Dynamics of Structures
Continuous Systems – Beam

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Introduction

- We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system.
- Known as *distributed* or *continuous systems*, it is not possible to identify discrete masses, dampers, or springs.
- We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate.
- This is why a continuous system is also called a *system of infinite degrees of freedom*.
- We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate.

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Part 1 - Beam Free Vibration – Exact Solution

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Vibration of Beams – Equation of motion

Consider the free-body diagram of an element of a beam shown in Fig., where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam.

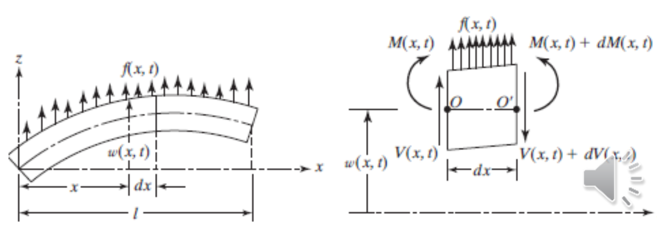
Since the inertia force acting on the element of the beam is

$$\rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

Apply the equilibrium equation based on Newton's Law



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Equation of motion

The moment equation of motion about the y-axis passing through point O, leads to

$$(M + dM) - (V + dV)\,dx + f(x,t)\,dx\,\frac{dx}{2} - M = 0$$

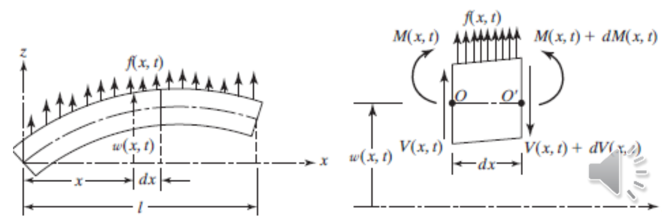
By writing

$$dV = \frac{\partial V}{\partial x}dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x}dx$$

By disregarding terms involving second powers in dx

$$-\frac{\partial V}{\partial x}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)$$

$$\frac{\partial M}{\partial x}(x,t) - V(x,t) = 0$$



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Equation of motion

$$-\frac{\partial V}{\partial x}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)$$

$$\frac{\partial M}{\partial x}(x,t) - V(x,t) = 0$$

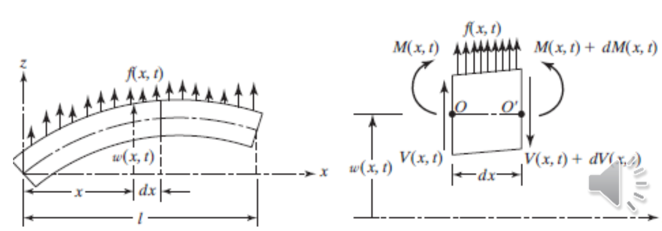
From the second equation $\rightarrow V = \frac{\partial M}{\partial x}$, then substitute in the 1st equation

$$-\frac{\partial^2 M}{\partial x^2}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as

$$M(x,t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)$$

where E is Young s modulus and $I(x)$ is the moment of inertia of the beam cross section about the y-axis



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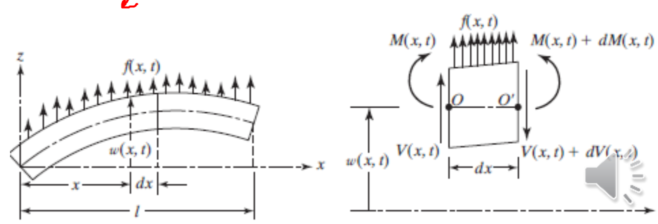
Equation of motion

$$-\frac{\partial^2 M}{\partial x^2}(x, t) + f(x, t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x, t) \qquad M(x, t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x, t)$$

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x, t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

This is the beam general equation of motion



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Equation of motion

For a uniform beam, it reduces to

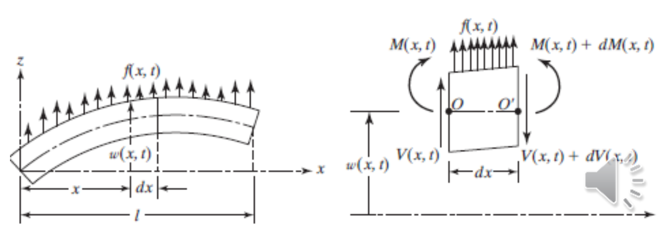
$$EI\frac{\partial^4 w}{\partial x^4}(x, t) + \rho A\frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2\frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$



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Equation of motion

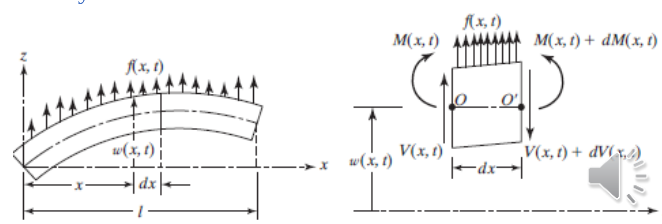
For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$

- In Mathematical-wise we are talking about ***initial-boundary value problem.***
- We assume the solution of the partial differential equation as being a linear combination of simple component functions, which also satisfy the equation and certain boundary conditions. This is a reasonable assumption provided the partial differential equation and the boundary conditions are linear.



