Dynamics of Structures
MDOF Systems

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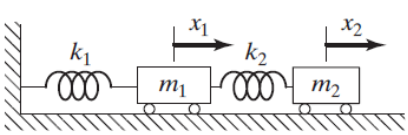
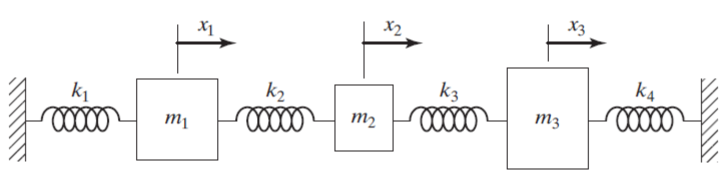
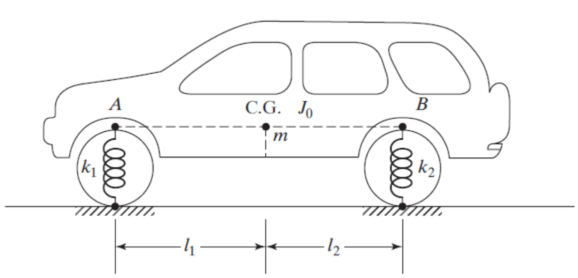
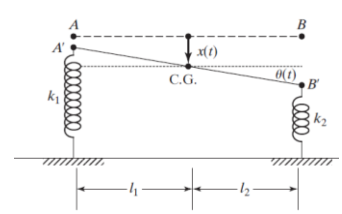
Introduction

- Systems that require two or more independent coordinates (variables) to describe their motion are called *multi-degree-of-freedom systems*.









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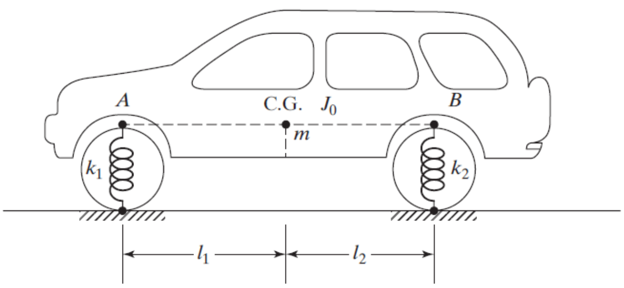
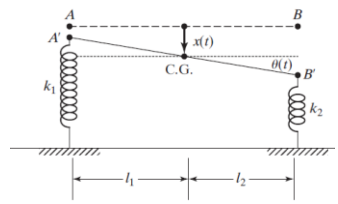
Introduction

Number of degrees of freedom of the system = Number of masses in the system × number of possible types of motion of each mass





2 DOF System

The system # DOF = the # government differential equations = the # system natural frequencies = the # system modes

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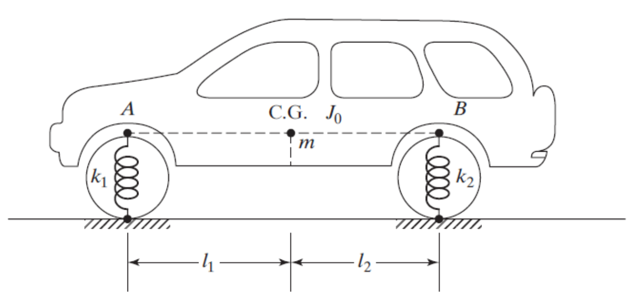
Introduction

- Thus, a two-degree-of-freedom system has two normal modes of vibration corresponding to two natural frequencies.
- If we give an arbitrary initial excitation to the system, the resulting free vibration will be a *superposition of the two normal modes* of vibration.
- However, if the system vibrates under the action of an external harmonic force, the resulting forced harmonic vibration takes place at the *frequency of the applied force*.
- Under harmonic excitation, resonance occurs, when the *forcing frequency* is equal to one of the *natural frequencies* of the system.



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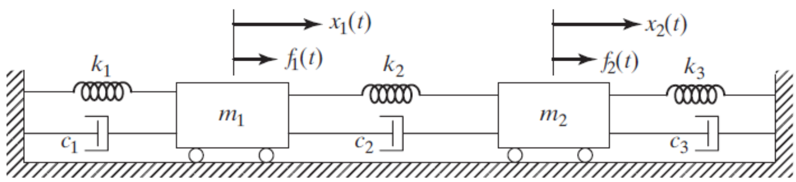
Equation of Motion

Consider a viscously damped two-degree-of-freedom spring-mass system, shown in Figure.
The motion of the system is completely described by the coordinates which define the positions of the masses and at any time t from the respective equilibrium positions.



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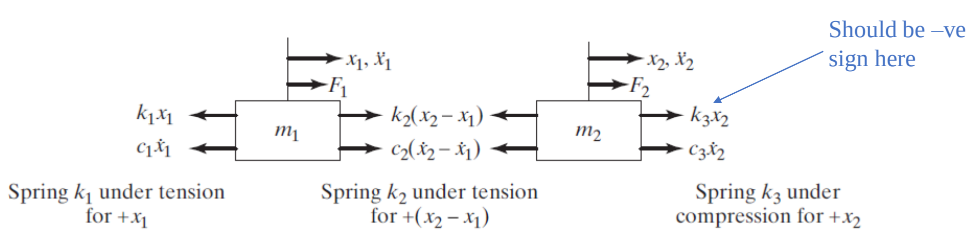
Equation of Motion

The application of Newton's second law of motion to each of the masses gives the equations of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$



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Equation of Motion

That represents two-coupled 2nd order nonhomogeneous differential equations

$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \qquad \vec{f}(t) = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

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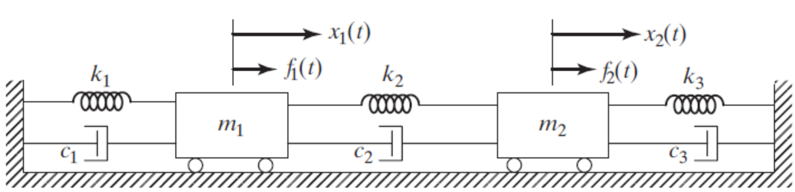
Free-Vibration Analysis of an Undamped System

For the system shown in figure, if we neglected the damping effect. The government equation for free vibration will have the form,

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$



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Free-Vibration Analysis of an Undamped System

