System Dynamics
Time Response

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System response

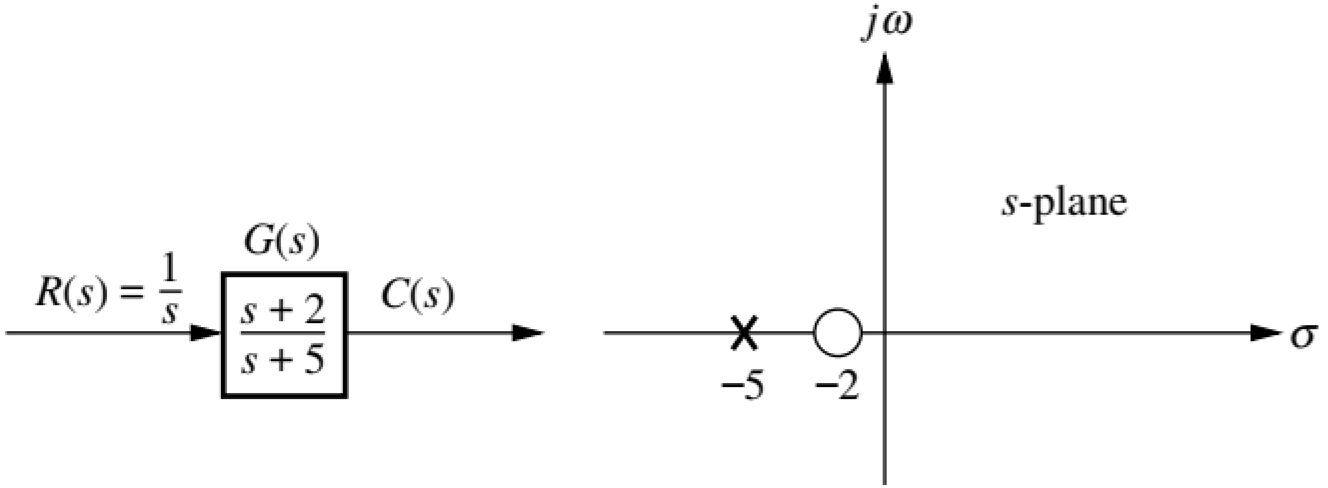
- The objective of the previous two chapters is to determine the system mathematical model.
- The next step is to study the system transient and steady state responses.
- The output response of a system is the sum of two responses: the forced response and the natural response.

System response – Definitions

<u>Poles of a Transfer Function</u> – the value of "*s*" that causes the transfer function to become infinite.

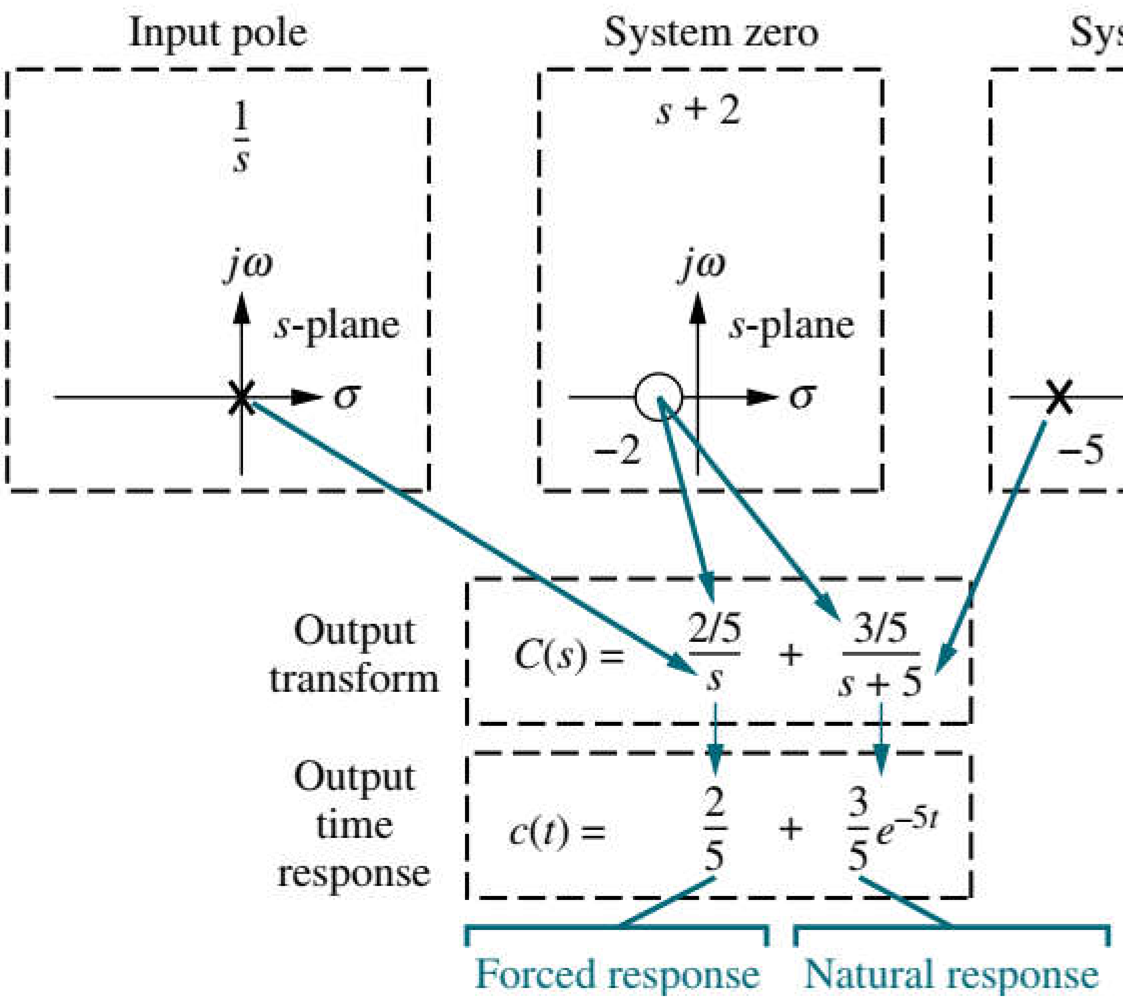
<u>Zeros of a Transfer Function</u> – the value of "*s*" that causes the transfer function to become zero.

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$



$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

System response – Definitions

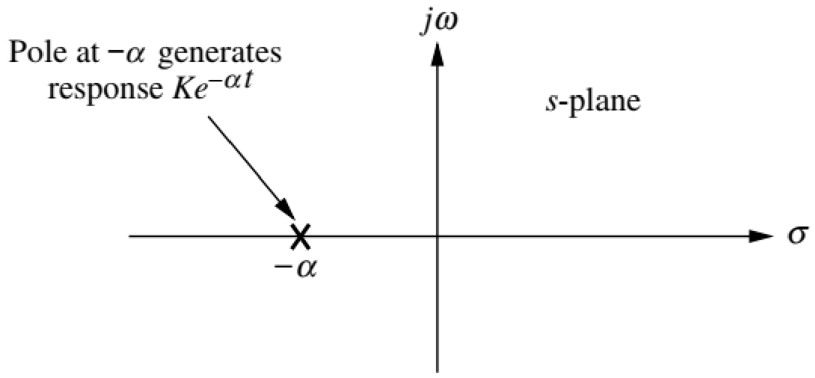


System response – Definitions

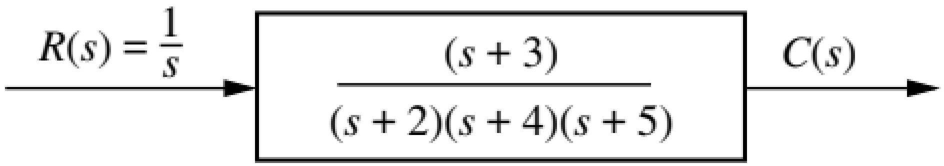
From the previous example, we can conclude that

- A *pole of the input* function generates the form of the forced response.
- A *pole of the transfer function* generates the form of the natural response.
- A *pole on the real axis* generates an exponential response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis.
- Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.

- The zeros and poles generate the amplitudes for both the forced and natural responses.



System response – Definitions



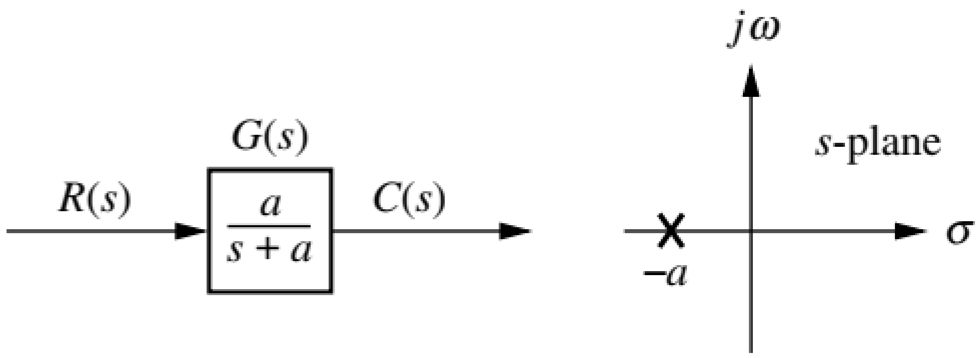
$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_{2e^{-2t}} + K_{3e^{-4t}} + K_{4e^{-5t}}}_{\text{Natural response}}$$

System response – First order system

A first-order system without zeros can be described by the transfer function shown

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$



Taking the inverse transform, the step response is given by

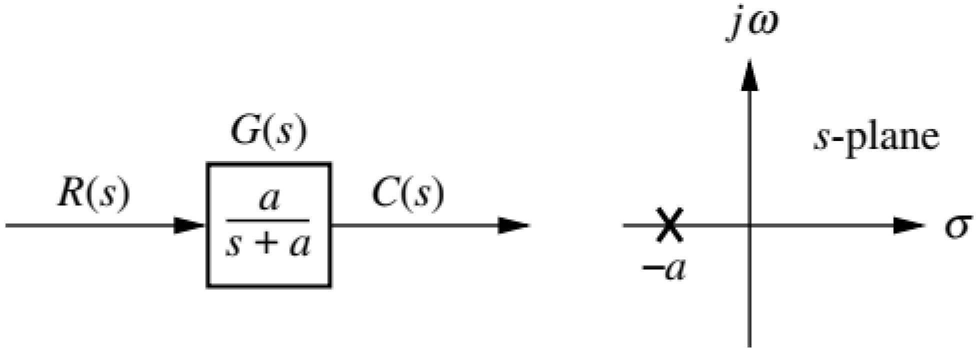
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

System response – First order system

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

Forced response

natural response



The parameter "a" is the only parameter needed to describe the first order system transient response.

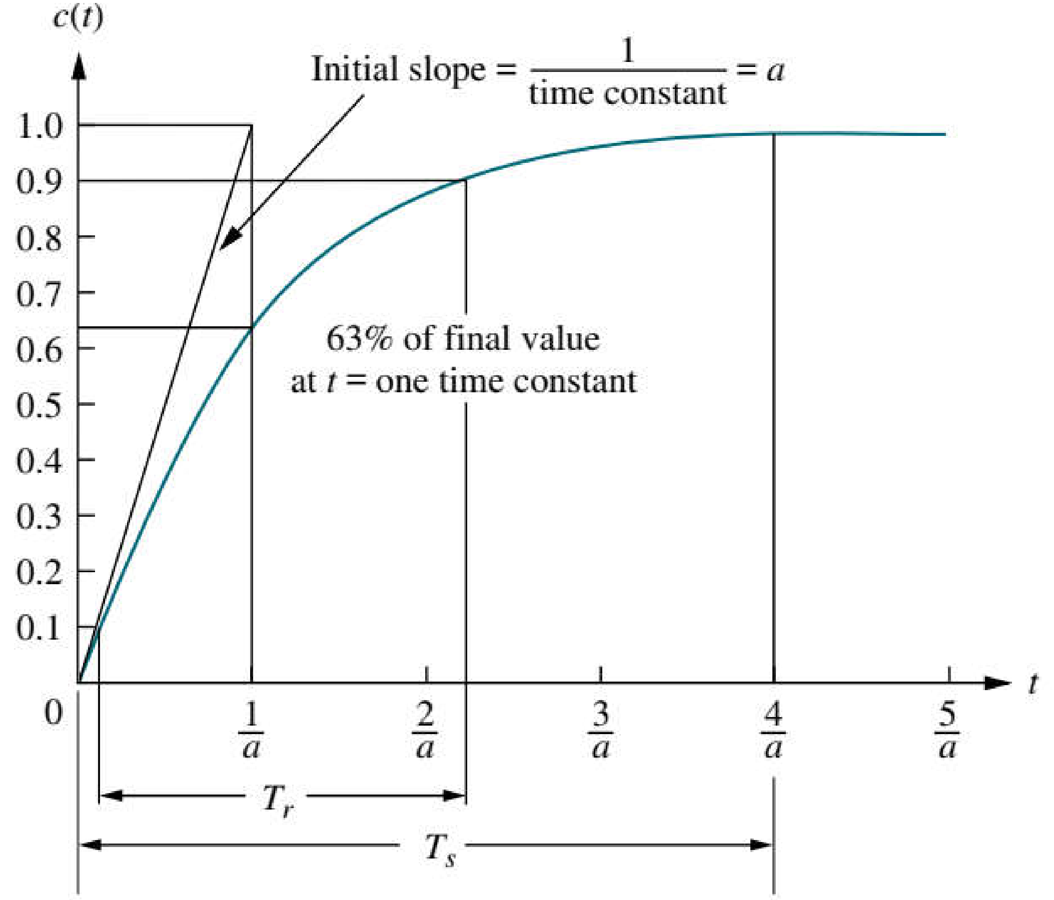
System response – First order system

Time Constant – We call *1/a* the time constant of the response. It is the time for e^{-at} to decay to 37% of its initial value Or the time it takes for the step response to rise to 63% of its final value