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$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

$$c = \sqrt{\frac{EI}{\rho A}}$$

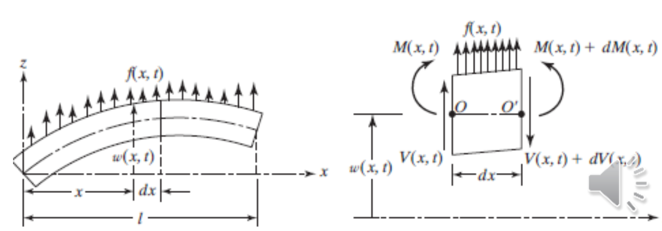
Free Vibration

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to *x*, <u>*two initial conditions*</u> and <u>*four boundary conditions*</u> are needed for finding a unique solution for *w*(*x*, *t*).

Usually, the values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at so that the initial conditions become

$$w(x, t = 0) = w_0(x)$$

$$\frac{\partial w}{\partial t}(x, t = 0) = \dot{w}_0(x)$$



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$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

$$c = \sqrt{\frac{EI}{\rho A}}$$

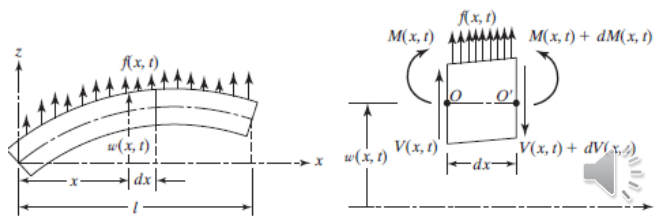
Free Vibration

The free-vibration solution can be found using the method of separation of variables as

$$w(x, t) = W(x)T(t)$$

$$\frac{c^2}{W(x)} \frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)} \frac{d^2T(t)}{dt^2} = a = \omega^2$$

where $a = \omega^2$ is a positive constant



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$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

$$c = \sqrt{\frac{EI}{\rho A}}$$

Free Vibration

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0$$

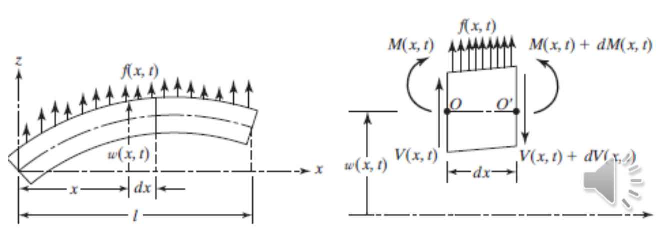
$$W(x) = Ce^{sx}$$

$$T(t) = A \cos \omega t + B \sin \omega t$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

Which represents two linear, homogenous, ordinary differential equations with constant coefficients.



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$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0$$

Free Vibration

$$W(x) = Ce^{sx}$$

where C and s are constants, and derive the auxiliary equation as

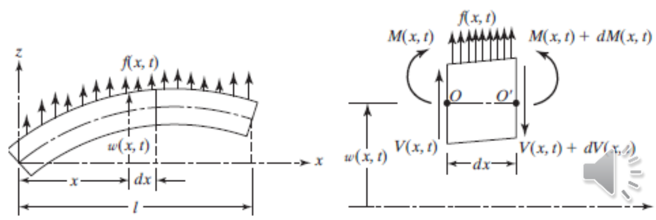
$$s^4 - \beta^4 = 0$$

The roots of this equation are

$$s_{1,2} = \pm\beta, \qquad s_{3,4} = \pm i\beta$$

Hence the solution of becomes

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$



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$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