Assume the masses m_1 *and* m_2 to have harmonic motion at the same frequency ω and same phase angle ϕ, then

$$x_1(t) = X_1 \cos(\omega t + \phi)$$

$$x_2(t) = X_2 \cos(\omega t + \phi)$$

where X_1 and X_2 are constants that denote the maximum amplitudes of $x_1(t)$ and $x_2(t)$, and ϕ is the phase angle. By substitute in the government equations,

$$[\{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2X_2]\cos(\omega t + \phi) = 0$$

$$[-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\}X_2]\cos(\omega t + \phi) = 0$$

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Free-Vibration Analysis of an Undamped System

$$[\{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2X_2]\cos(\omega t + \phi) = 0$$

$$[-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\}X_2]\cos(\omega t + \phi) = 0$$

Then,

$$\det\begin{bmatrix} \{-m_1\omega^2 + (k_1 + k_2)\} & -k_2 \\ -k_2 & \{-m_2\omega^2 + (k_2 + k_3)\} \end{bmatrix} = 0$$

$$(m_1m_2)\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$$

Which represents the frequency characteristics equation

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Free-Vibration Analysis of an Undamped System

The solution of this equation yields the system natural frequencies

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$

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Free-Vibration Analysis of an Undamped System

So, the system excitation in general will have the form,

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

Where c1 and c2 can be obtained from the initial conditions.

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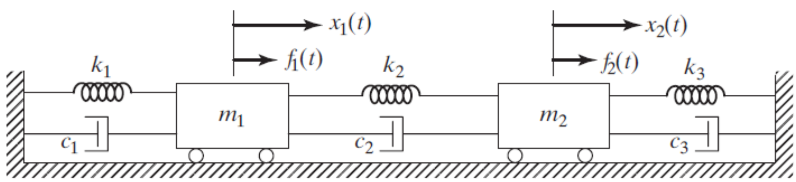
Free-Vibration Analysis of an Undamped System

Find the free-vibration response of the system shown in Fig. with $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, $m_1 = 10$, $m_2 = 1$, and $c_1 = c_2 = c_3 = 0$ for the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$.



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Free-Vibration Analysis of an Undamped System

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$10\omega^4 - 85\omega^2 + 150 = 0$$

from which the natural frequencies can be found as

$$\omega_1^2 = 2.5, \qquad \omega_2^2 = 6.0$$

or

$$\omega_1 = 1.5811, \qquad \omega_2 = 2.4495$$

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} X_1^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} X_1^{(2)}$$

The corresponding mode shapes

$$\left(1.5811, \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}\right) \ \& \ \left(2.4495, \begin{Bmatrix} -1 \\ 5 \end{Bmatrix}\right)$$

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Free-Vibration Analysis of an Undamped System

The free-vibration responses of the masses m_1 and m_2 are given by

$$x_1(t) = X_1^{(1)} \cos(1.5811t + \phi_1) + X_1^{(2)} \cos(2.4495t + \phi_2)$$

$$x_2(t) = 2X_1^{(1)} \cos(1.5811t + \phi_1) - 5X_1^{(2)} \cos(2.4495t + \phi_2)$$

From the initial conditions

$$x_1(t = 0) = 1 = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2$$

$$x_2(t = 0) = 0 = 2X_1^{(1)} \cos \phi_1 - 5X_1^{(2)} \cos \phi_2$$

$$\dot{x}_1(t = 0) = 0 = -1.5811 X_1^{(1)} \sin \phi_1 - 2.4495X_1^{(2)} \sin \phi_2$$

$$\dot{x}_2(t = 0) = -3.1622X_1^{(1)} + 12.2475X_1^{(2)} \sin \phi_2$$

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Free-Vibration Analysis of an Undamped System

Thus,

$$x_1(t) = \frac{5}{7} \cos 1.5811t + \frac{2}{7} \cos 2.4495t$$

$$x_2(t) = \frac{10}{7} \cos 1.5811t - \frac{10}{7} \cos 2.4495t$$

Try to plot it

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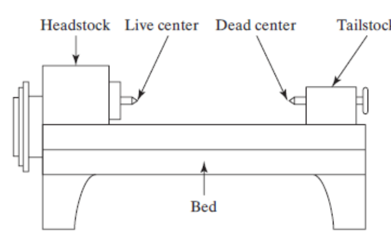
Generalized Coupling

- As stated earlier, an *n-degree-of-freedom* system requires *n independent coordinates* to describe its configuration.
- Usually, these coordinates are independent geometrical quantities measured from the equilibrium position of the vibrating body.
- However, it is possible to select some other set of *n* coordinates to describe the configuration of the system which is totally different than the original coordinates.
- Each of these sets of *n* coordinates is called the *generalized coordinates*.

Generalized Coupling – Lathe example

- As an example, consider the lathe shown in Fig.
- The lathe bed can be replaced by an elastic beam supported on short elastic columns and the headstock.
- Tailstock can be replaced by two lumped masses.

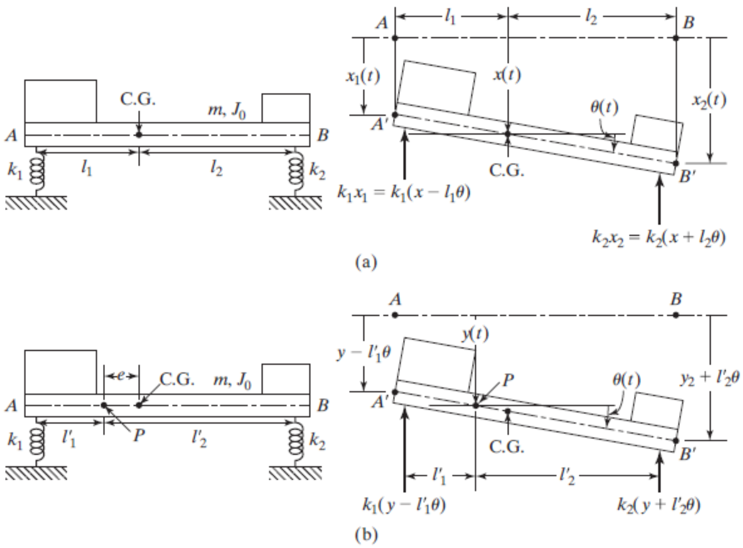


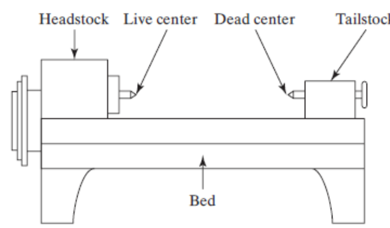
- Any of the following sets of coordinates can be used to describe the motion of this two-degree-of-freedom system:

1. Deflections $x_1(t)$ and $x_2(t)$ of the two ends of the lathe *AB*.
2. Deflection $x(t)$ of the C.G. and rotation $\theta(t)$.
3. Deflection of the end *A* $x_1(t)$ and rotation $\theta(t)$.
4. Deflection $y(t)$ of point *P* located at a distance *e* to the left of the C.G. and rotation $\theta(t)$.