When $t = 1/a$,

$$e^{-at}|_{t=1/a} = e^{-1} = 0.37$$

$$c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63$$



$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

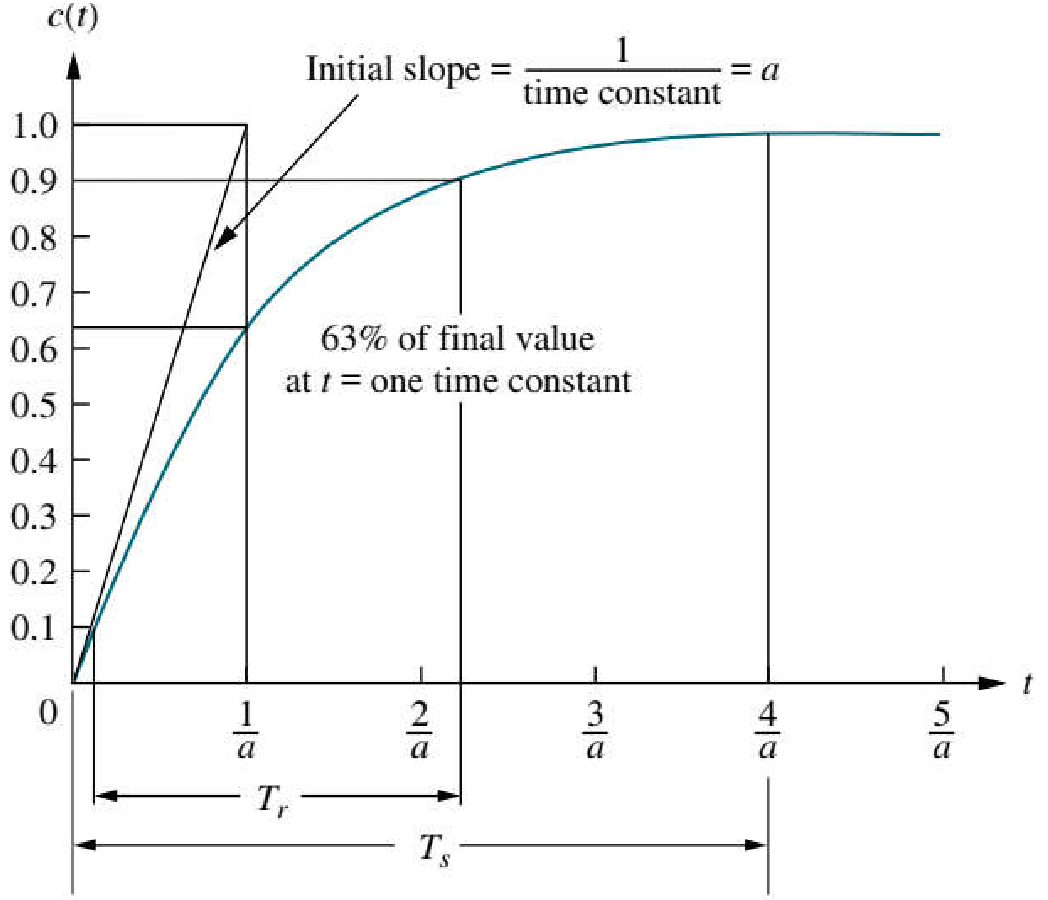
System response – First order system

Rise Time, T_r – the time for the waveform to go from 0.1 to 0.9 of its final value.

It can be obtained by solving for

$$c(t) = 0.9 \; and \; c(t) = 0.1$$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$



$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

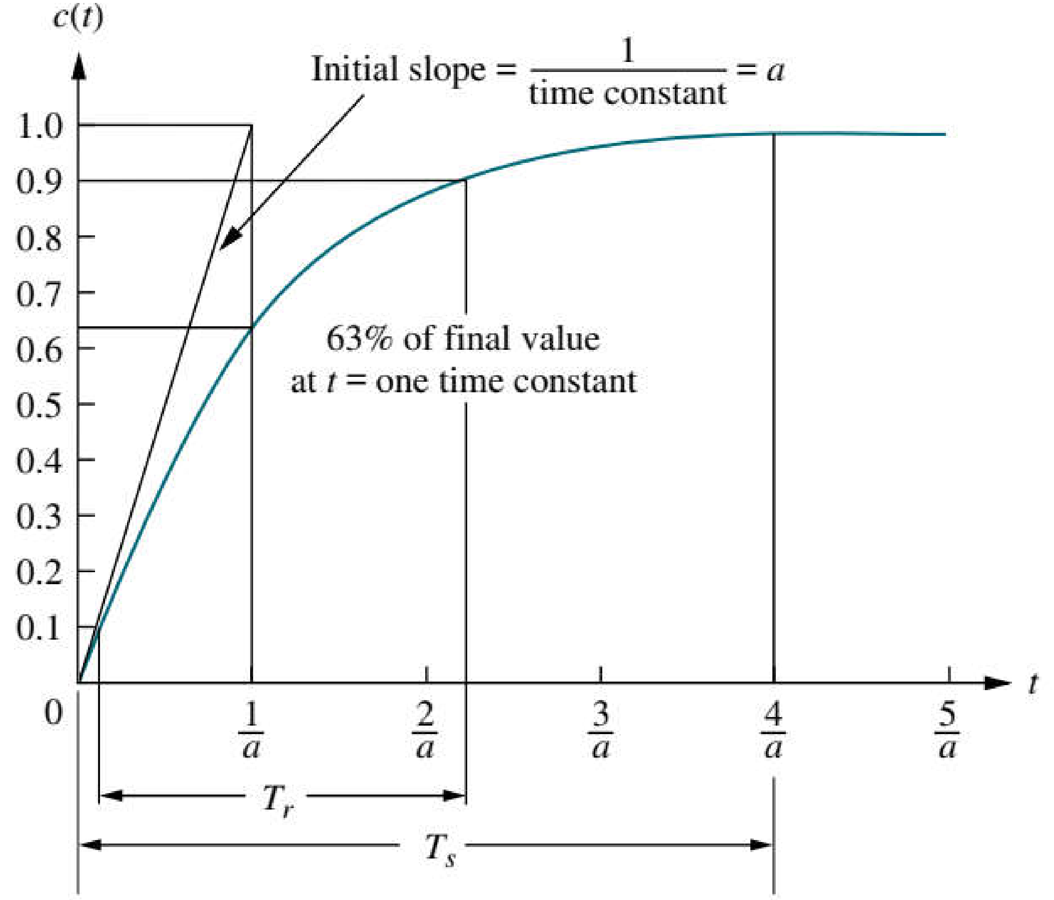
System response – First order system

Settling Time, T_r – the time for the response to reach, and stay within, 2% of its final value.

It can be obtained by solving for

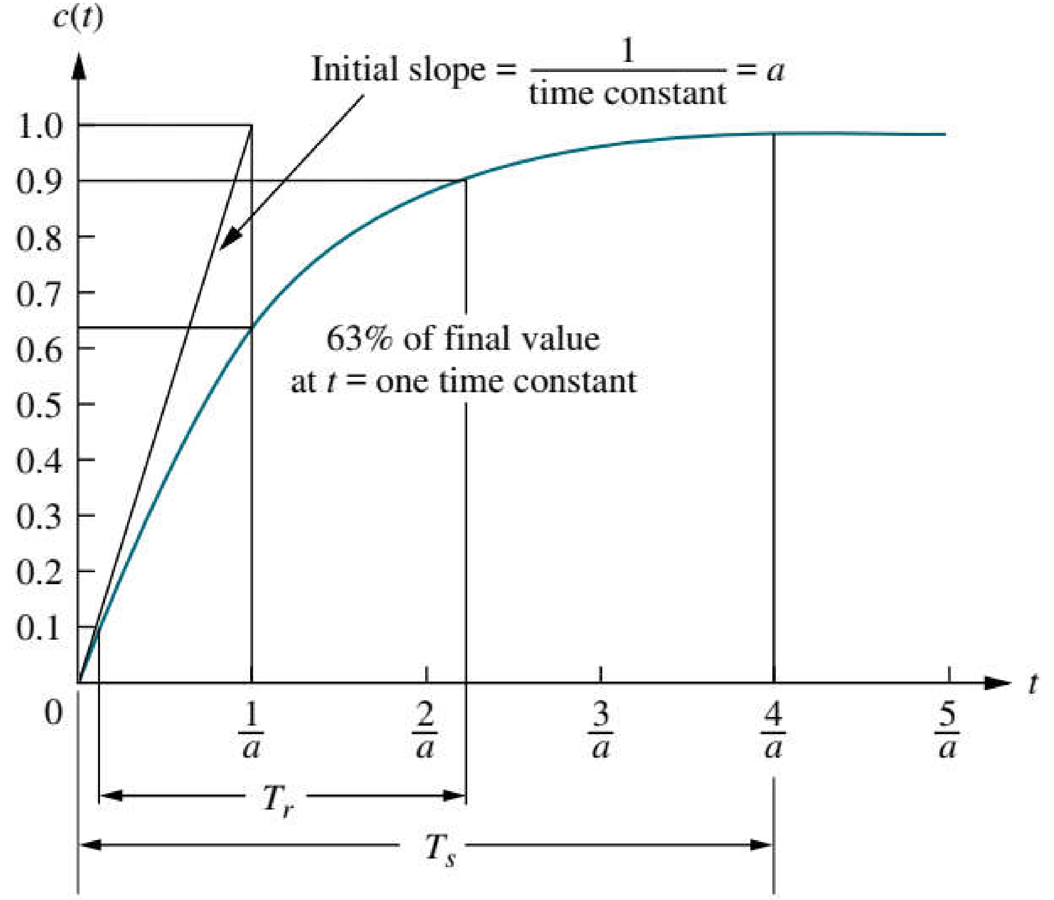
$$c(t) = 0.98$$

$$T_s = \frac{4}{a}$$



$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

System response – First order system



$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

First-Order Transfer Functions via Testing

- Often it is not possible to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- The system's step response can lead to the system performance.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

System response – First order system Example

First-Order Transfer Functions via Testing

Consider a simple first-order system, $G(s) = K/(s+a)$

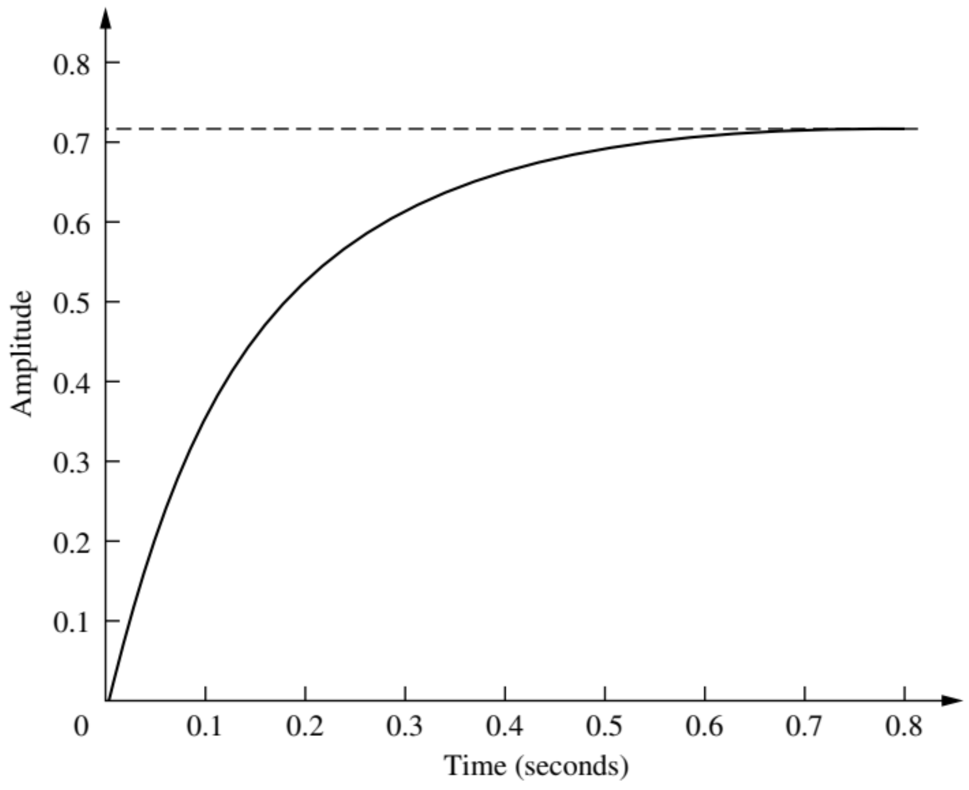
$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