Free Vibration

$$c = \sqrt{\frac{EI}{\rho A}}$$

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

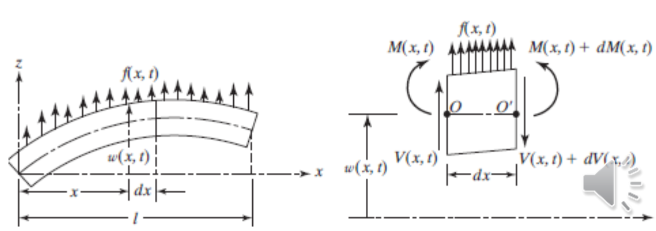
$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

or

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

The natural frequencies of the beam are

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$



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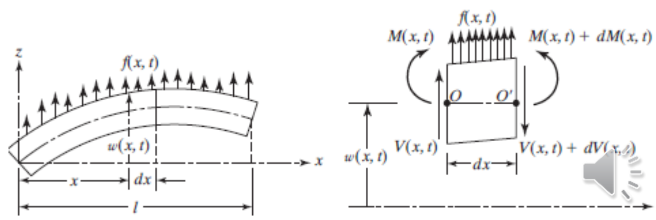
Free Vibration

- The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and is called the *natural frequency of vibration*.
- For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode.
- The unknown constants C_1 to C_4 and the value of β can be determined from the boundary conditions of the beam.



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Free Vibration

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$

The common boundary conditions are as follows:

1. *Free end:*

$$\text{Bending moment} = EI \frac{\partial^2 w}{\partial x^2} = 0$$

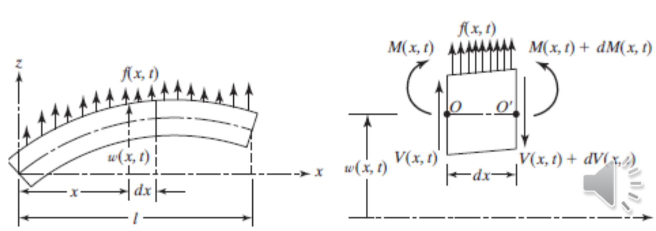
$$\text{Shear force} = \frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = 0$$

2. *Simply supported (pinned) end:*

$$\text{Deflection} = w = 0, \qquad \text{Bending moment} = EI \frac{\partial^2 w}{\partial x^2} = 0$$

3. *Fixed (clamped) end:*

$$\text{Deflection} = 0, \qquad \text{Slope} = \frac{\partial w}{\partial x} = 0$$



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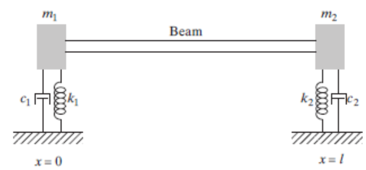
Free Vibration

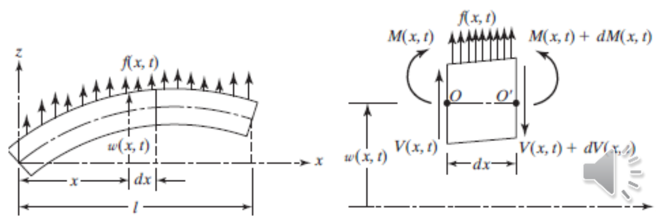
4. *End connected to a linear spring, damper, and mass* (Fig. 8.16(a)): When the end of a beam undergoes a transverse displacement w and slope $\partial w/\partial x$. with velocity $\partial w/\partial t$ and acceleration $\partial^2 w/\partial t^2$, the resisting forces due to the spring, damper, and mass are proportional to w, $\partial w/\partial t$, and $\partial^2 w/\partial t^2$, respectively. This resisting force is balanced by the shear force at the end. Thus

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = a\left[kw + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2}\right] \tag{8.97}$$

where $a = -1$ for the left end and $+1$ for the right end of the beam. In addition, the bending moment must be zero; hence

$$EI\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.98}$$





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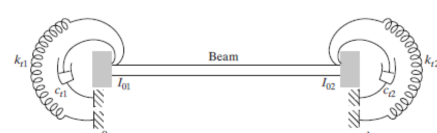
Free Vibration

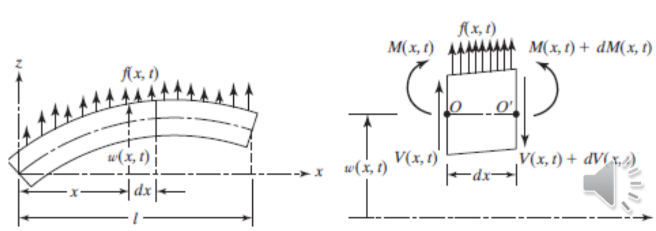
5. *End connected to a torsional spring, torsional damper, and rotational inertia* (Fig. 8.16(b)): In this case, the boundary conditions are

$$EI\frac{\partial^2 w}{\partial x^2} = a\left[k_t\frac{\partial w}{\partial x} + c_t\frac{\partial^2 w}{\partial x \partial t} + I_0\frac{\partial^3 w}{\partial x \partial t^2}\right] \tag{8.99}$$