Thus any set of these coordinates—(x_1, x_2), (x, θ), (x_1, θ), and (y, θ)—represents the generalized coordinates of the system. Now we shall derive the equations of motion of the lathe using two different sets of coordinates to illustrate the concept of coordinate coupling.

Generalized Coupling – Lathe example





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The Lathe Equation of Motion

Equations of Motion Using x(t) and $\theta(t)$

From the free-body diagram shown in Fig., with the positive values of the motion variables as indicated, the force equilibrium equation in the vertical direction can be written as

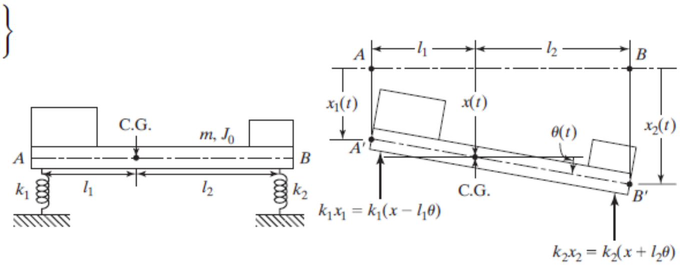
$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta)$$

and the moment equation about the C.G. can be expressed as

$$J_0\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2$$

In matrix Form,

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & (k_1l_1^2 + k_2l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



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The Lathe Equation of Motion

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

It can be seen that each of these equations contain x and θ. They become independent of each other if the coupling term $(k_1 l_1 - k_2 l_2)$ is equal to zero—that is, if $k_1 l_1 = k_2 l_2$. If $k_1 l_1 \neq k_2 l_2$, the resultant motion of the lathe AB is both translational and rotational when either a displacement or torque is applied through the C.G. of the body as an initial condition. In other words, the lathe rotates in the vertical plane and has vertical motion as well, unless $k_1 l_1 = k_2 l_2$. This is known as *elastic* or *static coupling*.

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Forced Vibration

The equations of motion of a general two-degree-of-freedom system under external forces can be written as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$F_j(t) = F_{j0} e^{i\omega t}, \qquad j = 1, 2$$

where ω is the forcing frequency. We can write the steady-state solutions as

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2$$

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Forced Vibration

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

we define the mechanical impedance $Z_{rs}(i\omega)$ as

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs}, \qquad r, s = 1, 2$$

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Forced Vibration

$$[Z(i\omega)]\vec{X} = \vec{F}_0$$

where

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega) \end{bmatrix} = \text{Impedance matrix}$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

and

$$\vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

$$\vec{X} = [Z(i\omega)]^{-1}\vec{F}_0$$

where the inverse of the impedance matrix is given by

$$[Z(i\omega)]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix}$$

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Forced Vibration

$$X_1(i\omega) = \frac{Z_{22}(i\omega)F_{10} - Z_{12}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

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Equations of Motion from Lagrange's Equations

- The use of Lagrange's equations is the standard method for setting up the equations of motion of multi-DOF systems.
- If the kinetic energy does not depend upon the displacements of the masses, and damping is ignored, then Lagrange's equations can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = Q_i \quad (i = 1, 2)$$

where T is the total kinetic energy of the system, and U the total potential energy. The q_i (q_1 and q_2 in this case) are generalized coordinates, and the Q_i are generalized forces.

Certain restrictions apply to the choice of these generalized quantities:

(a) The coordinates must be independent of each other;

(b) The generalized forces, Q_i, must do the same work as the actual forces, F_1 and F_2.

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Equations of Motion from Lagrange's Equations

These requirements are satisfied if we let $q_1 = z_1$, $q_2 = z_2$, $Q_1 = F_1$ and $Q_2 = F_2$. Writing down the kinetic energy, T, and the potential energy, U:

$$T = \tfrac{1}{2}m_1\dot{z}_1^2 + \tfrac{1}{2}m_2\dot{z}_2^2$$

$$U = \tfrac{1}{2}k_1 z_1^2 + \tfrac{1}{2}k_2(z_2 - z_1)^2 = \tfrac{1}{2}k_1 z_1^2 + \tfrac{1}{2}k_2 z_2^2 - k_2 z_1 z_2 + \tfrac{1}{2}k_2 z_1^2$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_1}\right) = m_1\ddot{z}_1 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial q_2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_2}\right) = m_2\ddot{z}_2$$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial z_1} = k_1 z_1 - k_2 z_2 + k_2 z_1 \qquad \frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial z_2} = k_2 z_2 - k_2 z_1$$

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Equations of Motion from Lagrange's Equations

Substituting these into Eq. (6.6), together with $F_1 = Q_1$ and $F_2 = Q_2$:

$$m_1\ddot{z}_1 + k_1 z_1 - k_2 z_2 + k_2 z_1 = F_1$$

$$m_2\ddot{z}_2 + k_2 z_2 - k_2 z_1 = F_2$$

or expressed in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

which is, of course, the same as both Eqs (6.2) and (6.5).

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Modal Coordinates

- In a multi-DOF system, it is possible to define modal coordinates instead, which define the amplitude of a deflection shape known as a mode.
- Thus, changing a single modal coordinate can affect the displacement over all or part of a structure.
- *There are two important classes of modes:*

1) The first type are assumed, or arbitrary, shapes. Assumed modes are used in component mode methods, often combined with normal modes of the components, and in the finite element method, the 'displacement functions' are essentially assumed modes defined between grid points.
2) The other class of modes is that of normal modes. These have the important property of making the system matrices diagonal, i.e. separating the freedoms so that they can be treated as a series of single degrees of freedom, and therefore much easier to deal with than a coupled system. Calculation of normal modes, however, requires the use of eigenvalues and eigenvectors.

