# System Dynamics
# Modeling in time domain

Mohamed Abdou Mahran Kasem, Ph.D.
Aerospace Engineering Department
Cairo University

# Modeling in time domain

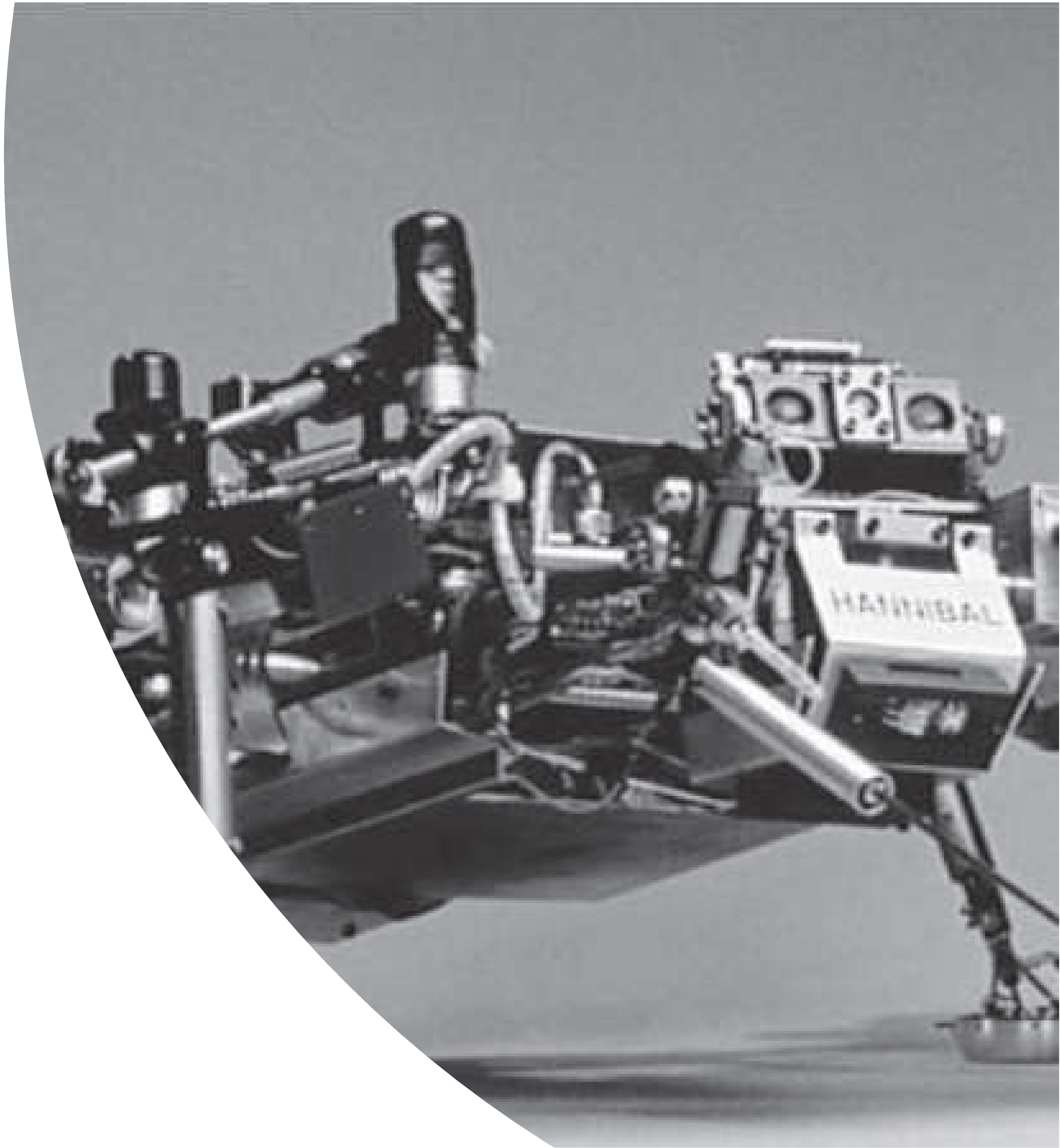
Two approaches are available for the analysis and design of feedback control systems:

- The first is known as the classical, or *frequency-domain*, technique.
  - This approach is based on converting a system's *differential equation* to a *transfer function*, thus generating a mathematical model of the system that algebraically relates a representation of the output to a representation of the input.
  - Replacing a differential equation with an algebraic equation not only simplifies the representation of individual subsystems but also simplifies modeling interconnected subsystems.
- The primary disadvantage of the classical approach is its *limited applicability*: It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

# Modeling in time domain

The state space approach or time domain approach:

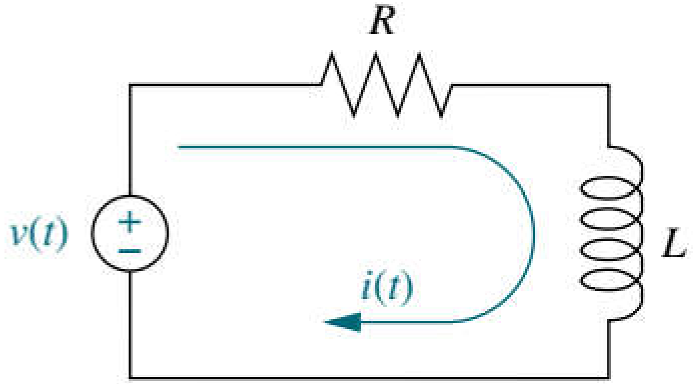
- Can be used to model a wide range of systems such as nonlinear systems, and systems with nonzero initial conditions, in addition to the systems that can be modeled using the classical approach.



# State space representation

The loop equation takes the form

$$L\frac{di}{dt} + Ri = v(t)$$




The output equation takes the form

$$v_R(t) = Ri(t)$$

Both the state equation and the output equation are called *state space representation.*

# State space representation

Rules

- The minimum number of *state variables* required to describe a system equals the order of the *differential equation* (mathematical model).

- The minimal state *variables* must be *linearly independent*, i.e. one cannot be written as a linear combination from others.

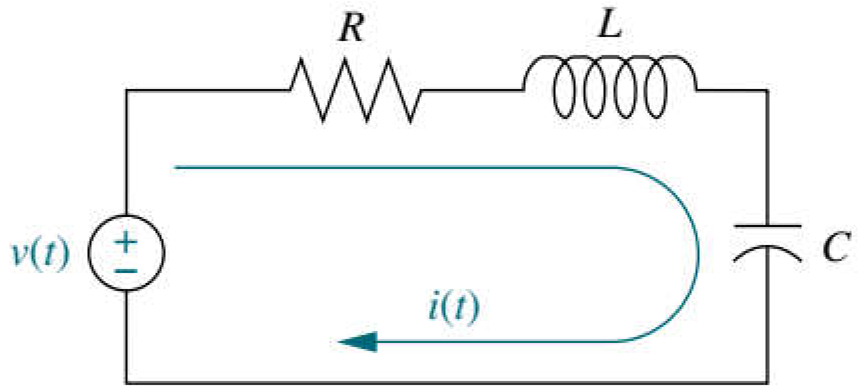
for instance, in the previous example, one cannot define both the voltage and current as state variables set because they are linearly dependent.

# State space representation

For the given electric system

The loop equation takes the form,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$




Converting to charge, using $i(t) = dq/dt$, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

Which represents a second order D.E. that can be transformed into two first order D.E's with two state variables

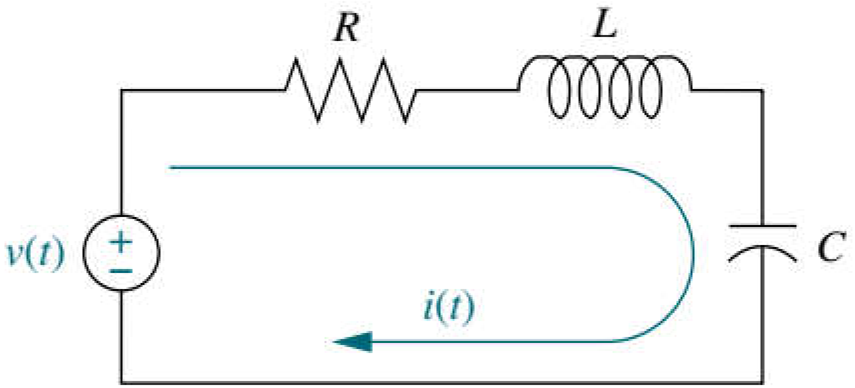
$$\frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t)$$

