Dynamics of Structures
Continuous Systems – Beam

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Part 2 – Approximate Solution

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Hamilton’s Principal

“ Among the several ways that a system can go from state A to B, the system will choose the way at which the action (energy) is minimum”

Thus, Hamilton’s Principal defines the system motion, if the system configuration is given at two-time t_1 and t_2.

As a result Hamilton’s Principle states that

$$\int_{t_1}^{t_2} \delta(\mathcal{L} + W)\, dt = 0$$

Where Hamilton’s Principal is equivalent to the Newton’s Law in integral form, and $\mathcal{L}$ denotes the <u>*Kinetic Potential Function*</u> $\mathcal{L} = T - U$.

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Lagrange’s Equations

Aversion of Hamilton’s Principal applied to discrete systems in terms of generalized coordinates,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i, \quad i = 1,2, \ldots$$

q_i is the generalized coordinates and Q_i is the generalized force.

Which can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = \frac{\partial W_{ex}}{\partial q_i}, \quad i = 1,2, \ldots$$

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The Rayleigh-Ritz Method

Rayleigh-Ritz method can be used to transform a continuous structure into discrete one by assuming the system response as,

$$w(y,t) = \sum_{j=1}^{n} \psi_j(y)\, q_j(t)$$

Where $\psi_j(y)$ are selected functions that are continuous and satisfy the geometric constraints *i.e.*

- Any rigid-body motion
- Symmetric conditions
- Essential BC's

The functions satisfies these conditions are called "admissible functions"

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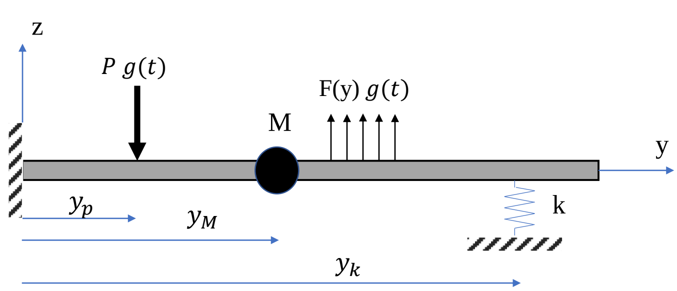
The Rayleigh-Ritz Method

Assume the general beam shown in figure

$$m_{ij} = \int_0^l m\, \psi_i\, \psi_j dy + \sum_n M_n \psi_i(y_M)\, \psi_j(y_M)$$

$$k_{ij} = \int_0^l EI\, \psi''_i\, \psi''_j dy + \sum_n k_n \psi_i(y_k)\, \psi_j(y_k)$$

$$F_i(t) = \left[\int_0^l F(y)\psi_i dy + \sum_n P_n \psi_i(y_p)\right] g(t)$$



$$\mathbf{m\ddot{q} + kq = F}$$

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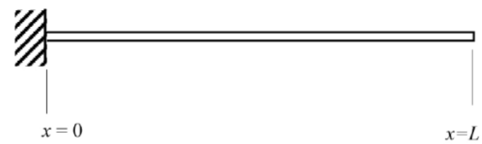
Admissible Functions - Examples

Cantilever beam



Geometric constraints:

At $x = 0 \; f = 0$

At $x = 0 \; df / dx = f' = 0$

Admissible functions:

$$f(x) = (x/L)^i \text{ for } i = 2, 3, 4 \ldots$$

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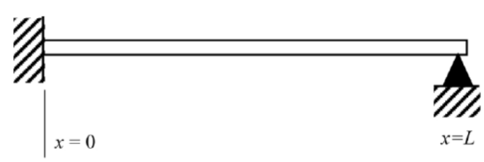
Admissible Functions - Examples

Propped cantilever beam



Geometric constraints:

At $x = 0 \; f = 0$

At $x = 0 \; df / dx = f' = 0$

At $x = L \; f = 0$

Admissible functions:

$$f(x) = (x/L)^i (1 - (x/L)) \text{ for } i = 2, 3, 4 \ldots$$

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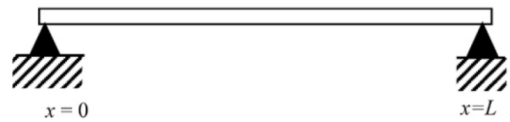
Admissible Functions - Examples

Pinned-pinned beam



Geometric constraints:

At $x = 0$ $f = 0$

At $x = L$ $f = 0$

Admissible functions:

$f(x) = (x/L)^i(1-(x/L))$ for $i = 1, 2, 3, 4...$

$f(x) = \sin(i\pi x/L)$ for $i = 1, 2, 3, 4...$

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Admissible Functions - Examples

Restrained cantilever

Geometric constraints:

At $x = 0$ $f = 0$

At $x = 0$ $df/dx = f' = 0$

Admissible functions:

$f(x) = (x/L)^i$ for $i = 2, 3, 4...$

Same as for a cantilever. The restraint is not a full constraint.