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Transformation from Global to Modal Coordinates

Suppose that we wish to transform the equations in global coordinates, say,

$$[M]\{\ddot{z}\} + [K]\{z\} = \{F\}$$

into an alternate form in modal coordinates, say,

$$[\underline{M}]\{\ddot{q}\} + [\underline{K}]\{q\} = \{Q\}$$

The first step is to define the transformation giving the relationship between the global displacements, $\{z\}$, and the modal displacements, $\{q\}$:

$$\{z\} = [X]\{q\}$$

from which

$$\{\dot{z}\} = [X]\{\dot{q}\}$$

$$\{z\}^{\mathrm{T}} = \{q\}^{\mathrm{T}}[X]^{\mathrm{T}}$$

and

$$\{\dot{z}\}^{\mathrm{T}} = \{\dot{q}\}^{\mathrm{T}}[X]^{\mathrm{T}}$$

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Transformation from Global to Modal Coordinates

Transformation of the mass matrix

$$T = \tfrac{1}{2}\{\dot{z}\}^{\mathrm{T}}[M]\{\dot{z}\}$$

$$T = \tfrac{1}{2}\{\dot{z}\}^{\mathrm{T}}[M]\{\dot{z}\} = \tfrac{1}{2}\{\dot{q}\}^{\mathrm{T}}[X]^{\mathrm{T}}[M][X]\{\dot{q}\} = \tfrac{1}{2}\{\dot{q}\}^{\mathrm{T}}[\underline{M}]\{\dot{q}\}$$

$$[\underline{M}] = [X]^{\mathrm{T}}[M][X]$$

$$U = \tfrac{1}{2}\{z\}^{\mathrm{T}}[K]\{z\}$$

From Eqs (6.22), (6.26) and (6.28):

$$U = \tfrac{1}{2}\{z\}^{\mathrm{T}}[K]\{z\} = \tfrac{1}{2}\{q\}^{\mathrm{T}}[X]^{\mathrm{T}}[K][X]\{q\} = \tfrac{1}{2}\{q\}^{\mathrm{T}}[\underline{K}]\{q\}$$

where the new stiffness matrix, in the modal system, is $[\underline{K}]$, given by:

$$[\underline{K}] = [X]^{\mathrm{T}}[K][X]$$

Transformation from Global to Modal Coordinates

Transformation of external forces

$$W = \{\hat{z}\}^{\mathrm{T}}\{F\}$$

If $\{Q\}$ is the set of external modal forces giving the same loading as $\{F\}$, and $\{\hat{q}\}$ is the corresponding set of virtual displacements, the same work is done, and

$$W = \{\hat{z}\}^{\mathrm{T}}\{F\} = \{\hat{q}\}^{\mathrm{T}}\{Q\}$$

Therefore

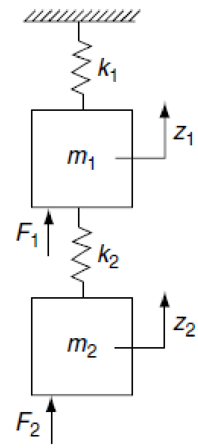
$$\{Q\} = [X]^{\mathrm{T}}\{F\}$$

which is the required expression relating the actual external forces, $\{F\}$, to the generalized or modal forces, $\{Q\}$.

Example

For the system shown in figure,

a. Transform the government equations into a form having no stiffness coupling
b. Show how actual input forces, F1 and F2, can be applied to the transformed equations as generalized modal forces, and how generalized modal displacements can be converted back to actual local displacements.



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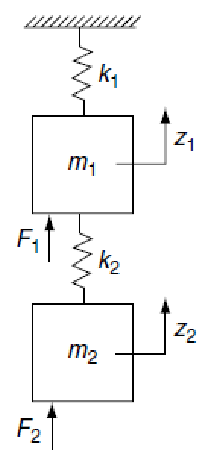
Example

Part a. The equation of motion can be obtained as,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



These equations were formed by taking the displacements, z_1 and z_2, of the individual masses, m_1 and m_2, as the generalized coordinates. This resulted in a coupled stiffness matrix, but an uncoupled mass matrix.

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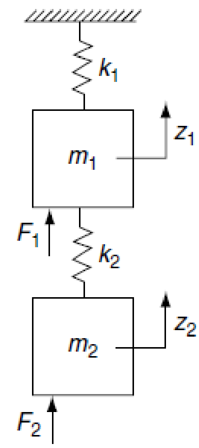
Example

Let us choose to represent the motion of the system by two coordinates, q_1 and q_2, where the actual displacements, z_1 and z_2, of the masses m_1 and m_2 are given by the following transformation, corresponding to

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

The columns of the 2×2 matrix are simple examples of assumed modes; we have arbitrarily decided that the first mode is $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, i.e. the two masses move one unit upwards together, and the second mode is $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, i.e. m_1 does not move at all, and m_2 moves one unit upwards. From Fig. 6.3, we see that the first mode compresses k_1, but not k_2, and that the second mode compresses only k_2, not k_1. We would therefore expect that using these modes would remove the stiffness coupling.



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Example

$$[M]\{\ddot{z}\} + [K]\{z\} = \{F\}$$

to the form of Eq. (6.25):

$$[\underline{M}]\{\ddot{q}\} + [\underline{K}]\{q\} = \{Q\}$$

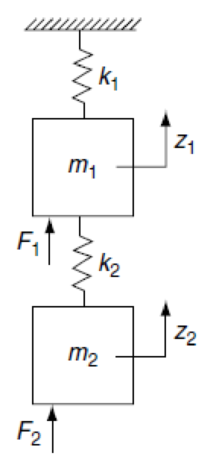
$$[\underline{M}] = [X]^{\mathrm{T}}[M][X]$$

and

$$[\underline{K}] = [X]^{\mathrm{T}}[K][X]$$

where

$$[X] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



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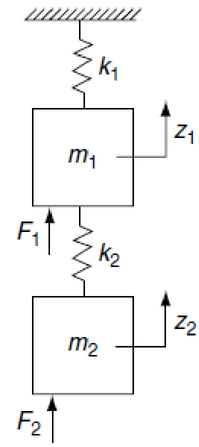
Example



$$[X]^{\mathrm{T}}[M][X]\{\ddot{q}\} + [X]^{\mathrm{T}}[K][X]\{q\} = \{Q\}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

Multiplied out, these become

$$\begin{bmatrix} (m_1 + m_2) & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