1. We determine that it has the first-order characteristics when there is no overshoot and nonzero initial slope.

2. From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value.

$$c(T_s) = 0.63 * 0.72 = 0.45 \;\Rightarrow T_s = 0.13\ sec.$$

Hence, a = 1/0.13 = 7.7



System response – First order system

First-Order Transfer Functions via Testing

Consider a simple first-order system, $G(s) = K/(s+a)$

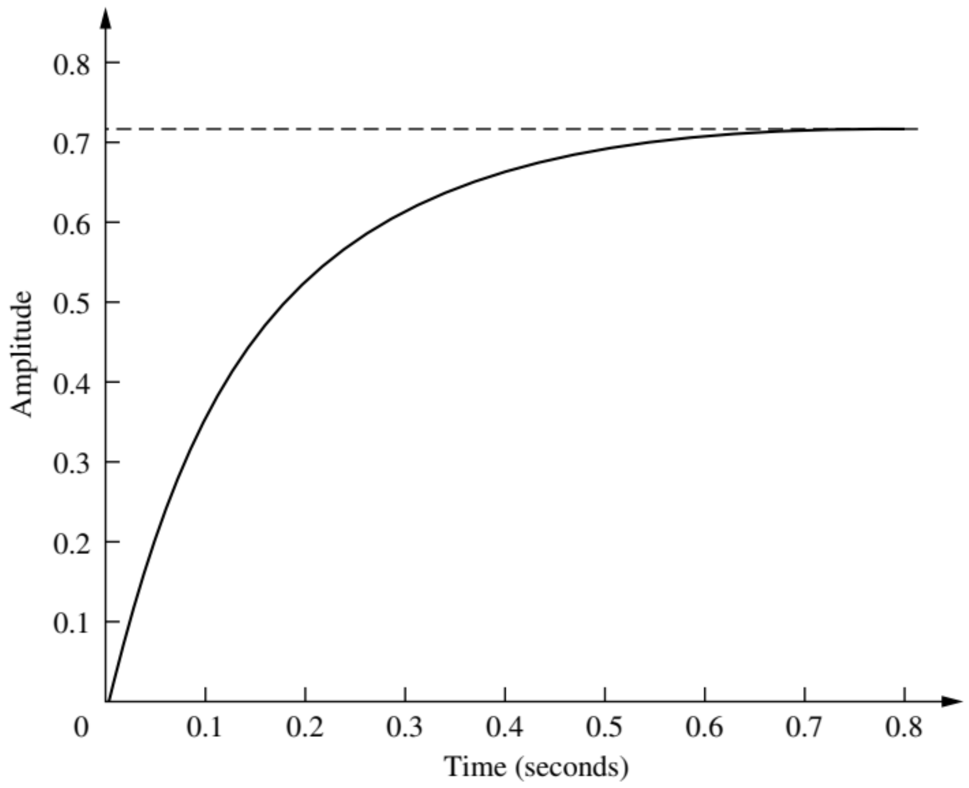
$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

3. *The forced response reaches a steady state value at*

$$\frac{K}{a} = 0.72 \Rightarrow K = 5.54$$

4. The system T.F. is

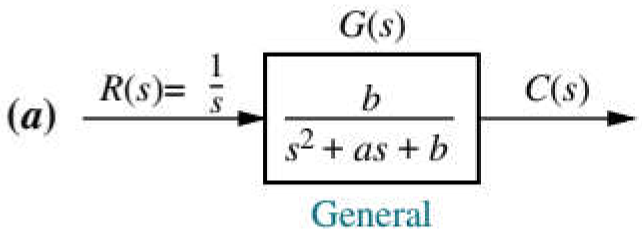
$$G(s) = \frac{5.54}{s+7.7}$$

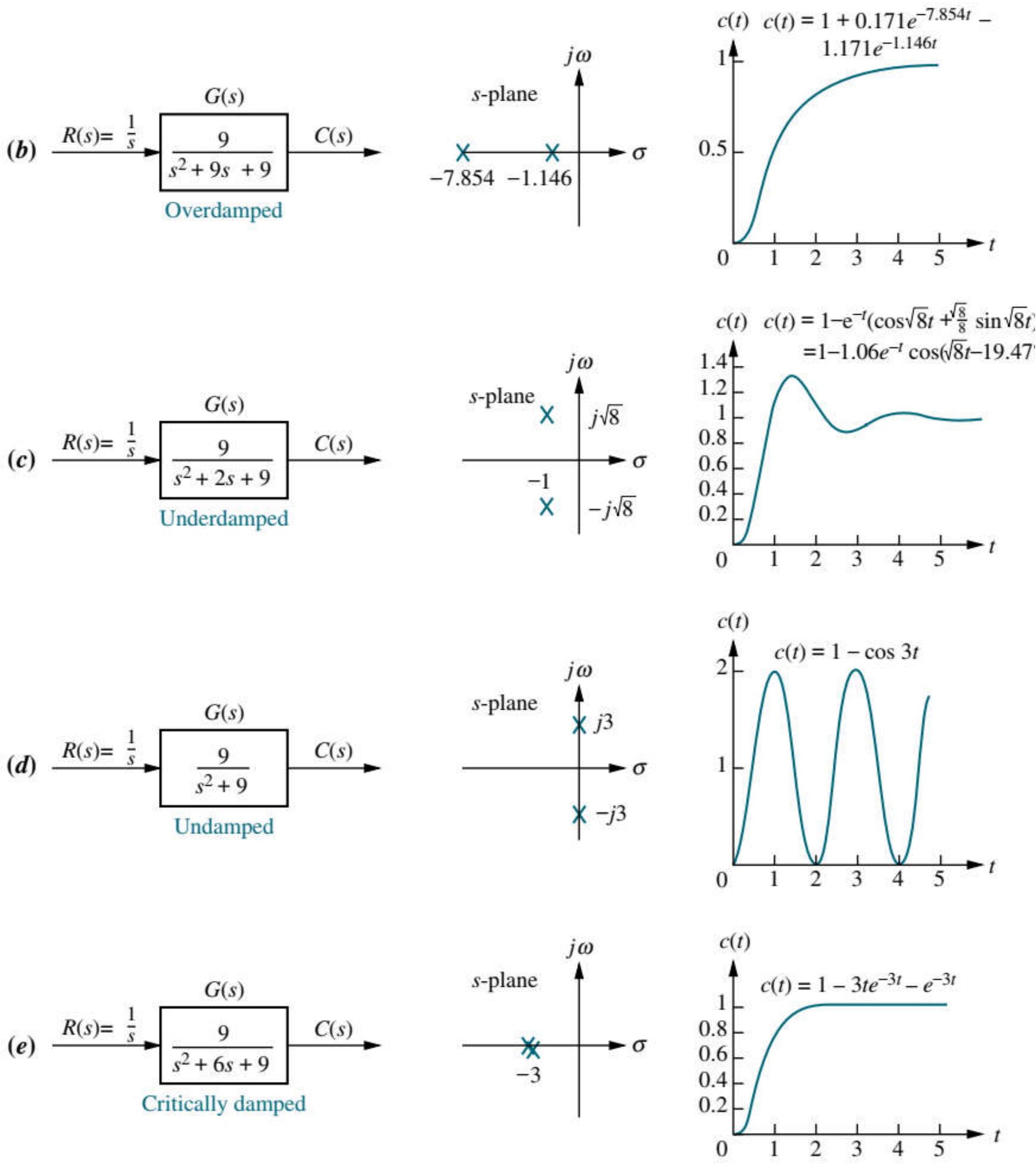


System response – Second order system

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described (shown in Figure).

- A second-order system can display characteristics much like a first-order system, or, depending on component values, display damped or pure oscillations for its transient response.



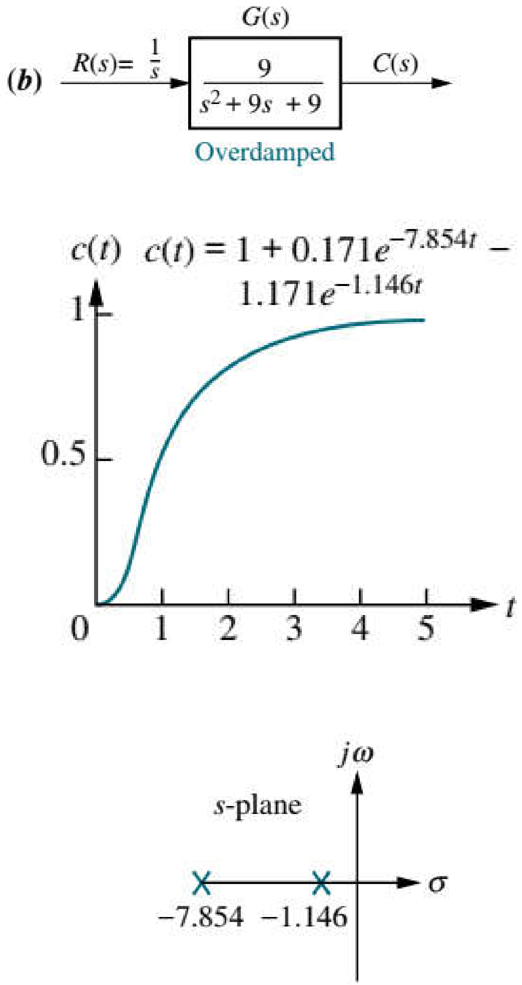


System response – Second order system
1. *Overdamped response*

- This function has a pole at the origin that comes from the unit step input and two real poles that come from the system.
- The input pole at the origin generates the constant forced response.
- Each of the two system poles on the real axis generates an exponential natural response whose exponential frequency is equal to the pole location.

$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$$

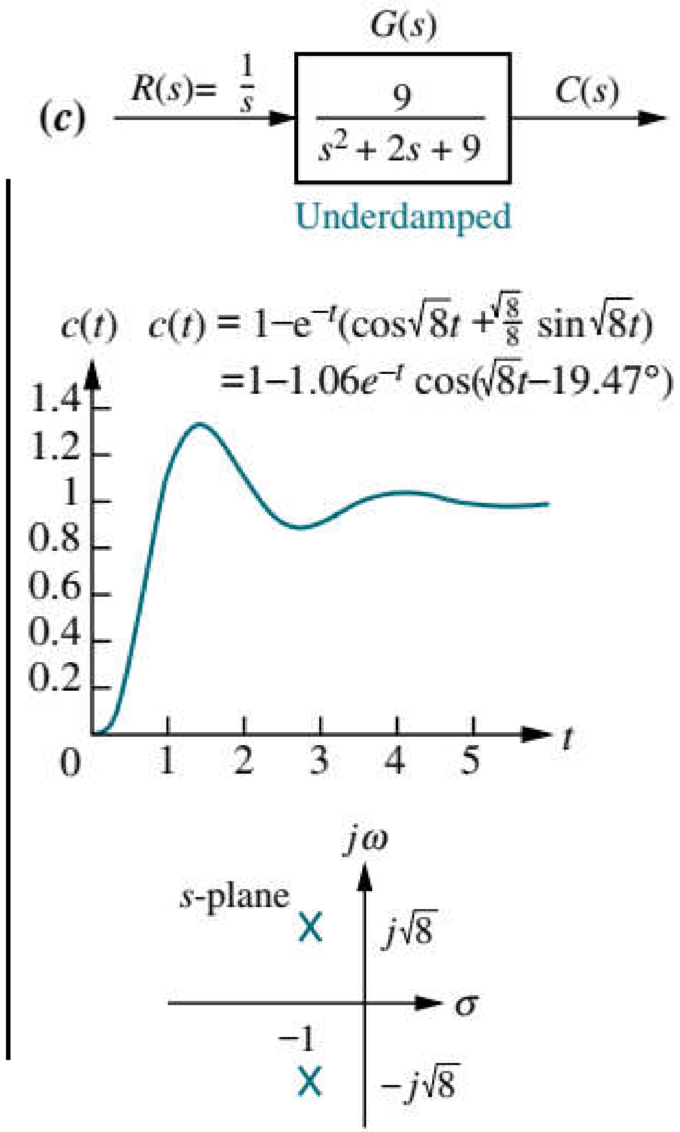
So, one can determine the system response and nature without solving the governing equation.



System response – Second order system
2. *Underdamped response*

- This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system.