# State space representation

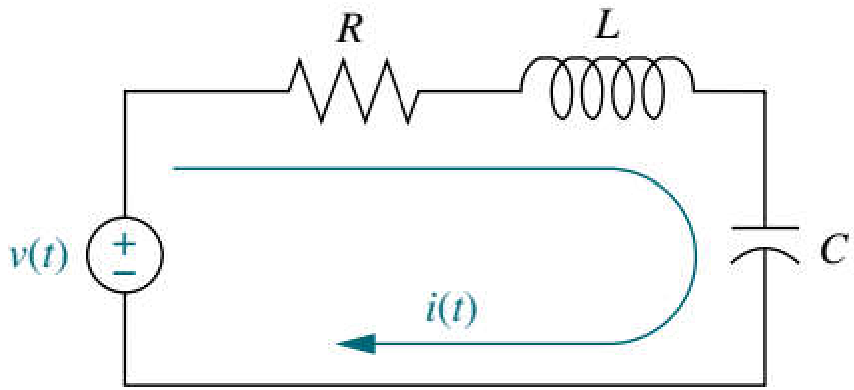
From these two state variable, one can solve for other network variables,

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$




This equation is known as the *output equation*; we say that $v_L(t)$ is a linear combination of the state variables, *q(t)* and *i(t)*, and the input, *v(t)*.

# State space representation




If the system is linear, the state and output equations can be written in matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\dot{\mathbf{x}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}; \; u = v(t)$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$y = v_L(t); \quad \mathbf{C} = [-1/C \quad -R]; \quad \mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = 1; \quad u = v(t)$$

# State space representation

A state space representation consists of:

1. A system of first order differential equation in the state variables *(state equation)*.

2. the algebraic *output equation* from which all other system variables can be found.

A state-space representation is not unique, since a different choice of state variables leads to a different representation of the same system.

# State space representation

Definitions:

- *Linear combination*: A linear combination of $n$ variables, $x_i$, for $i = 1{:}n$, is given by the following sum, S:

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

where each $K_i$ is a constant.

- *Linear independence:* A set of variables is said to be linearly independent if none of the variables can be written as a *linear combination* of the others.
- *System variable:* Any variable that responds to an input or initial conditions in a system.
- *State variables:* The smallest set of linearly independent system variables

# State space representation

Definitions:

- *State vector:* A vector whose elements are the *state variables*.
- *State space*. The n-dimensional space whose axes are the state variables.
- *State equations*. A set of n simultaneous, first-order differential equations with *n* variables, where the *n* variables to be solved are the *state variables*.
- *Output equation*. The algebraic equation that expresses the output variables of a system as *linear combinations* of the *state variables* and the inputs.

# State space representation

Definitions:

- So a system can be represented in state space as,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

for $t \geq t_0$ and initial conditions, $\mathbf{x}(t_0)$, where

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

# State space representation

Definitions:

- The ***state vector*** must be selected such that,

1. A minimum number of state variables must be selected as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system.
2. The components of the state vector (that is, this minimum number of state variables) must be linearly independent.