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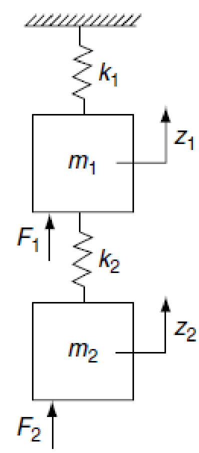
Example



Part b.

$$\{Q\} = [X]^{\mathrm{T}}\{F\}$$

to convert them to modal external forces $\{Q\}$. In this example:

$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{z\} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

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Orthogonality of mode shapes

Let $\{\boldsymbol{\varphi}\}_i$ be the i^{th} mode shape of free vibration of a system associated with the i^{th} natural frequency ω_i. Let $\{\boldsymbol{\varphi}\}_j$ be the j^{th} mode shape of free vibration of a system associated with the j^{th} natural frequency ω_j. Then, it can be shown that, $for\ i \neq j$

$$\lfloor\boldsymbol{\varphi}\rfloor_i[\boldsymbol{m}]\{\boldsymbol{\varphi}\}_j = 0\ , and \qquad \lfloor\boldsymbol{\varphi}\rfloor_i[\mathbf{k}]\{\boldsymbol{\varphi}\}_j = 0$$

Which means that the mode shapes of free vibration are orthogonal with respect to both the mass and stiffness matrices. On the other hand, if $i = j$

$$\lfloor\boldsymbol{\varphi}\rfloor_i[\boldsymbol{m}]\{\boldsymbol{\varphi}\}_i = M_i\ , and \qquad \lfloor\boldsymbol{\varphi}\rfloor_i[\mathbf{k}]\{\boldsymbol{\varphi}\}_i = K_i = M_i\omega_i^2$$

So, the mode shapes of free vibration can be used directly to decouple the system of differential equation "Diagonalization".

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

- ***Eigenvalues*** and ***eigenvectors*** are mathematical concepts, and their use is not confined to vibration theory.
- The words incidentally are derived from the German word *eigen*, meaning '*own*', so the eigenvalues of a set of equations are its own values, and the eigenvectors are its own vectors.
- In vibration the system eigenvalues defines the natural frequencies, and the system eigenvectors defines the mode shapes.

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$[M]\{\ddot{z}\} + [K]\{z\} = \{F\}$$

Since the eigenvalues and eigenvectors are unaffected by the applied forces, they can be obtained from the 'homogeneous' equations given by omitting the external forces:

$$[M]\{\ddot{z}\} + [K]\{z\} = 0$$

or, say,

$$\begin{bmatrix} m_{11} & m_{12} & \dots \\ m_{21} & m_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & k_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ \vdots \end{Bmatrix} = 0$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Since there is no damping, if the system is assumed to have been set in motion at frequency ω, all the elements of the vector $\{z\}$ will vibrate in phase, or exact antiphase, with each other. The responses at all the points z_1, z_2, etc. can be considered as horizontal or vertical projections of a rotating, complex unit vector, $e^{i\omega t}$, multiplied by the *real* constants $\bar{z}_1$, $\bar{z}_2$, etc. Then the responses of all the points, z_1, z_2, etc., are given by:

$$\{z\} = \begin{Bmatrix} z_1 \\ z_2 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \end{Bmatrix} e^{i\omega t} = \{\bar{z}\}e^{i\omega t}$$

The vector of accelerations, $\{\ddot{z}\}$, is given by differentiating Eq. (6.42) twice with respect to time:

$$\{\ddot{z}\} = \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \end{Bmatrix} = -\omega^2 \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \end{Bmatrix} e^{i\omega t} = -\omega^2\{\bar{z}\}e^{i\omega t}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$-\omega^2 \begin{bmatrix} m_{11} & m_{12} & \dots \\ m_{21} & m_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & k_{22} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \end{Bmatrix} = 0$$

or

$$\left(-\omega^2[M] + [K]\right)\{\bar{z}\} = 0$$

or writing $\lambda = \omega^2$

$$(-\lambda[M] + [K])\{\bar{z}\} = 0$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

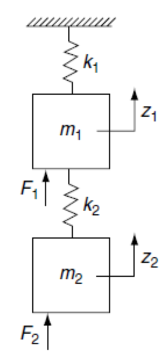
(a) Find the eigenvalues and eigenvectors of the undamped system shown in Fig. with:

$m_1 = 1\,\text{kg}; \quad m_2 = 2\,\text{kg}; \quad k_1 = 10\,\text{N/m} \quad k_2 = 10\,\text{N/m}$

Scale the eigenvectors so that the largest absolute element in each column is set to unity.

(b) Demonstrate that a transformation to modal coordinates using the eigenvectors as modes enables the equations to be written as uncoupled single-DOF systems.

(c) Rescale the eigenvectors so that the mass matrix, in normal mode coordinates, is a unit matrix.



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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Inserting the given numerical values:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Omitting the applied forces F_1 and F_2 gives the homogeneous equations, from which the eigenvalues and eigenvectors can be found:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = 0$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$\left(\begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0$$

or

$$\begin{bmatrix} (20 - \lambda) & -10 \\ -10 & (10 - 2\lambda) \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} = 0$$

$$\begin{vmatrix} (20 - \lambda) & -10 \\ -10 & (10 - 2\lambda) \end{vmatrix} = 0,$$

or

$$(20 - \lambda)(10 - 2\lambda) - 100 = 0,$$

giving

$$\lambda^2 - 25\lambda + 50 = 0.$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Solving this quadratic equation by the formula $\lambda = \frac{25 \pm \sqrt{25^2 - 200}}{2}$ gives the lower eigenvalue, $\lambda_1 = 2.1922$, and the higher eigenvalue, $\lambda_2 = 22.807$.