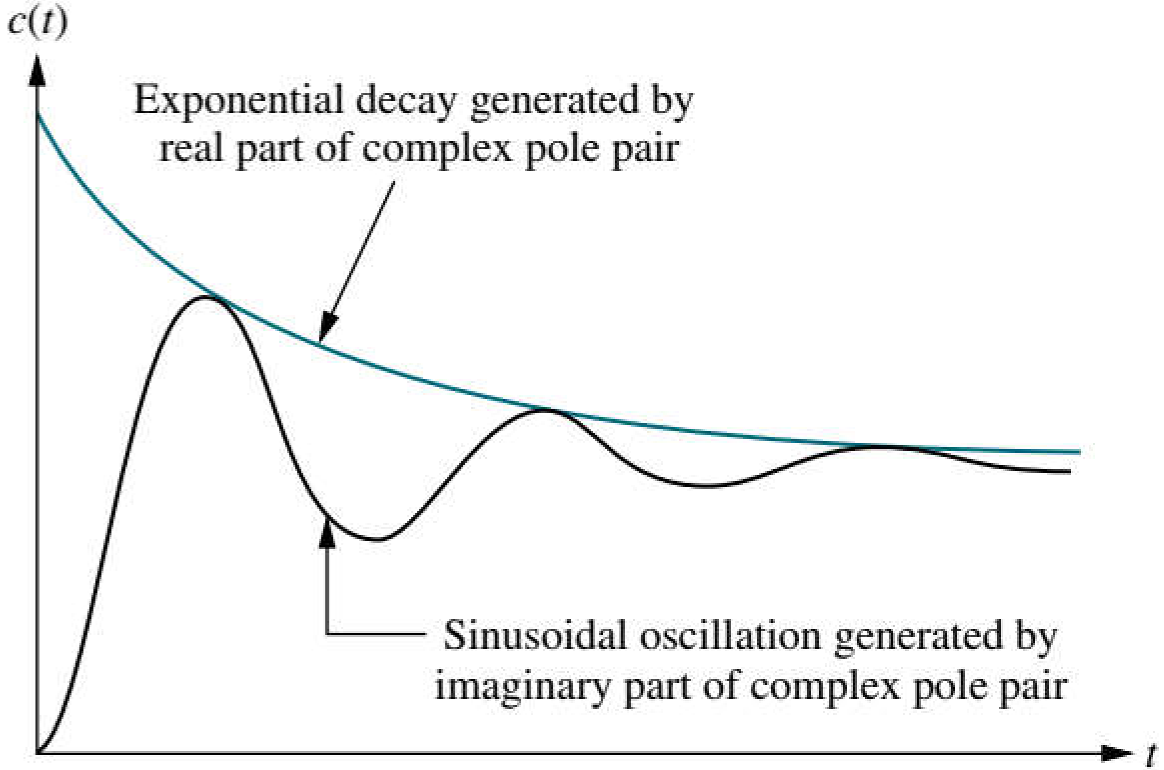
$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$

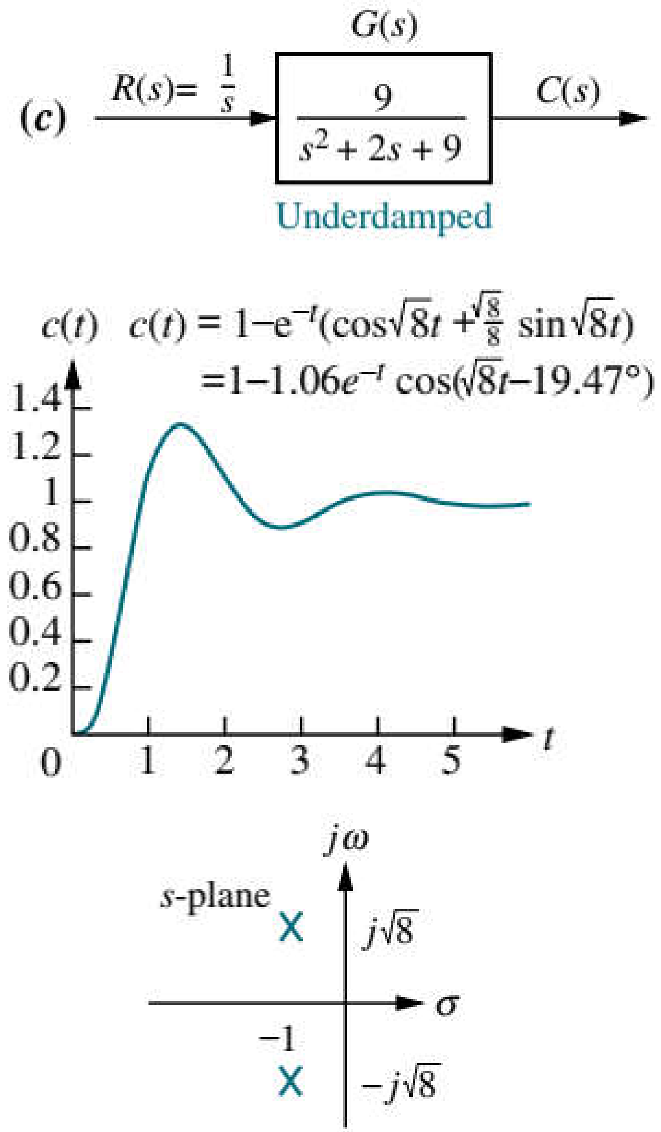


System response – Second order system
2. *Underdamped response*

- The real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

damped sinusoidal response for a second order system.





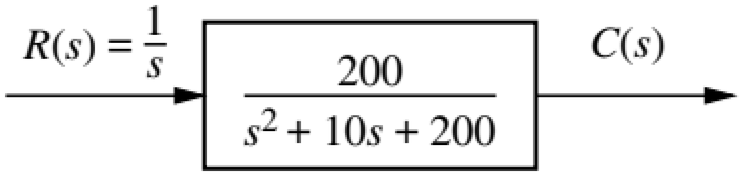
System response – Second order system
2. *Underdamped response*

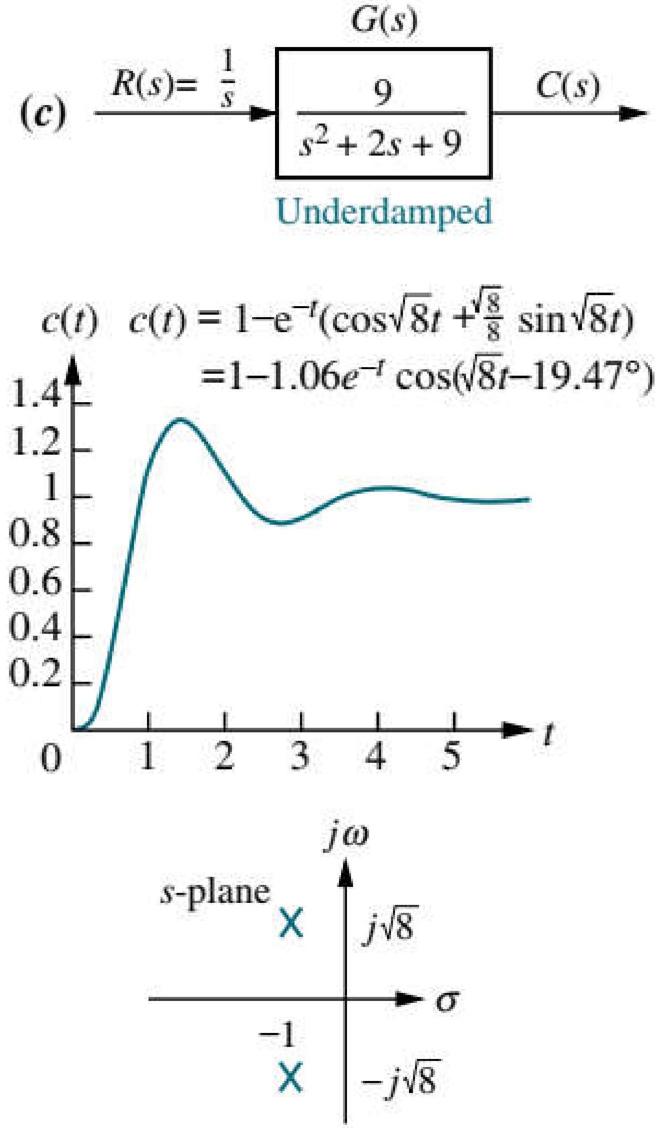
Steps to determine the under damped response:

1. Factoring the denominator of the transfer function in the shown Figure.

$$s = -5 \pm j13.23$$

2. The real part, -5, is the exponential frequency for the damping. It is also the reciprocal of the time constant of the decay of the oscillations.





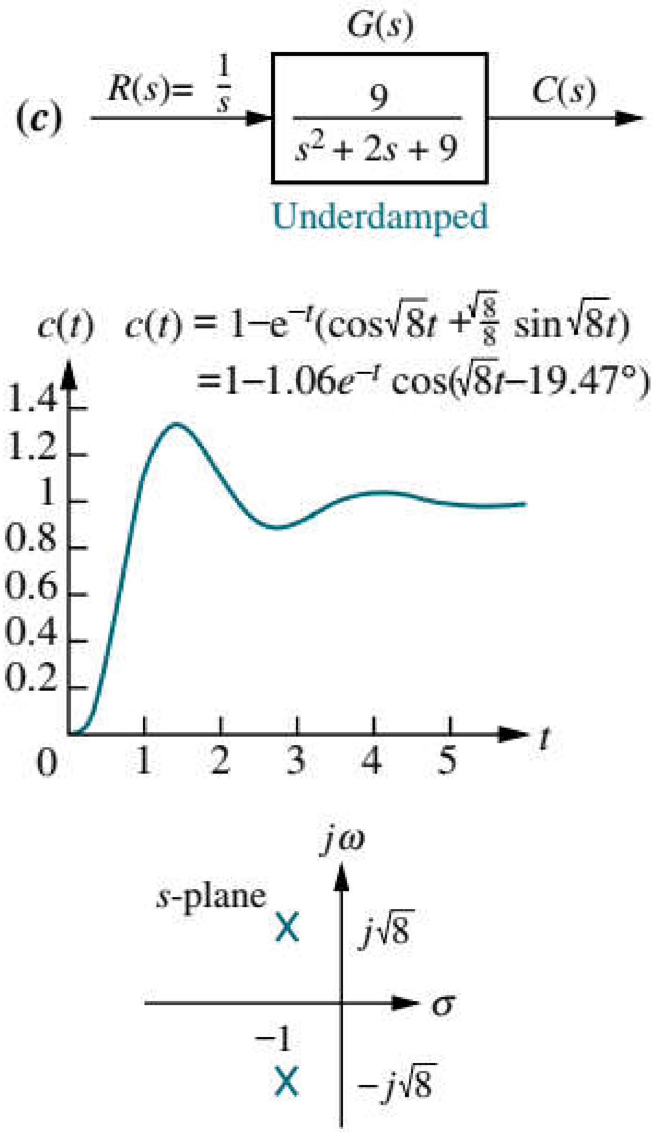
System response – Second order system
2. *Underdamped response*

Steps to determine the under damped response:

3. The imaginary part, 13.23, is the radian frequency for the sinusoidal oscillations.

$$c(t) = K_1 + e^{-5t}(K_2 \cos 13.23t + K_3 \sin 13.23t) = K_1 + K_4 e^{-5t}(\cos 13.23t - \phi),$$

$$\phi = \tan^{-1} K_3/K_2,\ K_4 = \sqrt{K_2^2 + K_3^2},$$



System response – Second order system

3. *Undamped response*

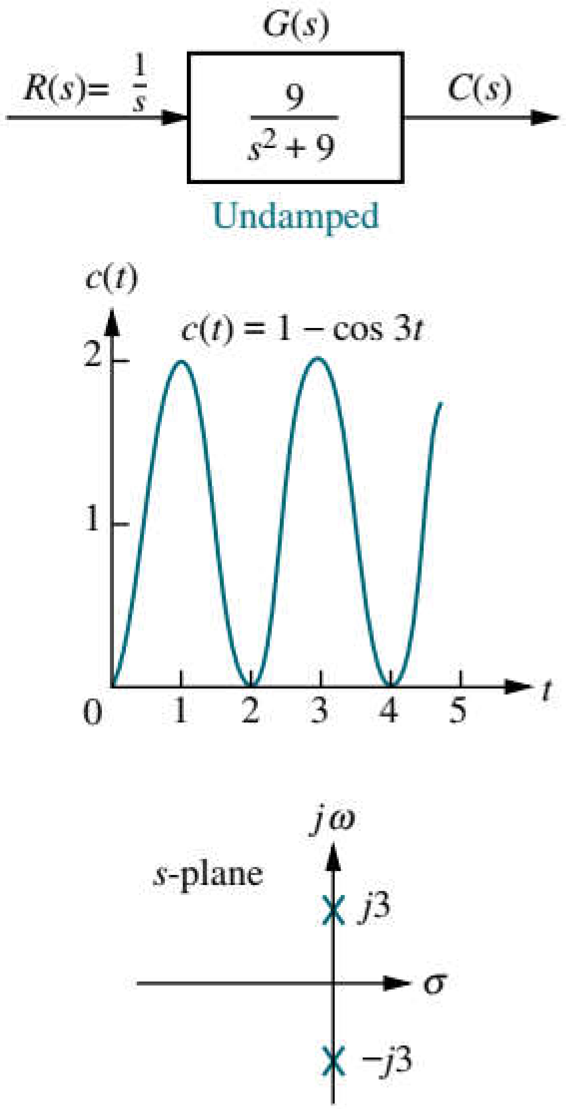
- This function has a pole at the origin that comes from the unit step input and two imaginary poles that come from the system.

- The input pole at the origin generates the constant forced response, and the two system poles on the imaginary axis at $\mp 3j$ generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles.

$$c(t) = K_1 + K_4 \cos(3t - \phi)$$

The response does not decay because of the absence of the real part. $e^{-0t} = 1.$

$$C(s) = \frac{9}{s(s^2+9)}$$



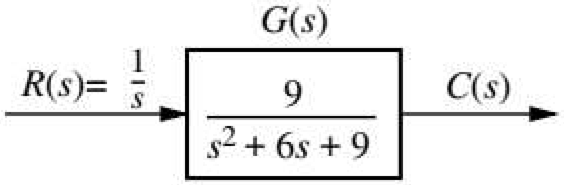
System response – Second order system

4. *Critically damped response*

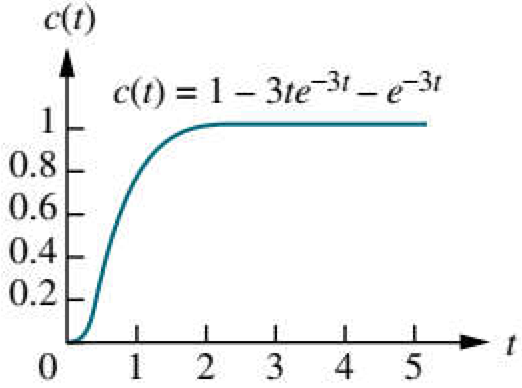
- This function has a pole at the origin that comes from the unit step input and two multiple real poles that come from the system.
- The input pole at the origin generates the constant forced response, and the two poles on the real axis at 3 generate a natural response consisting of an exponential and an exponential multiplied by time.
- the exponential frequency is equal to the location of the real poles.