# State space representation – Example 1

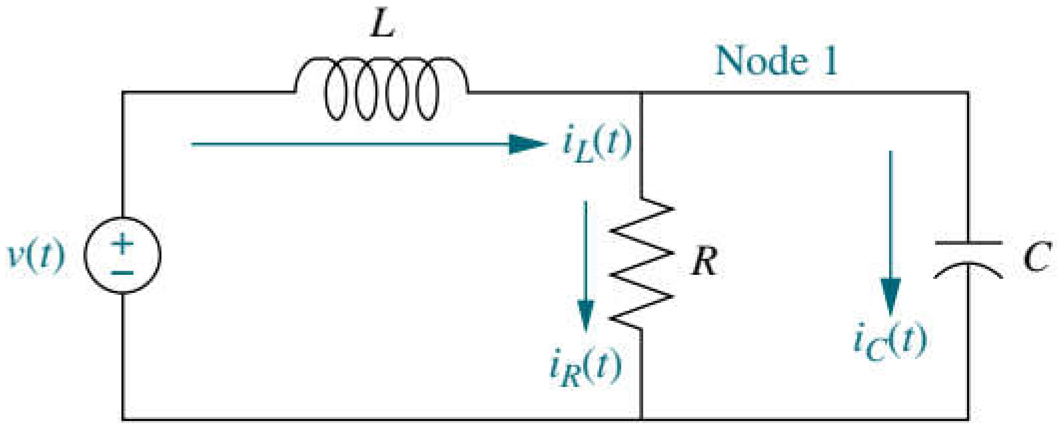
Given the electrical network shown in Figure, find a state-space representation if the output is the current through the resistor.

*Step 1:* Label all of the branch currents in the network. These include $i_L, i_R, and i_C$.

*Step 2* Select the state variables by writing the derivative equation for all energy storage elements, that is, the inductor and the capacitor. Thus,

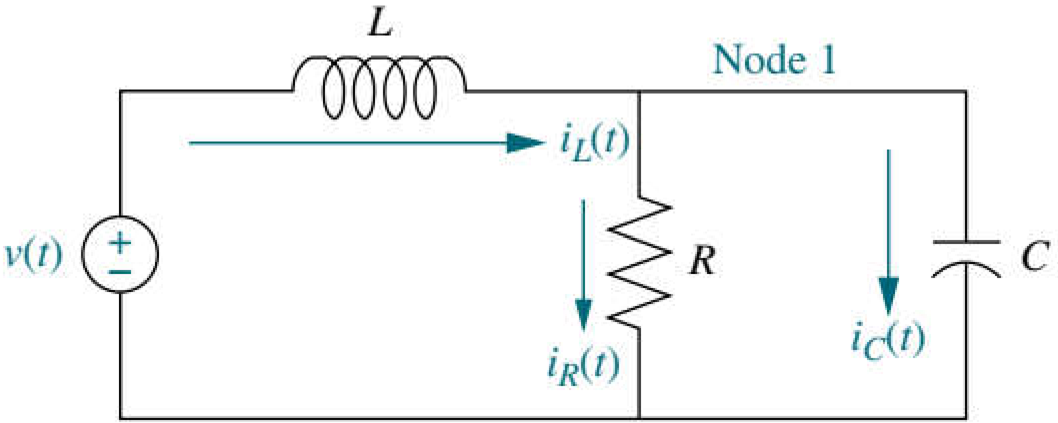
$$C\frac{dv_C}{dt} = i_C$$

$$L\frac{di_L}{dt} = v_L$$




The state variables can be defined as the differentiated quantities $i_L, and v_C$.

# State space representation – Example 1




_Step 3_ Express the other system variables ($i_C$, $and v_L$) in terms of the state space variables ($i_L$, $and v_C$).

Apply network theory, such as Kirchhoff's voltage and current laws, to obtain $i_C$ and $v_L$ in terms of the state variables, $v_C$ and $i_L$. At Node 1,

$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R}v_C + i_L \end{aligned}$$