where $a = +1$ for the left end and -1 for the right end of the beam, and

$$\frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0 \tag{8.100}$$





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Free Vibration

As an exercise try to obtain these values

End Conditions of Beam	Frequency Equation	Mode Shape (Normal Function)	Value of $\beta_n l$
Pinned-pinned	$\sin \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sin \beta_n x + \sinh \beta_n x + \alpha_n(\cos \beta_n x + \cosh \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l - \sinh \beta_n l}{\cosh \beta_n l - \cos \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sinh \beta_n x_n x - \sin \beta_n x + \alpha_n(\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\frac{\sinh \beta_n l - \sin \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free	$\cos \beta_n l \cdot \cosh \beta_n l = -1$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x - \alpha_n(\cos \beta_n x - \cosh \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l + \sinh \beta_n l}{\cos \beta_n l + \cosh \beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x + \alpha_n(\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l - \sinh \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x + \alpha_n \sinh \beta_n x]$ where $\alpha_n = \left(\frac{\sin \beta_n l}{\sinh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

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Free Vibration

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

or

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

Determine the natural frequencies of vibration of a uniform beam fixed at $x = 0$ and simply supported at $x = l$,

Solution: The boundary conditions can be stated as

$$W(0) = 0$$

$$\frac{dW}{dx}(0) = 0$$

$$W(l) = 0$$

$$EI \frac{d^2W}{dx^2}(l) = 0 \quad \text{or} \quad \frac{d^2W}{dx^2}(l) = 0$$

W(0) = 0 Then $C_1 + C_3 = 0$

$$\left.\frac{dW}{dx}\right|_{x=0} = \beta[-C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x]_{x=0} = 0$$

$$\beta[C_2 + C_4] = 0$$

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Free Vibration

The solution becomes

$$W(x) = C_1(\cos \beta x - \cosh \beta x) + C_2(\sin \beta x - \sinh \beta x)$$

From the two other conditions

$$C_1(\cos \beta l - \cosh \beta l) + C_2(\sin \beta l - \sinh \beta l) = 0$$

$$-C_1(\cos \beta l + \cosh \beta l) - C_2(\sin \beta l + \sinh \beta l) = 0$$

For a nontrivial solution of C_1 and C_2, the determinant of their coefficients must be zero.

$$\begin{vmatrix} (\cos \beta l - \cosh \beta l) & (\sin \beta l - \sinh \beta l) \\ -(\cos \beta l + \cosh \beta l) & -(\sin \beta l + \sinh \beta l) \end{vmatrix} = 0$$

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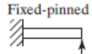
Fixed-pinned



$\tan \beta_n l - \tanh \beta_n l = 0$

$$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x + \alpha_n (\cosh \beta_n x - \cos \beta_n x)]$$

where

$$\alpha_n = \left(\frac{\sin \beta_n l - \sinh \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$$

$\beta_1 l = 3.926602$
$\beta_2 l = 7.068583$
$\beta_3 l = 10.210176$
$\beta_4 l = 13.351768$

Free Vibration

Expanding the determinant gives the frequency equation

$$\cos \beta l \sinh \beta l - \sin \beta l \cosh \beta l = 0$$

or

$$\tan \beta l = \tanh \beta l$$

The roots of this equation, $\beta_n l$, give the natural frequencies of vibration

$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho A l^4}\right)^{1/2}, \qquad n = 1, 2, \ldots$$

$$C_{2n} = -C_{1n}\left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l}\right)$$

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Free Vibration

$$W_n(x) = C_{1n}\left[(\cos\beta_n x - \cosh\beta_n x) - \left(\frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l}\right)(\sin\beta_n x - \sinh\beta_n x)\right]$$

The normal modes of vibration can be obtained as

$$w_n(x, t) = W_n(x)\,(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

The general or total solution of the fixed-simply supported beam can be expressed by the sum of the normal modes:

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x, t)$$

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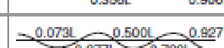
Beam Modes – Free beam

Natural frequencies		Normal modes		
$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n^a	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.00000	0.2323	

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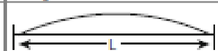
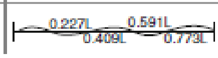
Beam Modes – Fixed beam

Natural frequencies		Normal modes		
$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n^a	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.00000	0.2323	

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Beam Modes – Cantilever beam

Natural frequencies		Normal modes		
$\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$ $\quad \sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$				
n	$C_n = (a_n L)^2$	σ_n	I_n^a	Shape
1	3.5160	0.734096	0.7830	
2	22.0345	1.018466	0.4340	
3	61.6972	0.999225	0.2589	
4	120.0902	1.000033	0.0017	
5	199.8600	1.000000	0.0707	

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Homework – Problem 1

Solve the differential equation for beam free vibration and derive and plot the first three natural frequencies for

1. Cantilever beam
2. Free-free beam
3. Simply supported beam