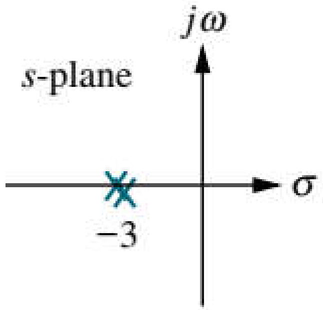
$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2}$$

$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}.$$



Critically damped

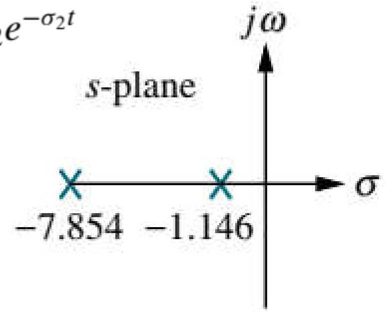




System response – Second order system All together

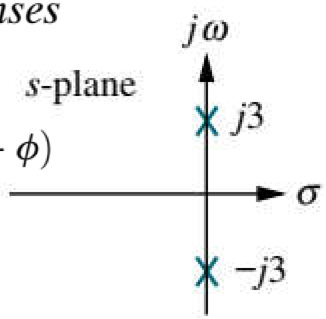
Overdamped responses

$c(t) = K_1e^{-\sigma_1 t} + K_2e^{-\sigma_2 t}$



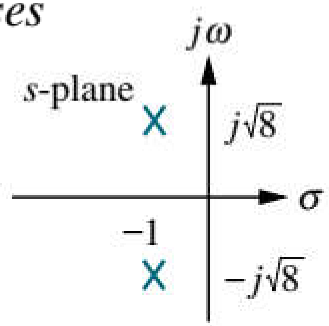
Undamped responses

$c(t) = A\cos(\omega_1 t - \phi)$

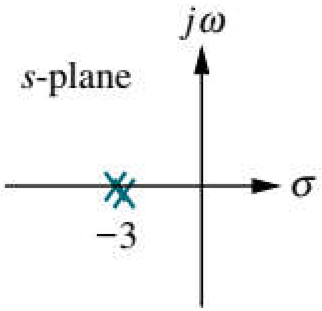


Underdamped responses

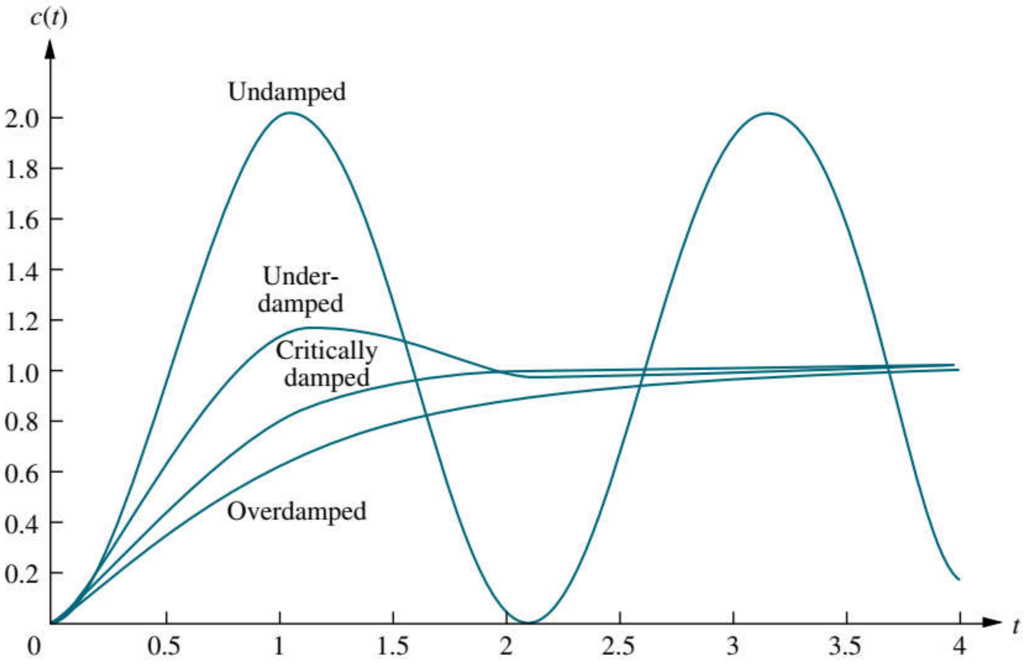
$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$



Critically damped responses



$c(t) = K_1e^{-\sigma_1 t} + K_2te^{-\sigma_1 t}$



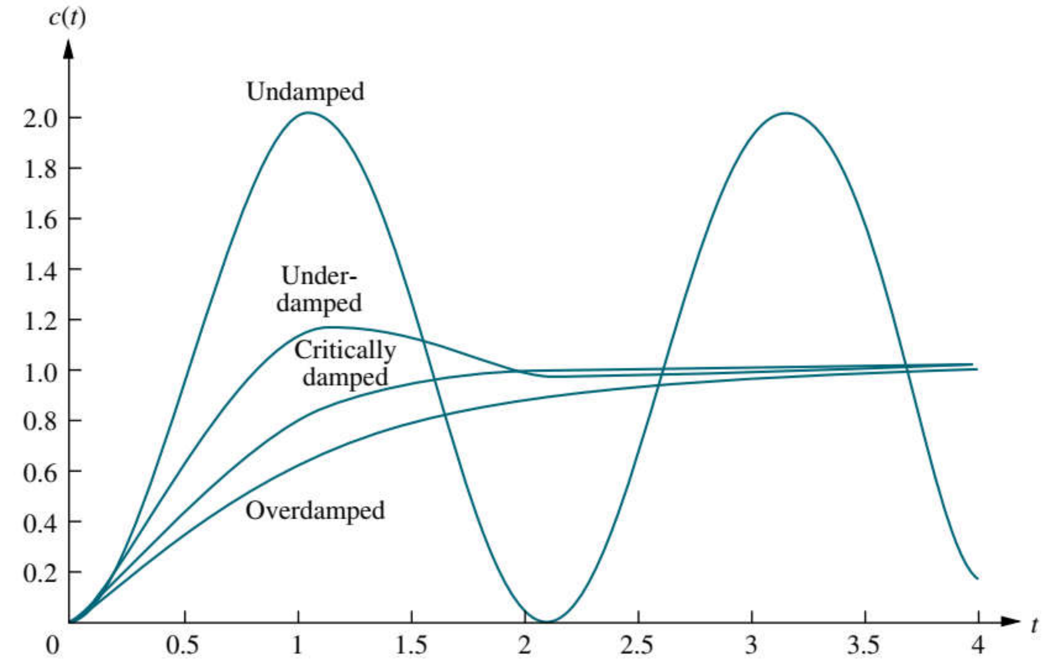
System response – Second order system

There are mainly two physical quantities have been used to descript the second order systems:

- Natural frequency, ω_n: is the frequency of oscillation of the system without damping.

- Damping ration, ξ:

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}$$



System response – Second order system

There are mainly two physical quantities have been used to descript the second order systems:

- Natural frequency, ω_n: is the frequency of oscillation of the system without damping.

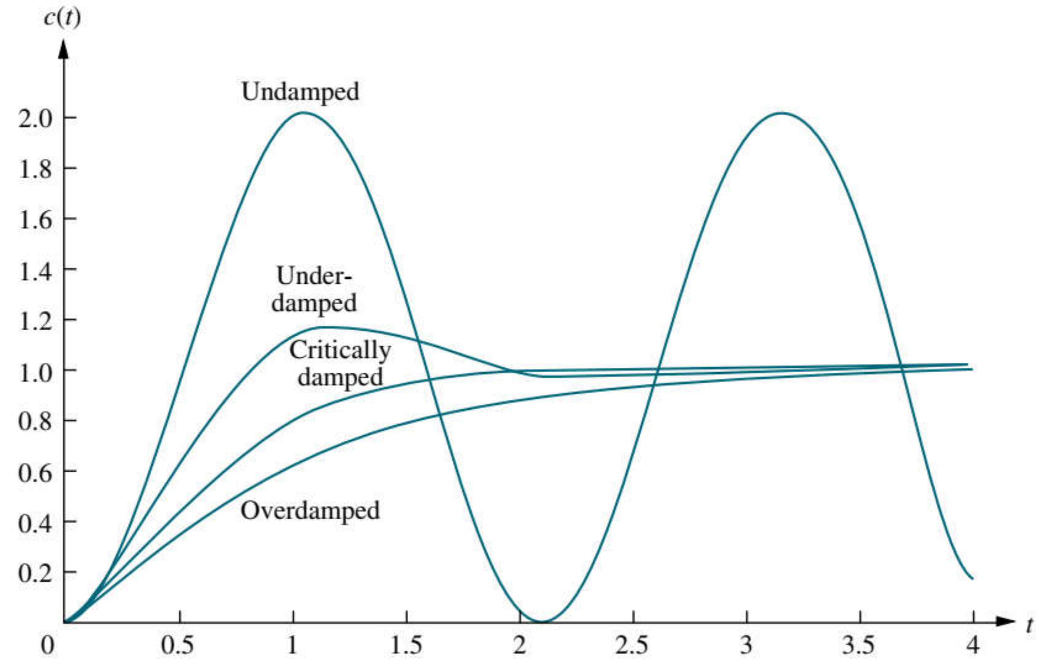
- Damping ration, ξ:

System equation in terms of the natural frequency and damping coefficient.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

