution. Moreover, the proposed distribution is flexible and can be used quite effectively to analyze positively skewed lifetime. Furthermore, characteristics of the proposed model have been introduced; namely the survival function, the failure rate function or hazard function. The moments and the moment generating function have been derived. For estimating the parameters of the *WTD*, LSE and MLE methods have been used and a real data example was studies. The results showed that the weighted three parameter weighted Weibull distribution is acting better that the Weibull and the weighted exponential distribution. Furthermore, its performance is slightly better than the weighted two parameter Weibull distribution.

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