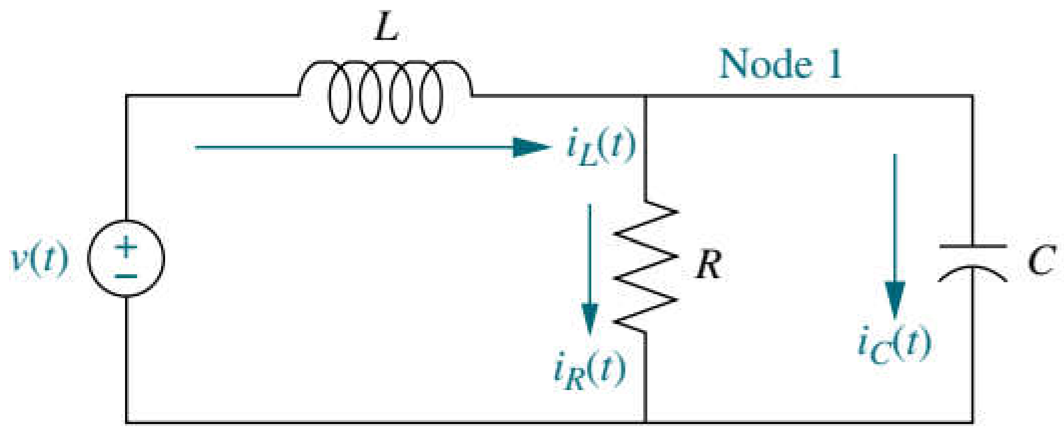
Around the outer loop,

$$v_L = -v_C + v(t)$$

# State space representation – Example 1

*Step 4* Define the state equation

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$
$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t)$$




*Step 5* Define the output equation

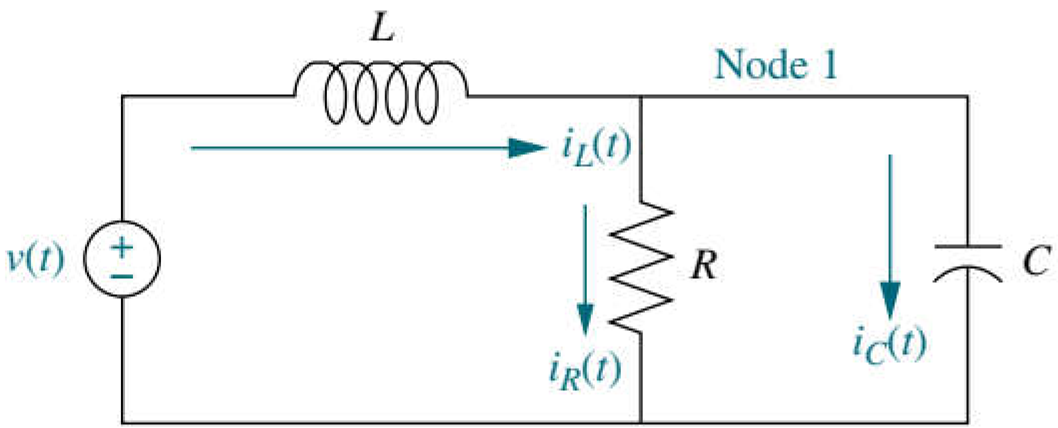
$$i_R = \frac{1}{R}v_C$$

# State space representation – Example 1

All in matrix form

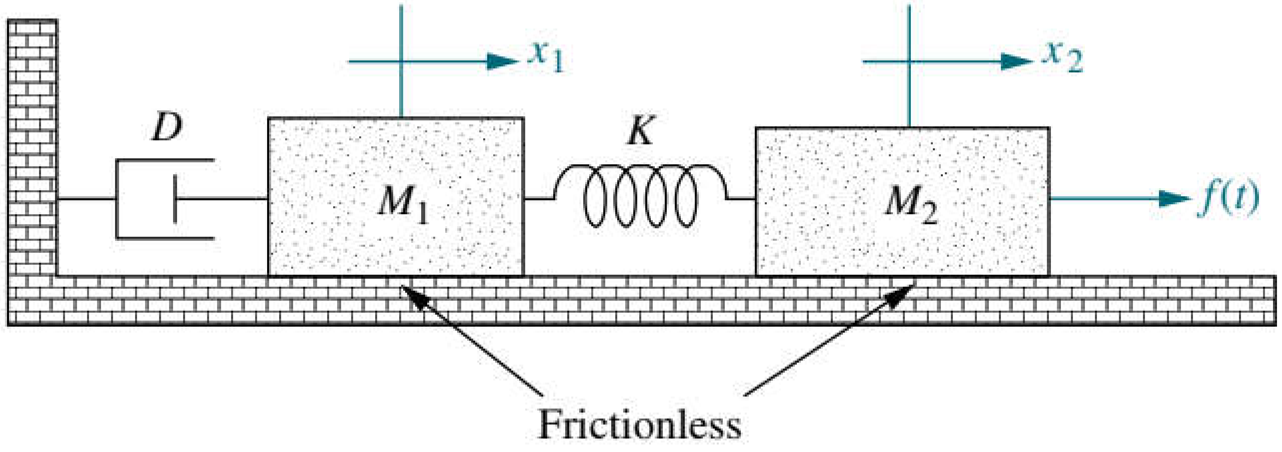
$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$i_R = [1/R \quad 0] \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$