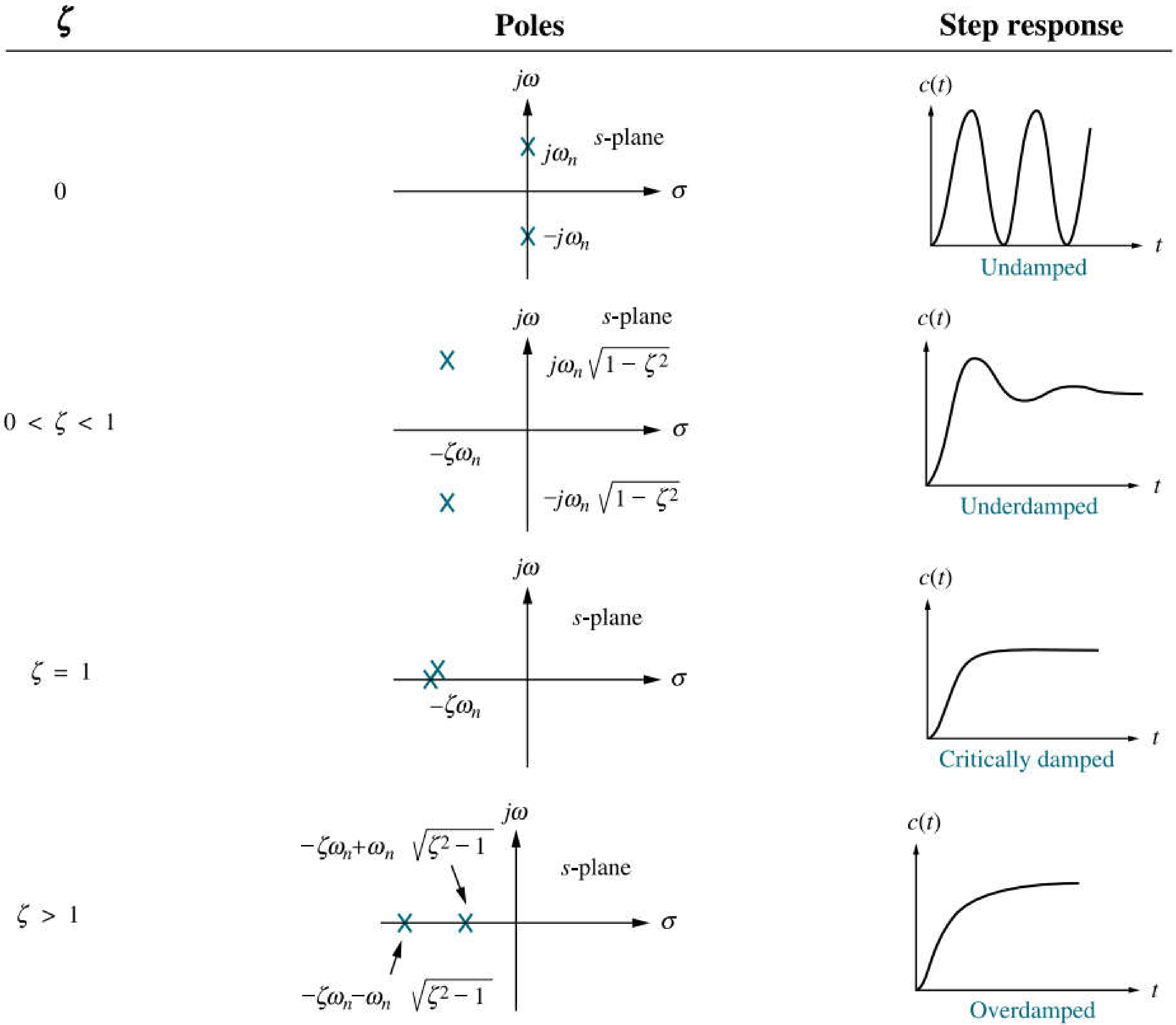
The system poles are

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



$$G(s) = \frac{b}{s^2 + as + b}$$

System response – Second order system



Underdamped Second-Order Systems

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

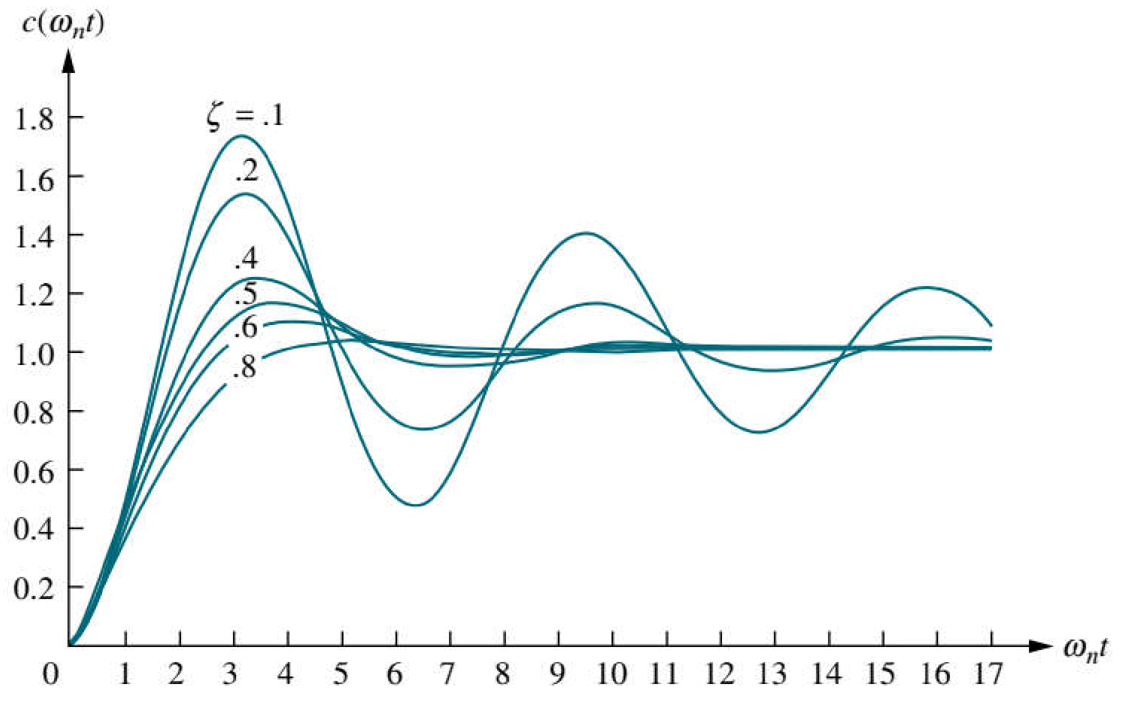
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$$

$$\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2}).$$

Underdamped Second-Order Systems

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\,t\right)$$

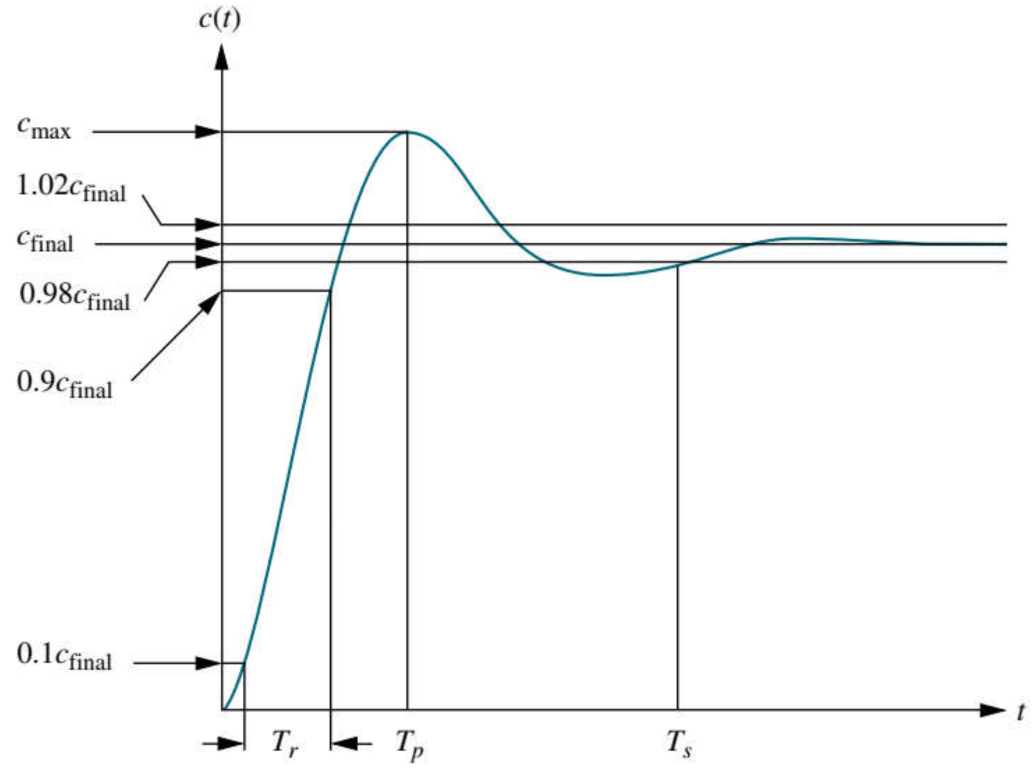
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\,t - \phi)$$



Underdamped Second-Order Systems

- The underdamped second order system is the common model for physical problems.
- There are other important parameters associated with the second order system: rise time, peak time, percent overshoot, and settling time.

1. *Rise time, T_r.* The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time, T_P.* The time required to reach the first, or maximum, peak.
3. *Percent overshoot, %OS.* The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time, T_s.* The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

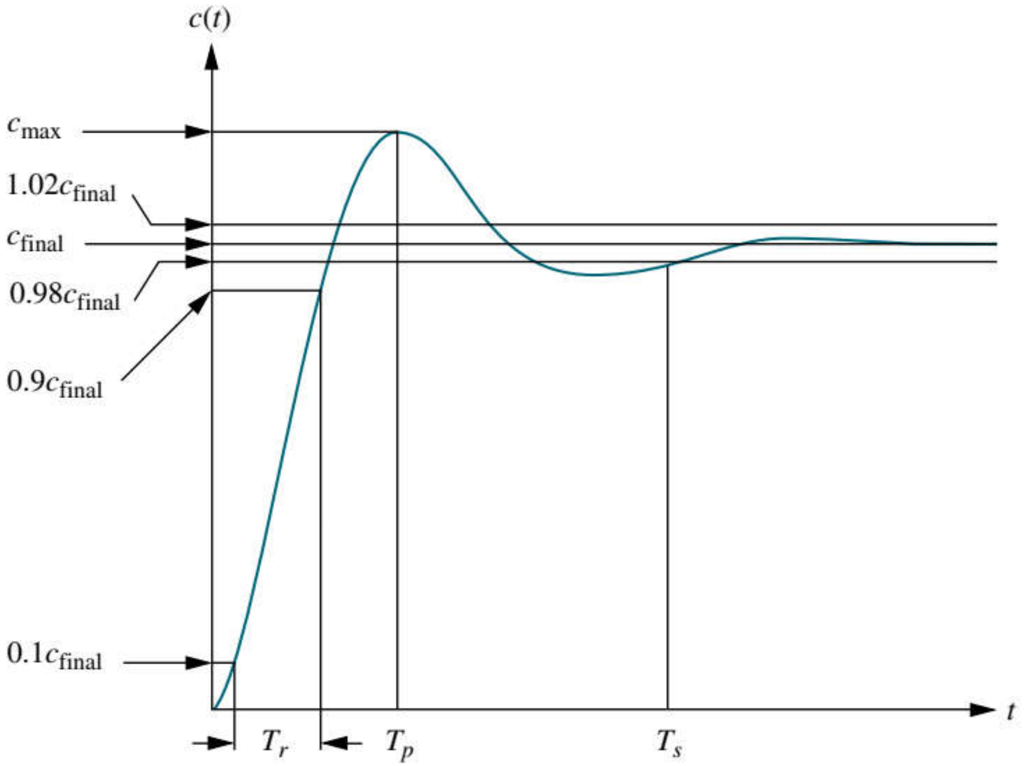


Underdamped Second-Order Systems

- Notice that the definitions for settling time and rise time are basically the same as the definitions for the first-order response.

- Rise time, peak time, and settling time yield information about the speed of the transient response.

- The formulas describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros.

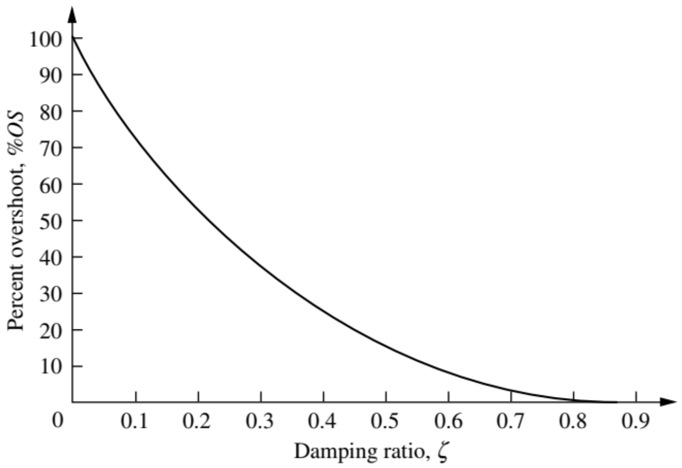


Underdamped Second-Order Systems

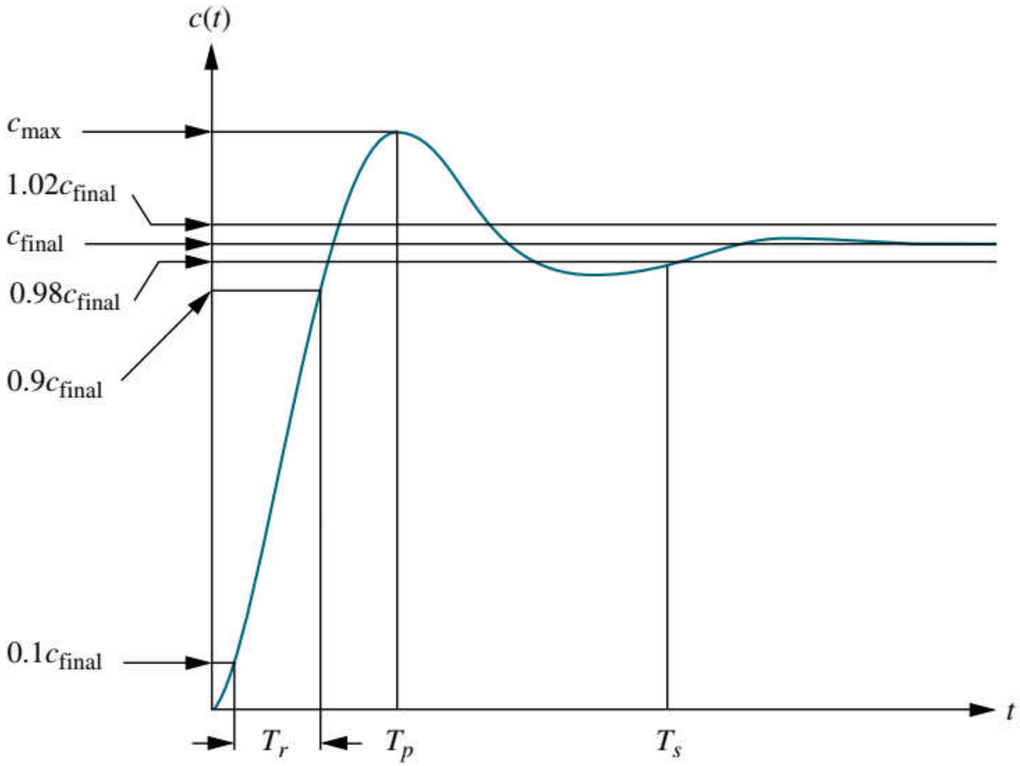
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