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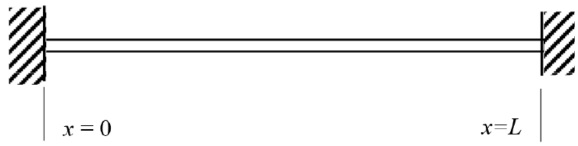
Admissible Functions - Examples

Clamped-clamped beam



Geometric constraints:

At $x = 0$ $f = 0$

At $x = 0$ $df / dx = f' = 0$

At $x = L$ $f = 0$

At $x = L$ $df / dx = f' = 0$

Admissible functions:

$f(x) = (1 - \cos(2i\pi x / L))$ for $i = 1, 2, 3, 4...$

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Admissible Functions - Examples

Rigid body motion $\varphi_i(x) = 1$, for $i = 1$

Linear deformation $\varphi_i(x) = \left(\frac{x}{L}\right)$, for $i = 2$

Quadratic deformation $\varphi_i(x) = \left(\frac{x}{L}\right)^2$, for $i = 3$

Sin or Cosin $\varphi_i(x) = \cos\frac{(i-3)\pi x}{L}$, for $i = 4, 5, \ldots n$

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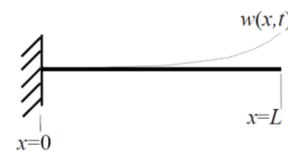
Example – Dynamic analysis of cantilever beam

Consider the lateral vibration of a cantilever beam of length L, mass per unit length m and flexural rigidity EI as shown in Figure



Let the lateral dynamic displacement be $w(x,t)=f(x)\sin(\omega t+\phi)$, where

$$f(x) = G_1\,(x/L)^2 + G_2\,(x/L)^3$$

Both functions $(x/L)^2$ and $(x/L)^3$ are admissible, since they are continuous for $0\le x\le L$ and satisfy the geometric constraints that $f(0) = 0$ and $f'(0) = 0$.

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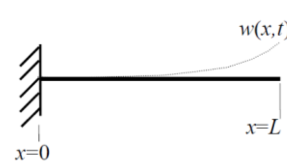
Example – Dynamic analysis of cantilever beam

The first step is to transform the continuous beam, into equivalent discrete model using Lagrange's equation and Ritz approximation. So we need to determine both the mass and stiffness matrices to write the model in the form $\mathcal{M}\ddot{\mathbf{x}} + \mathbf{Kx} = \mathbf{F}$

$$m_{ij} = \int_0^l m\,\psi_i\,\psi_j dx + \sum_n M_n \psi_i(y_M)\,\psi_j(y_M)$$

No concentrated masses in the present problem, i.e M = 0. Then we can write the mass matrix in matrix form,



$$\mathcal{M} = \int_0^L m\,\boldsymbol{\psi}^T\boldsymbol{\psi}\,dy \qquad \text{Where,} \qquad \boldsymbol{\psi} = \left[\left(\frac{x}{L}\right)^2 \quad \left(\frac{x}{L}\right)^3\right]$$

$$\mathcal{M} = m\int_0^L \begin{bmatrix} \left(\frac{x}{L}\right)^4 & \left(\frac{x}{L}\right)^5 \\ \left(\frac{x}{L}\right)^5 & \left(\frac{x}{L}\right)^6 \end{bmatrix} dx = mL\begin{bmatrix} \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

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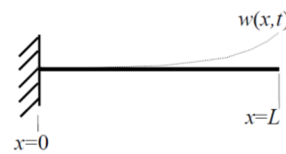
Example – Dynamic analysis of cantilever beam

The stiffness matrix can be calculated as

$$k_{ij} = \int_0^l EI\, \psi''_i\, \psi''_j dx + \sum_n k_n \psi_i(y_k)\, \psi_j(y_k)$$



No springs are attached to the beam, so k = 0, the stiffness equation in matrix form

$$\mathcal{K} = \int_0^L EI \boldsymbol{\psi}''^T \boldsymbol{\psi}''\, dx \qquad \text{Where,} \qquad \boldsymbol{\psi}'' = \begin{bmatrix} \frac{2}{L^2} & \frac{6}{L^3}x \end{bmatrix}$$

$$\mathcal{K} = EI \int_0^L \begin{bmatrix} \frac{4}{L^4} & \frac{12}{L^5}x \\ \frac{12}{L^5}x & \frac{36}{L^6}x^2 \end{bmatrix} dx = \frac{EI}{L^3}\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}$$

Then the natural frequencies and mode shapes can be obtained as we did in the discrete system

Homework – Problem 2

A steel simply supported beam of diameter 2 cm and length 1 m is subjected to an exponentially decaying force $\mathrm{F} = 100e^{-0.1t}$N at the middle of the beam, as shown in Figure. Assume the density $\rho = 7500\,\frac{kg}{m^3}$ and Young's modulus of steel as $210x10^9\,\frac{N}{m^2}$, respectively. Determine:

1. the beam natural frequencies and modes using exact, closed form solution method.
2. the beam natural frequencies and modes using Lagrange's approximation. Compare them with the exact solution.
3. the beam response using Lagrange's approximation