The two natural frequencies are

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{2.1922} = 1.480\,\text{rad/s, and}$$
$$\omega_2 = \sqrt{\lambda_2} = \sqrt{22.807} = 4.775\,\text{rad/s}$$

$$\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{10}{20 - \lambda} \end{pmatrix} \text{ or } \begin{pmatrix} \dfrac{10 - 2\lambda}{10} \end{pmatrix}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

The eigenvectors can now be called 'normal modes', and it is usual to write them as vectors:

$$\{\phi\}_1 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_1 \quad \{\phi\}_2 = \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix}_2$$

Using the first of the standard methods for normalizing the eigenvectors (making the largest absolute element in each column equal to unity), these are

$$\{\phi\}_1 = \begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} \quad \{\phi\}_2 = \begin{Bmatrix} 1 \\ -0.2807 \end{Bmatrix}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$\{z\} = [X]\{q\}$$

is now formed by *using the eigenvectors of the original equations as its columns.* The transformation matrix $[X]$, in this case, is therefore:

$$[X] = [\{\phi\}_1\{\phi\}_2] = \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix}$$

From Eq. (6.31), the new mass matrix $[\underline{M}]$, in modal coordinates, is given by:

$$[\underline{M}] = [X]^{\mathrm{T}}[M][X]$$

or numerically:

$$[\underline{M}] = \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix} = \begin{bmatrix} 2.315 & 0 \\ 0 & 1.157 \end{bmatrix}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

the new stiffness matrix, $[\underline{K}]$, in modal coordinates, is given by:

$$[\underline{K}] = [X]^{\mathrm{T}}[K][X]$$

or numerically:

$$[\underline{K}] = \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} 0.5615 & 1 \\ 1 & -0.2807 \end{bmatrix} = \begin{bmatrix} 5.075 & 0 \\ 0 & 26.40 \end{bmatrix}$$

The complete equations in normal mode coordinates are now given by substituting

$$[\underline{M}]\{\ddot{q}\} + [\underline{K}]\{q\} = \{Q\}$$

giving

$$\begin{bmatrix} 2.315 & 0 \\ 0 & 1.157 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 5.075 & 0 \\ 0 & 26.40 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

The use of the eigenvectors as modes in the transformation has made the mass and stiffness matrices diagonal.

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

$$\begin{bmatrix} 2.315 & 0 \\ 0 & 1.157 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 5.075 & 0 \\ 0 & 26.40 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

The previous equation now consists of two completely independent single-DOF equations:

$$\underline{m}_{11}\ddot{q}_1 + \underline{k}_{11}q_1 = Q_1$$

and

$$\underline{m}_{22}\ddot{q}_2 + \underline{k}_{22}q_2 = Q_2$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

The other standard method for normalizing the eigenvectors is to scale them so that the new mass matrix, $[\underline{M}]$, in modal coordinates, is equal to the unit matrix, $[I]$.

$$[\underline{M}] = \begin{bmatrix} \underline{m}_{11} & 0 \\ 0 & \underline{m}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

The eigenvectors are then described as *weighted normal* or *orthonormal*, and the corresponding stiffness matrix, $[\underline{K}]$, then becomes

$$[\underline{K}] = \begin{bmatrix} \underline{k}_{11} & 0 \\ 0 & \underline{k}_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

that is, a diagonal matrix of the squares of the natural frequencies. Scaling the eigenvectors so that the mass and stiffness matrices take these simple forms can be achieved in two ways.

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Method – 1

$$\alpha_i = \frac{1}{\sqrt{\underline{m}_{ii}}} \qquad \alpha_1 = \frac{1}{\sqrt{2.315}} \quad \text{and} \quad \alpha_2 = \frac{1}{\sqrt{1.157}}$$

$$\{\phi\}_1^{\text{new}} = \frac{1}{\sqrt{2.315}}\begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.3690 \\ 0.6572 \end{Bmatrix}$$

$$\{\phi\}_2^{\text{new}} = \frac{1}{\sqrt{1.15767}}\begin{Bmatrix} 1 \\ -0.2807 \end{Bmatrix} = \begin{Bmatrix} -0.9294 \\ 0.2609 \end{Bmatrix}$$

$$[X]^{\text{new}} = [\{\phi\}_1\{\phi\}_2] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}$$

Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Method – 1

$$[\underline{M}] = [X]^{\text{T}}[M][X] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}^{\text{T}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $[\underline{M}]$ becomes a unit matrix and

$$[\underline{K}] = [X]^{\text{T}}[K][X] = \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}^{\text{T}} \begin{bmatrix} 20 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} 0.3690 & -0.9294 \\ 0.6572 & 0.2609 \end{bmatrix}$$
$$= \begin{bmatrix} 2.192 & 0 \\ 0 & 22.807 \end{bmatrix}$$

where the diagonal terms are now seen to be equal to the squares of the natural frequencies, ω_1^2 and ω_2^2.

Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Method – 2

$$\underline{m}_{ii} = \{\underline{\phi}\}_i^{\mathrm{T}}[M]\{\underline{\phi}\}_i$$

where $\{\underline{\phi}\}_i$ represents eigenvector i in any arbitrary form, for example, in the non-standard form:

$$\{\underline{\phi}\}_1 = \begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} \quad \text{and} \quad \{\underline{\phi}\}_2 = \begin{Bmatrix} -3.561 \\ 1 \end{Bmatrix}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Method – 2

and $[M]$ is the original mass matrix in global coordinates, in this example given as

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Equations (P) and (U) can be combined, giving the more usual expression for α_i:

$$\alpha_i = \left(\frac{1}{\{\underline{\phi}\}_i^{\mathrm{T}}[M]\{\underline{\phi}\}_i}\right)^{\frac{1}{2}}$$

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Transformation using Eigen-Modes
Eigenvalues and eigenvectors

Method – 2

Substituting the numerical values above into Eq. (W) gives $\alpha_1 = 0.6572$ and $\alpha_2 = 0.2609$. The orthonormal eigenvectors are then:

$$\{\phi\}_1 = \alpha_1\{\underline{\phi}\}_1 = 0.6572\begin{Bmatrix} 0.5615 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.3690 \\ 0.6572 \end{Bmatrix}$$

$$\{\phi\}_2 = \alpha_2\{\underline{\phi}\}_2 = 0.260958\begin{Bmatrix} -3.561 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.9294 \\ 0.2609 \end{Bmatrix}$$

which are seen to be the same as the eigenvectors given in Eqs (Q_1) and (Q_2) by the first method.

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Eigenvalues and Eigenvectors from Flexible Matrix

$$(-\lambda[M] + [K])\{\bar{z}\} = 0$$

where $\lambda = \omega^2$. The eigenvalues, λ_i, were then found as the roots of the characteristic equation given by expanding the determinant $|[K] - \lambda[M]|$, or $|[M]^{-1}[K] - \lambda[I]|$, and the eigenvectors were found by substituting the eigenvalues back into the equations of motion.