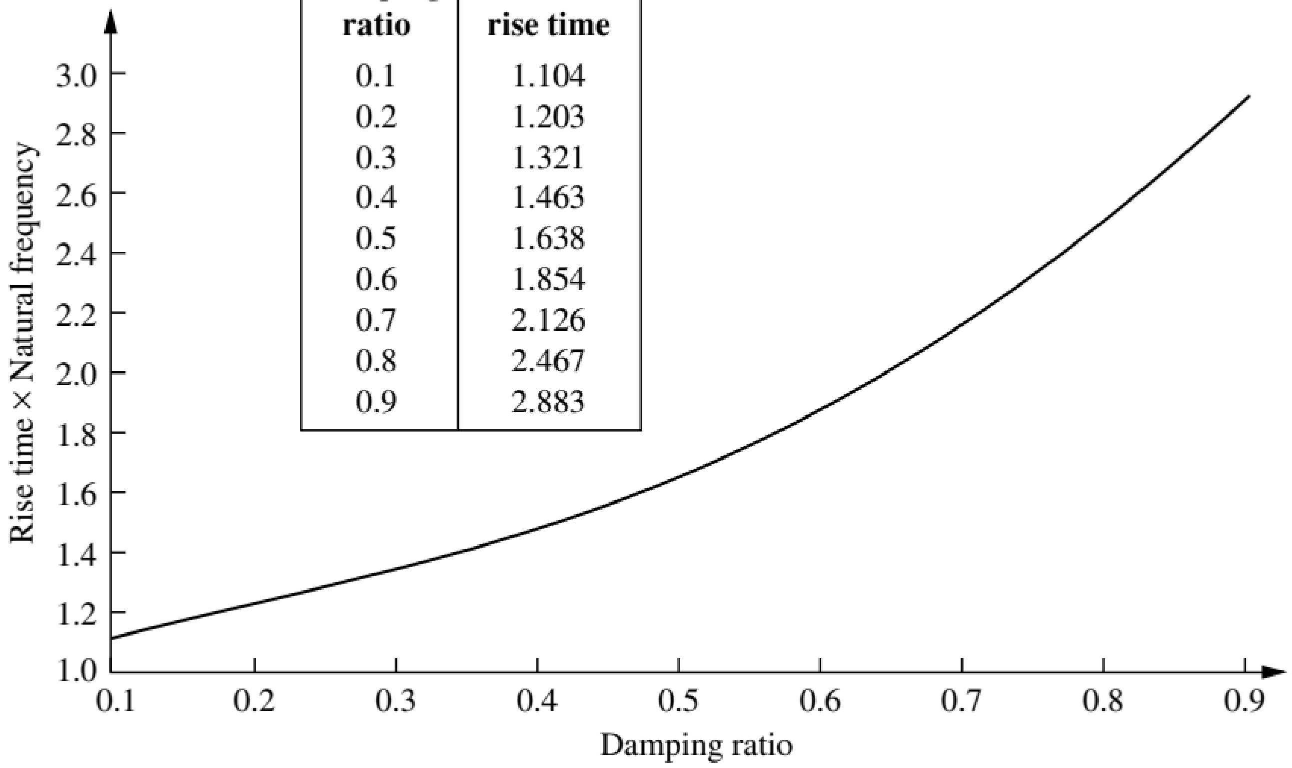
$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



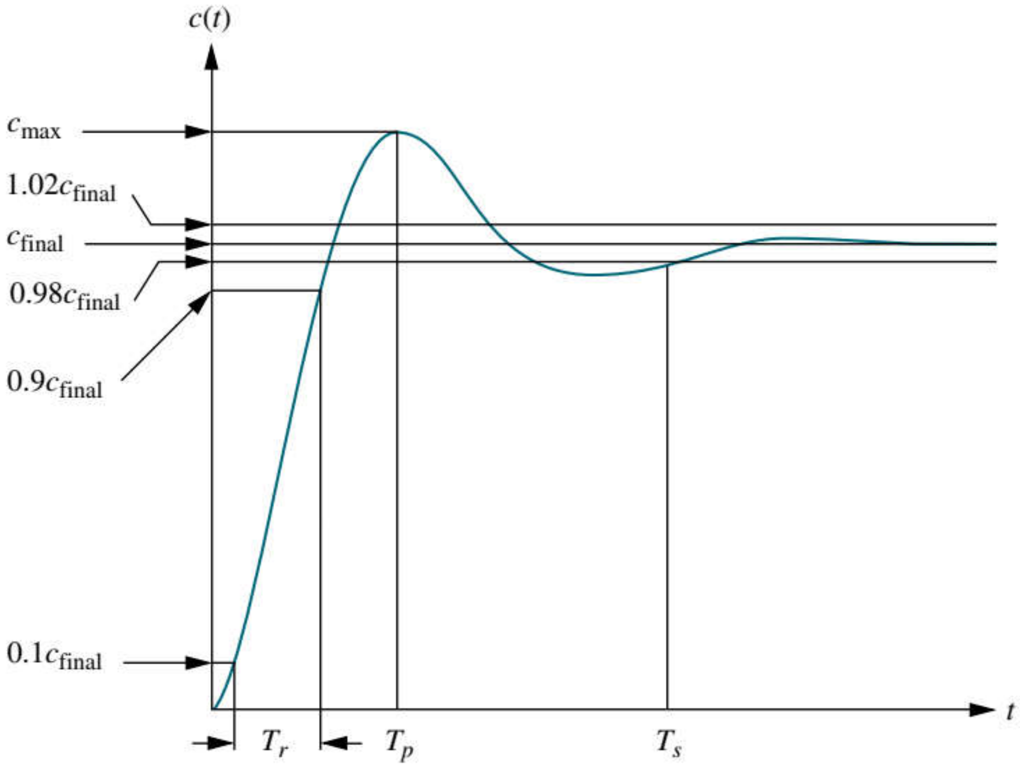
$$T_s = \frac{4}{\zeta\omega_n}$$



Underdamped Second-Order Systems



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Underdamped Second-Order Systems Example

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

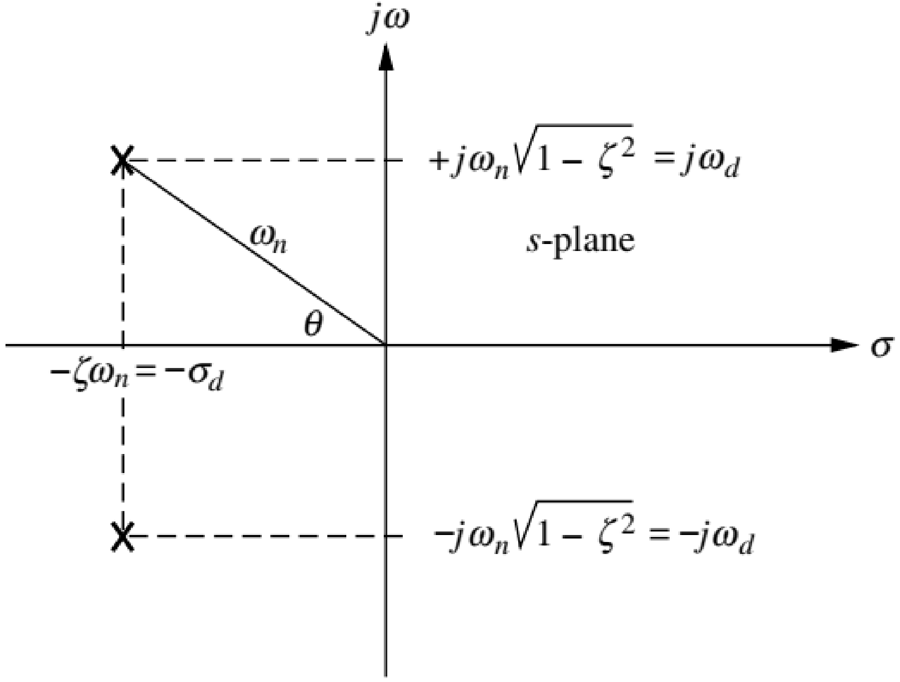
find T_p, $\%OS$, T_s, and T_r.

SOLUTION: ω_n and ζ are calculated as 10 and 0.75, respectively. Now substitute ζ and ω_n into Eqs. (4.34), (4.38), and (4.42) and find, respectively, that $T_p = 0.475$ second, $\%OS = 2.838$, and $T_s = 0.533$ second. Using the table in Figure 4.16, the normalized rise time is approximately 2.3 seconds. Dividing by ω_n yields $T_r = 0.23$ second. This problem demonstrates that we can find T_p, $\%OS$, T_s, and T_r without the tedious task of taking an inverse Laplace transform, plotting the output response, and taking measurements from the plot.

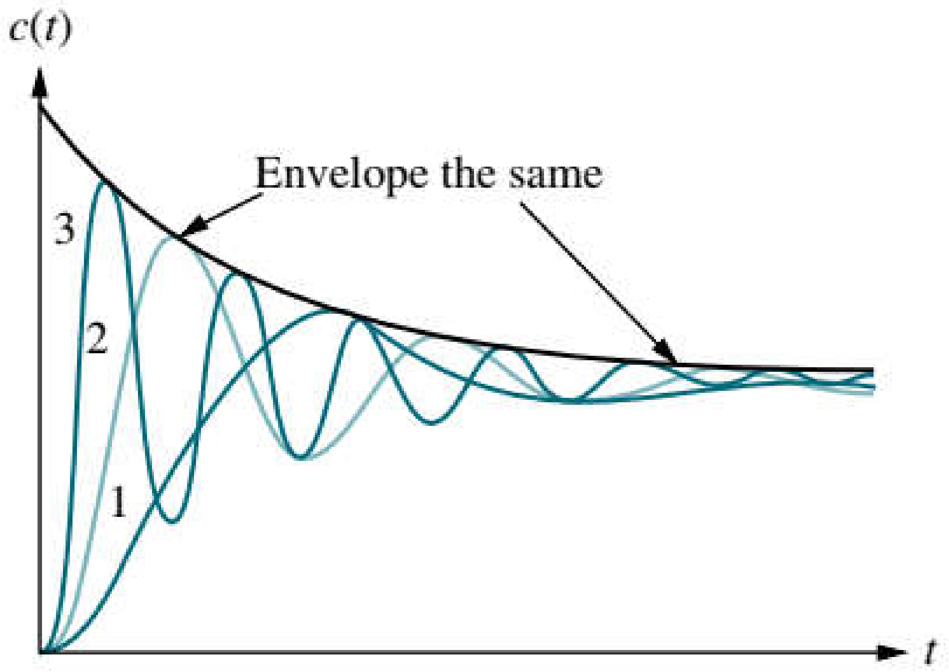
Underdamped Second-Order Systems

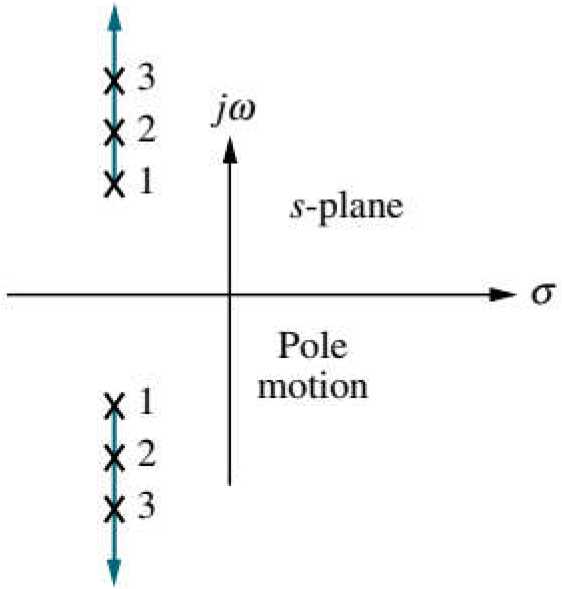
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{\pi}{\sigma_d}$$