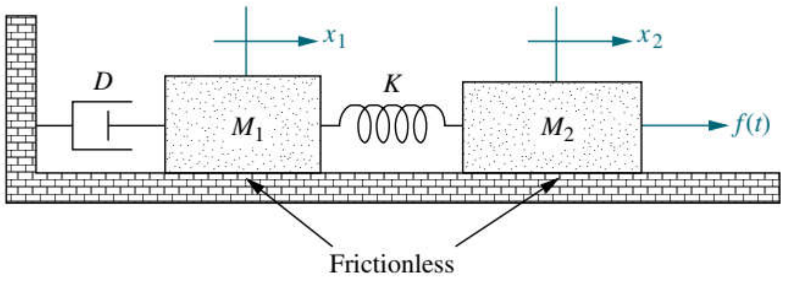
# State space representation – Example 2

Find the state equations for the translational mechanical system shown in Figure.



# State space representation – Example 2

First, write the system equation of motions,



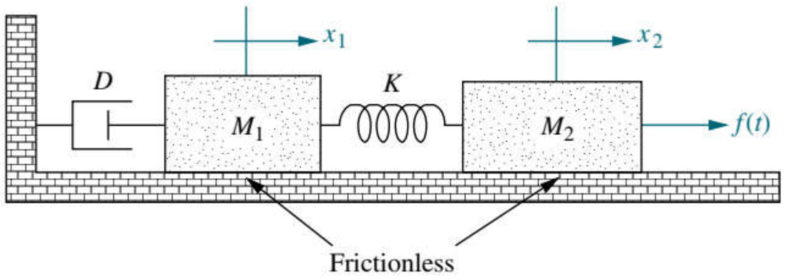

$$M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2 \frac{d^2 x_2}{dt^2} + Kx_2 = f(t)$$

Now let $d^2x_1/dt^2 = dv_1/dt$, and $d^2x_2/dt^2 = dv_2/dt$, and then select $x_1$, $v_1$, $x_2$, and $v_2$ as state variables.

# State space representation – Example 2

Then, the state equations will take the form



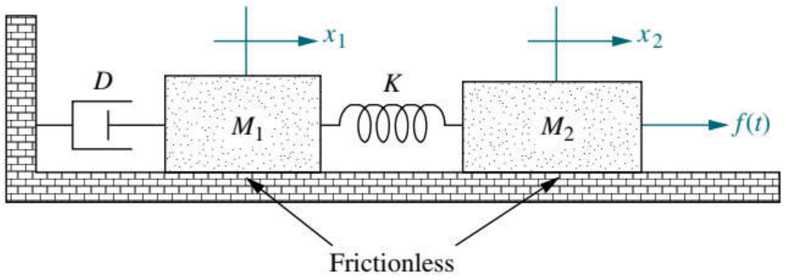

$$\frac{dx_1}{dt} = +v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt} = +v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

# State space representation – Example 2




In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

# Converting from state space to transfer function

# Converting from State Space to a Transfer Function

- Given the state space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- Take the Laplace transform assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

- Solving for X(s)

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

where **I** is the identity matrix.