Very often, the flexibility matrix, $[K]^{-1}$, is known rather than $[K]$. In this case, there is no need to invert $[K]^{-1}$ to give $[K]$, since the eigenvalues and vectors can be extracted equally well from the mass and flexibility matrices, as follows. Pre-multiplying Eq. (6.45) by $[K]^{-1}$ gives

$$-\lambda[K]^{-1}[M]\{\bar{z}\} + [K]^{-1}[K]\{\bar{z}\} = 0$$

or

$$\left([K]^{-1}[M] - \underline{\lambda}[I]\right)\{\bar{z}\} = 0 \tag{6.47}$$

where $\underline{\lambda} = 1/\lambda = 1/\omega^2$. The eigenvalues $\underline{\lambda}_i$ are then given by the roots of the characteristic equation:

$$\left|[K]^{-1}[M] - \underline{\lambda}[I]\right| = 0 \tag{6.48}$$

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Damping in multi-DOF systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{z}_i}\right)+\frac{\partial D}{\partial \dot{z}_i}+\frac{\partial U}{\partial z_i}=\overline{F}_i \quad (i=1,2)$$

where in this case the dissipation energy, D, analogous to the kinetic energy T and the potential energy, U, is given by:

$$D=\tfrac{1}{2}c_1\dot{z}_1^2+\tfrac{1}{2}c_2(\dot{z}_2-\dot{z}_1)^2$$

Whichever method is used, the equations of motion are the same as for the undamped system, but with additional terms due to the damping:

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Example

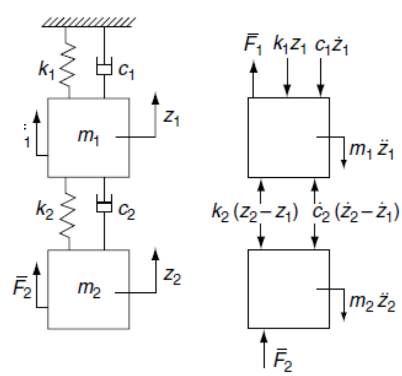
$$\overline{F}_1-m_1\ddot{z}_1-k_1z_1-c_1\dot{z}_1+c_2(\dot{z}_2-\dot{z}_1)+k_2(z_2-z_1)=0$$

$$\overline{F}_2-m_2\ddot{z}_2-c_2(\dot{z}_2-\dot{z}_1)-k_2(z_2-z_1)=0$$

or in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix}+\begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}=\begin{bmatrix} \overline{F}_1 \\ \overline{F}_2 \end{bmatrix}$$

$$[M]\{\ddot{z}\}+[C]\{\dot{z}\}+[K]\{z\}=\{\overline{F}\}$$



The damping matrix, $[C]$, is a square, symmetric matrix, comparable to the mass and stiffness matrices.

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Solution of damped systems

There are fundamentally two ways of solving an equation such as Eq. (6.65) for particular forms of the applied forces $\{\overline{F}\}$. These are

(1) Solution by the normal mode summation method.

(2) Direct solution of the global equations of motion.

Method (1), the normal mode summation method, is usually preferred, as it is completely analytic. Unfortunately, some forms of the damping matrix, $[C]$, can cause problems when we try to use it, and these are discussed in the next section.

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Solution of damped systems

Sometimes the damping matrix cannot be diagonalized

One solution to this problem would be to use the fact that any damping matrix can be made diagonal, together with the mass and stiffness matrices, if complex eigenvalues and eigenvectors are used. This is a considerable extra complication, and is avoided when possible, but there are some systems where this approach has to be used. Some examples are the following:

(1) Rotating systems such as helicopter rotors, where centrifugal and Coriolis forces can lead to damping coupling between the modes;

(2) Aircraft flutter analysis and response calculations using the p method, where large cross-damping terms can arise from aerodynamic forces. The original flutter solution method, now known as the k method, was, in fact, originally devised specifically to avoid the use of complex eigenvalues and eigenvectors.

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Proportional Damping

$$[M]\{\ddot{z}\} + [C]\{\dot{z}\} + [K]\{z\} = \{\overline{F}\}$$

$$[C] = a[M] + b[K]$$

$$[M]\{\ddot{z}\} + \mathrm{i}g[K]\{z\} + [K]\{z\} = \{\overline{F}\}$$

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Response to Harmonic Excitation

Assume 2-Dof damped system with the equation of motion

$$M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t)$$

$$\mathbf{F}(t) = \mathbf{F}e^{i\omega t}$$

where in general the mass, damping and stiffness matrices are given by

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}, \ C = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}, \ K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

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Response to Harmonic Excitation

$$\mathbf{x}(t) = \mathbf{X}(i\omega)e^{i\omega t}$$

By substitute in government equation

$$Z(i\omega)\mathbf{X}(i\omega) = \mathbf{F}$$

Where *Z* represents the impedance function

$$Z(i\omega) = -\omega^2 M + i\omega C + K$$

$$z_{ij}(i\omega) = -\omega^2 m_{ij} + i\omega c_{ij} + k_{ij},\ i, j = 1, 2$$

Then the solution will take the form

$$\mathbf{X}(i\omega) = Z^{-1}(i\omega)\mathbf{F}$$

Response to Harmonic Excitation

Then the solution will take the form

$$\mathbf{X}(i\omega) = Z^{-1}(i\omega)\mathbf{F}$$

where the inverse has the explicit form

$$Z^{-1}(i\omega) = \frac{1}{|Z(i\omega)|}\begin{bmatrix} z_{22}(i\omega) & -z_{12}(i\omega) \\ -z_{12}(i\omega) & z_{11}(i\omega) \end{bmatrix}$$

$$= \frac{1}{z_{11}(i\omega)z_{22}(i\omega) - z_{12}^2(i\omega)}\begin{bmatrix} z_{22}(i\omega) & -z_{12}(i\omega) \\ -z_{12}(i\omega) & z_{11}(i\omega) \end{bmatrix}$$

And finally

$$X_1(i\omega) = \frac{z_{22}(i\omega)F_1 - z_{12}(i\omega)F_2}{z_{11}(i\omega)z_{22}(i\omega) - z_{12}^2(i\omega)},\ X_2(i\omega) = \frac{-z_{12}(i\omega)F_1 + z_{11}(i\omega)F_2}{z_{11}(i\omega)z_{22}(i\omega) - z_{12}^2(i\omega)}$$