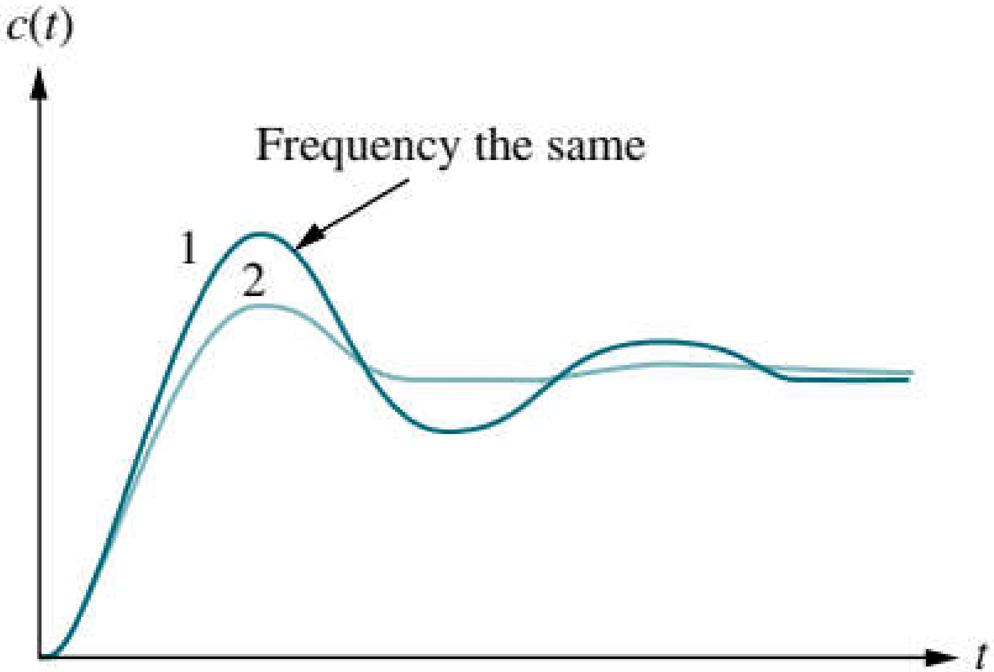


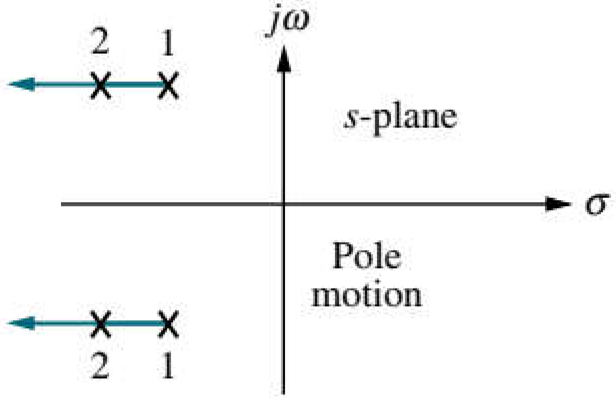
Underdamped Second-Order Systems



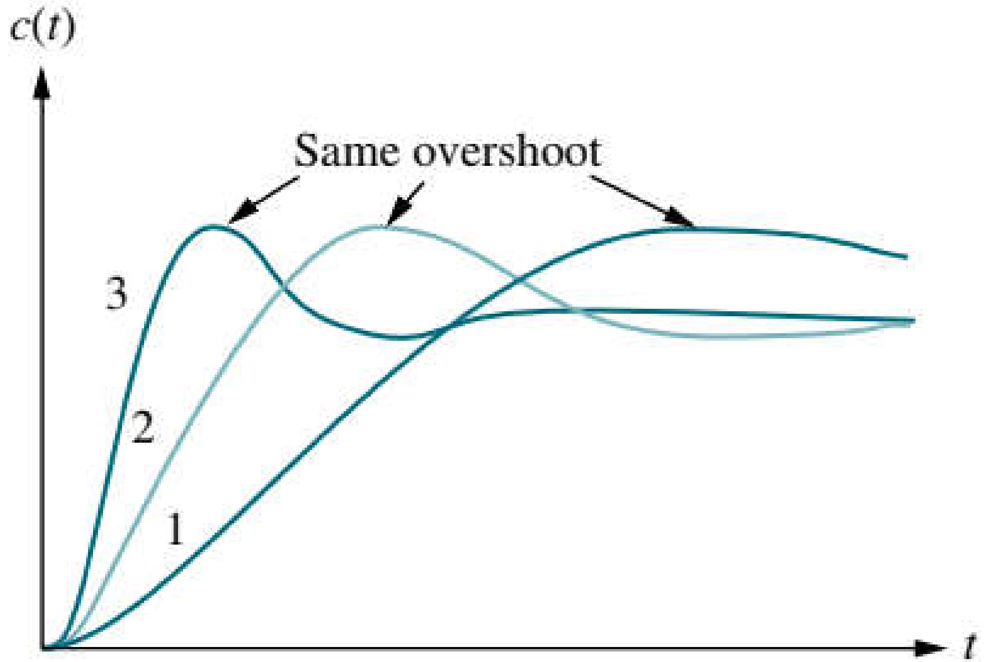


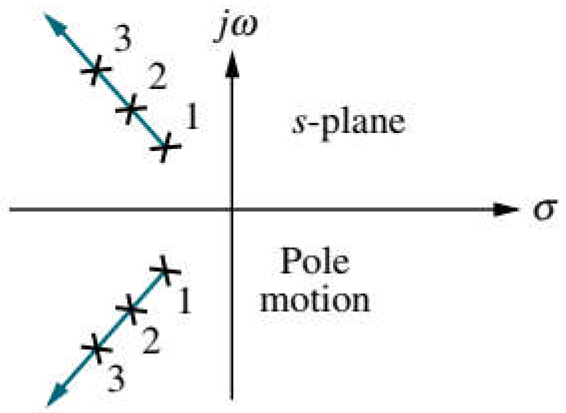
Underdamped Second-Order Systems





Underdamped Second-Order Systems





Underdamped Second-Order Systems Example

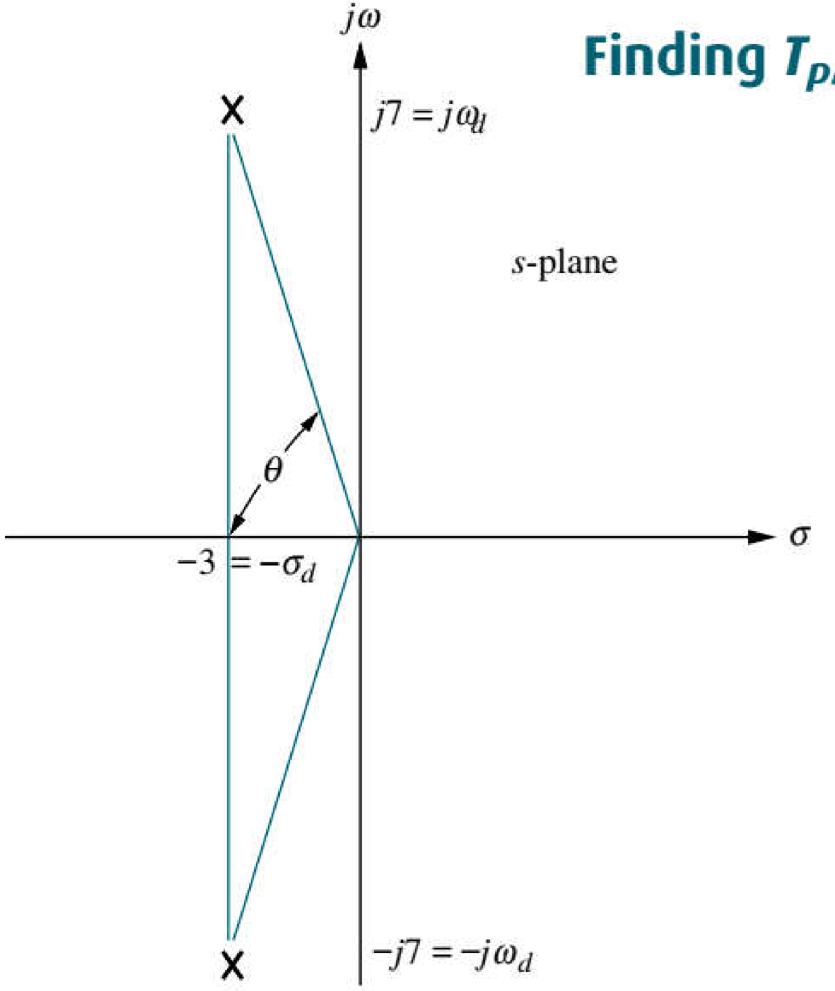


FIGURE 4.20 Pole plot for Example 4.6

Finding T_p, $\%OS$, and T_s from Pole Location

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

SOLUTION: The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second} \qquad (4.46)$$

The percent overshoot is

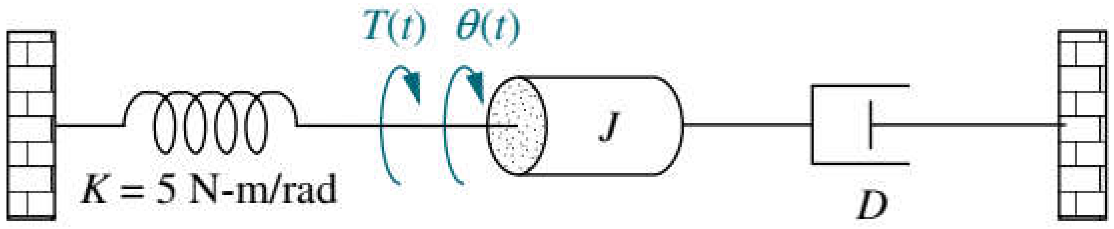
$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\% \qquad (4.47)$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds} \qquad (4.48)$$

Underdamped Second-Order Systems Example

PROBLEM: Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.



Underdamped Second-Order Systems Example

SOLUTION: First, the transfer function for the system is

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \tag{4.49}$$

From the transfer function,

$$\omega_n = \sqrt{\frac{K}{J}} \tag{4.50}$$

and

$$2\zeta\omega_n = \frac{D}{J} \tag{4.51}$$

But, from the problem statement,

$$T_s = 2 = \frac{4}{\zeta\omega_n} \tag{4.52}$$

or $\zeta\omega_n = 2$. Hence,

$$2\zeta\omega_n = 4 = \frac{D}{J} \tag{4.53}$$

Underdamped Second-Order Systems Example

Also, from Eqs. (4.50) and (4.52),

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \tag{4.54}$$

From Eq. (4.39), a 20% overshoot implies $\zeta = 0.456$. Therefore, from Eq. (4.54),

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456 \tag{4.55}$$

Hence,

$$\frac{J}{K} = 0.052 \tag{4.56}$$

From the problem statement, $K = 5$ N-m/rad. Combining this value with Eqs. (4.53) and (4.56), $D = 1.04$ N-m-s/rad, and $J = 0.26$ kg-m^2.

Systems with more than two poles

- The formulas describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros.
- If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.
- However, under certain conditions, a system with more than two poles or with zeros can be approximated as a second-order system that has just two complex dominant poles

Systems with more than two poles

Consider a system with two complex poles and one real pole.

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

System transformation and the eigenvalue problem

System transformation results in similar systems that have different state space representations, but the same transfer function and hence the same poles and eigenvalues.

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{P} = [\mathbf{U}_{\mathbf{z}_1} \mathbf{U}_{\mathbf{z}_2}] = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{P}\mathbf{z}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

$\mathbf{P}$ is a transformation matrix

System transformation – Example

Given the system represented in state space, transform the system to a new set of state variables, z, where the new state variables are related to the original state variables, x, as follows:

$$z_1 = 2x_1$$
$$z_2 = 3x_1 + 2x_2$$
$$z_3 = x_1 + 4x_2 + 5x_3$$

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \mathbf{x} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix}$$
$$= \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\mathbf{CP} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

System transformation – Example

Given the system represented in state space, transform the system to a new set of state variables, z, where the new state variables are related to the original state variables, x, as follows:

$$z_1 = 2x_1$$
$$z_2 = 3x_1 + 2x_2$$
$$z_3 = x_1 + 4x_2 + 5x_3$$

Therefore, the transformed system is

$$\dot{\mathbf{z}} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.55 & 0.4 & -6.2 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$
$$y = [0.5 \quad 0 \quad 0]\mathbf{z}$$

Diagonalization

- A diagonal system matrix has the advantage that each state equation is a function of only one state variable.
- Hence, each differential equation can be solved independently of the other equations.
- If we find the correct matrix, P, the transformed system matrix, $P^{-1}AP$, will be a diagonal matrix.

$$\mathbf{P} = [\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3, \ldots,\ \mathbf{x_n}]$$

$$\mathbf{AP} = \mathbf{PD}$$

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

Diagonalization

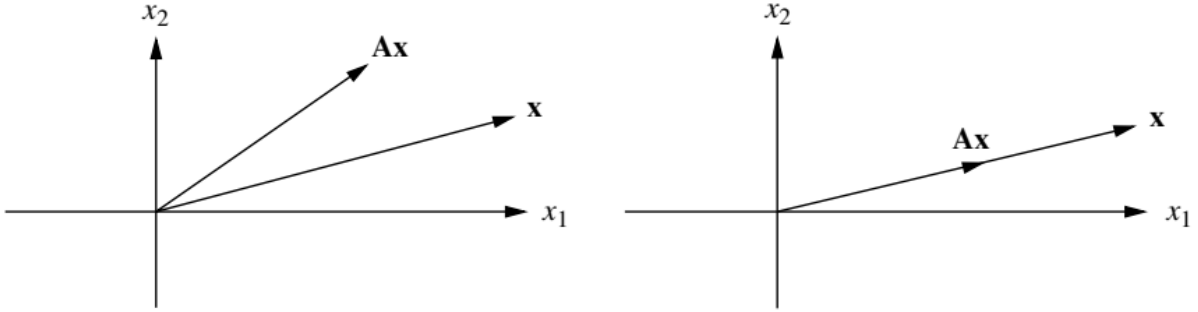
If the eigenvectors of the matrix **A** are chosen as the basis vectors of a transformation, **P**, the resulting system matrix will be diagonal.

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n]$$

Eigenvector. The eigenvectors of the matrix **A** are all vectors, $\mathbf{x_i} \neq \mathbf{0}$, which under the transformation **A** become multiples of themselves; that is,

$$\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i}$$

where λ_i's are constants.



Diagonalization

If the eigenvectors of the matrix **A** are chosen as the basis vectors of a transformation, **P**, the resulting system matrix will be diagonal.

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x_n}]$$

Eigenvalue. The eigenvalues of the matrix **A** are the values of λ_i that satisfy Eq. (5.80) for $\mathbf{x_i} \neq \mathbf{0}$.

To find the eigenvectors, we rearrange Eq. (5.80). Eigenvectors, $\mathbf{x_i}$, satisfy

$$\mathbf{0} = (\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x_i} \tag{5.81}$$

$$\det(\lambda_i \mathbf{I} - \mathbf{A}) = \mathbf{0}$$

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

Solution:

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8 \end{aligned}$$

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

Solution:

$$\begin{aligned}\det(\lambda\mathbf{I} - \mathbf{A}) &= \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\right| \\ &= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8\end{aligned}$$

from which the eigenvalues are $\lambda = -2$, and -4.

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

Solution:

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-3x_1 + x_2 = -2x_1$$
$$x_1 - 3x_2 = -2x_2$$

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

Solution:

from which $x_1 = x_2$. Thus,

$$\mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

Using the other eigenvalue, -4, we have

$$\mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\mathbf{CP} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

Diagonalization – Example

For the given system, find its the eigen-system, and transform it into its diagonal form.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} \mathbf{x}$$

Solution:

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -1 \end{bmatrix} \mathbf{z}$$