# Converting from State Space to a Transfer Function

- By substituting in the output equation

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

We call the matrix $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ the transfer function matrix

Then,

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

# Converting from State Space to a Transfer Function – Example

**PROBLEM:** Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u \qquad (3.74a)$$

$$y = [1 \quad 0 \quad 0]\mathbf{x} \qquad (3.74b)$$

# Converting from State Space to a Transfer Function – Example

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

# Converting from State Space to a Transfer Function – Example

Substituting $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ into Eq. (3.73), where

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

we obtain the final result for the transfer function:

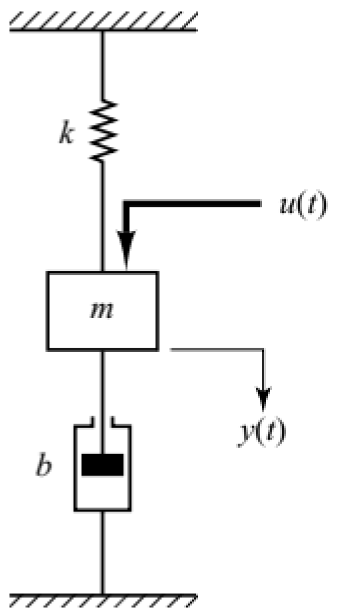
$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

# Converting from State Space to a Transfer Function – Example

**EXAMPLE 2–2** Consider the mechanical system shown in Figure 2–15. We assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u$$



# Converting from State Space to a Transfer Function – Example

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

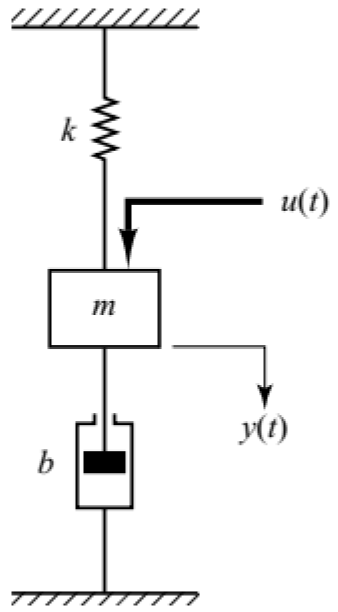
or

$$\dot{x}_1 = x_2 \tag{2–17}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{2–18}$$

The output equation is

$$y = x_1 \tag{2–19}$$



# Converting from State Space to a Transfer Function – Example

In a vector-matrix form, Equations (2–17) and (2–18) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad (2\text{–}20)$$

The output equation, Equation (2–19), can be written as

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2\text{–}21)$$

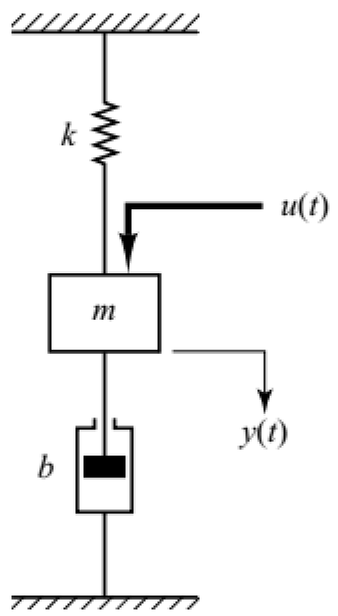
Equation (2–20) is a state equation and Equation (2–21) is an output equation for the system. They are in the standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$



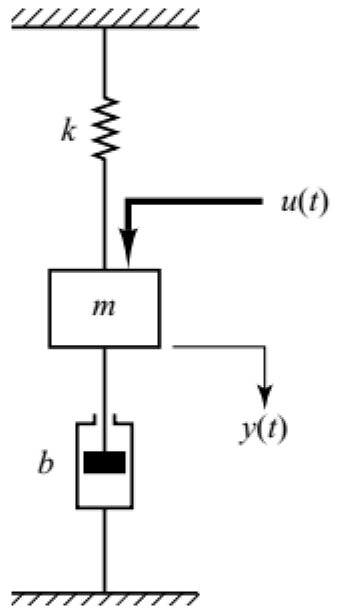
# Converting from State Space to a Transfer Function – Example

By substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $D$ into Equation (2–29), we obtain

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$



# Converting from State Space to a Transfer Function – Example

Note that

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

(Refer to Appendix C for the inverse of the 2 × 2 matrix.)
Thus, we have

$$G(s) = [1 \quad 0] \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2+bs+k}$$