Response to Harmonic Excitation

For undamped second order systems

$$z_{11}(\omega) = k_{11} - \omega^2 m_1,\ z_{22}(\omega) = k_{22} - \omega^2 m_2,\ z_{12}(\omega) = k_{12}$$

$$X_1(\omega) = \frac{(k_{22} - \omega^2 m_2)F_1 - k_{12}F_2}{(k_{11} - \omega^2 m_1)(k_{22} - \omega^2 m_2) - k_{12}^2}$$

$$X_2(\omega) = \frac{-k_{12}F_1 + (k_{11} - \omega^2 m_1)F_2}{(k_{11} - \omega^2 m_1)(k_{22} - \omega^2 m_2) - k_{12}^2}$$

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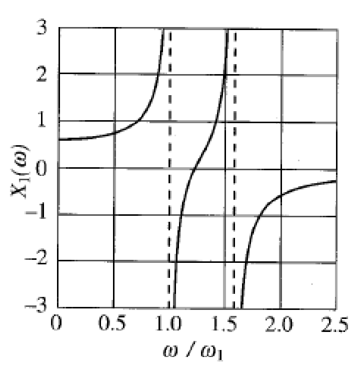
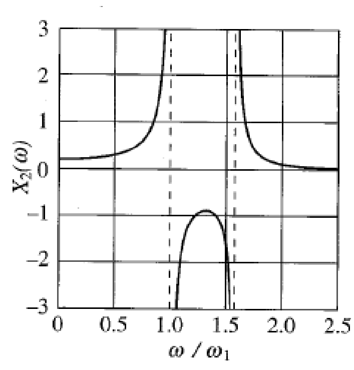
Response to Harmonic Excitation

For $F_2 = 0$

$$X_1(\omega) = \frac{F_1}{5k}\frac{3 - 2(\omega/\omega_1)^2}{[1 - (\omega/\omega_1)^2][1 - (\omega/\omega_2)^2]}$$

$$X_2(\omega) = \frac{F_1}{5k}\frac{1}{[1 - (\omega/\omega_1)^2][1 - (\omega/\omega_2)^2]}$$





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Response of multi-DOF systems by direct integration
Runge–Kutta Method

$$m\ddot{z} + c\dot{z} + kz = F$$

was split into two first-order equations

$$u = \dot{z}$$

and, since $\ddot{z} = \dot{u}$,

$$\dot{u} = \frac{F}{m} - \frac{c}{m}u - \frac{k}{m}z$$

$$\langle u \rangle = \tfrac{1}{6}\left(u_j^{(1)} + 2u_j^{(2)} + 2u_j^{(3)} + u_j^{(4)}\right)$$

and

$$\langle \dot{u} \rangle = \tfrac{1}{6}\left(\dot{u}_j^{(1)} + 2\dot{u}_j^{(2)} + 2\dot{u}_j^{(3)} + \dot{u}_j^{(4)}\right)$$

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Response of multi-DOF systems by direct integration
Runge–Kutta Method for multi-degree of freedom systems

$$\{u\} = \{\dot{z}\}$$

and, using Eq. (6.79):

$$\{\dot{u}\} = \{\ddot{z}\} = [M]^{-1}(\{F\} - [C]\{u\} - [K]\{z\})$$

Equations (6.80a) and (6.80b) are usually combined, giving:

$$\begin{Bmatrix} \{u\} \\ \{\dot{u}\} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} \{z\} \\ \{u\} \end{Bmatrix} + \begin{bmatrix} 0 \\ [M]^{-1}\{F\} \end{bmatrix}$$

now become vectors of average slopes for the multi-DOF version:

$$\{\langle u \rangle\} = \tfrac{1}{6}\left(\{u_j\}^{(1)} + 2\{u_j\}^{(2)} + 2\{u_j\}^{(3)} + \{u_j\}^{(4)}\right)$$

$$\{\langle \dot{u} \rangle\} = \tfrac{1}{6}\left(\{\dot{u}_j\}^{(1)} + 2\{\dot{u}_j\}^{(2)} + 2\{\dot{u}_j\}^{(3)} + \{\dot{u}_j\}^{(4)}\right)$$

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Response of multi-DOF systems by direct integration

Runge–Kutta Method for multi-degree of freedom systems

$$\{z_{j+1}\} = \{z_j\} + \frac{h}{6}\left(\{u_j\}^{(1)} + 2\{u_j\}^{(2)} + 2\{u_j\}^{(3)} + \{u_j\}^{(4)}\right)$$

$$\{u_{j+1}\} = \{u_j\} + \frac{h}{6}\left(\{\dot{u}_j\}^{(1)} + 2\{\dot{u}_j\}^{(2)} + 2\{\dot{u}_j\}^{(3)} + \{\dot{u}_j\}^{(4)}\right)$$

Definition of Average Slopes in Fourth-order Runge–Kutta Method

$\{z_j\}^{(1)} = \{z_j\}$	$\{u_j\}^{(1)} = \{u_j\}$	$\{\dot{u}_j\}^{(1)} = [M]^{-1}\left(\{F_j\} - [C]\{u_j\}^{(1)} - [K]\{z_j\}^{(1)}\right)$
$\{z_j\}^{(2)} = \{z_j\} + \frac{h}{2}\{u_j\}^{(1)}$	$\{u_j\}^{(2)} = \{u_j\} + \frac{h}{2}\{\dot{u}_j\}^{(1)}$	$\{\dot{u}_j\}^{(2)} = [M]^{-1}\left(\{F_j\} - [C]\{u_j\}^{(2)} - [K]\{z_j\}^{(2)}\right)$
$\{z_j\}^{(3)} = \{z_j\} + \frac{h}{2}\{u_j\}^{(2)}$	$\{u_j\}^{(3)} = \{u_j\} + \frac{h}{2}\{\dot{u}_j\}^{(2)}$	$\{\dot{u}_j\}^{(3)} = [M]^{-1}\left(\{F_j\} - [C]\{u_j\}^{(3)} - [K]\{z_j\}^{(3)}\right)$
$\{z_j\}^{(4)} = \{z_j\} + h\{u_j\}^{(3)}$	$\{u_j\} = \{u_j\} + h\{\dot{u}_j\}^{(3)}$	$\{\dot{u}_j\}^{(4)} = [M]^{-1}\left(\{F_j\} - [C]\{u_j\}^{(4)} - [K]\{z_j\}^{(4)}\right)$

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