Airframe Design and Construction

Bending stresses – Beam theory

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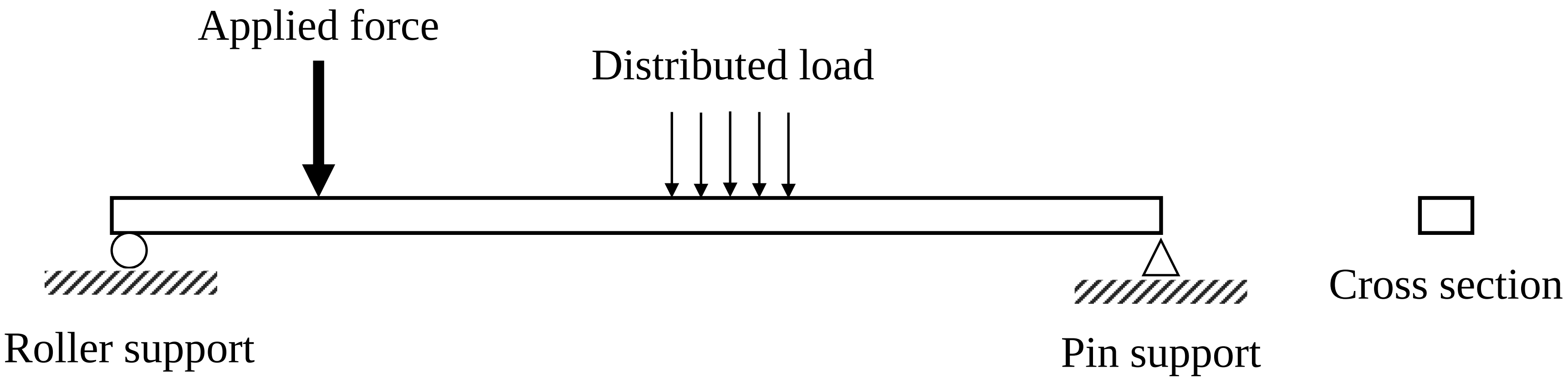
Beam theory

The word beam refers to an element that

- Has a section dimensions that are small relative to its length
- Designed to support transverse loads



Beam theory

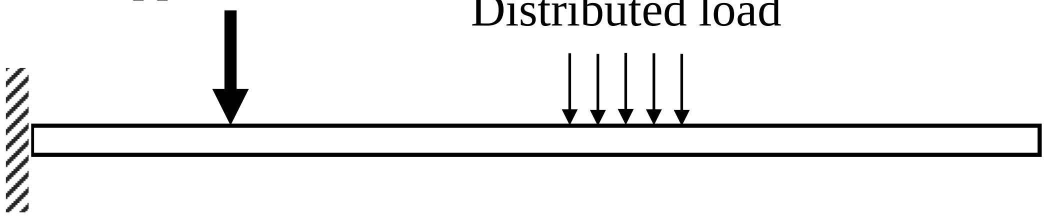
- In the present course, wing stresses are calculated based on the beam theory which treats the wing as a cantilevered beam based on the following assumptions:

- Transverse sections of the wing that are originally plane remains plane after bending; which means that strain variation is linear.
- This assumption neglects strain due shear stresses in skin (shear-lag effect).
- This assumption is true except near major cut-outs and concentrated loads.

- The stress distribution is directly proportional to strain and is also linear. This assumption is corrected by using the so-called effective section.

Beam theory

- $M = EI \frac{d^2v}{dx^2}$
- $V = \frac{dM}{dx} = EI \frac{d^3v}{dx^3}$
- $q = \frac{dV}{dx} = EI \frac{d^4v}{dx^4}$



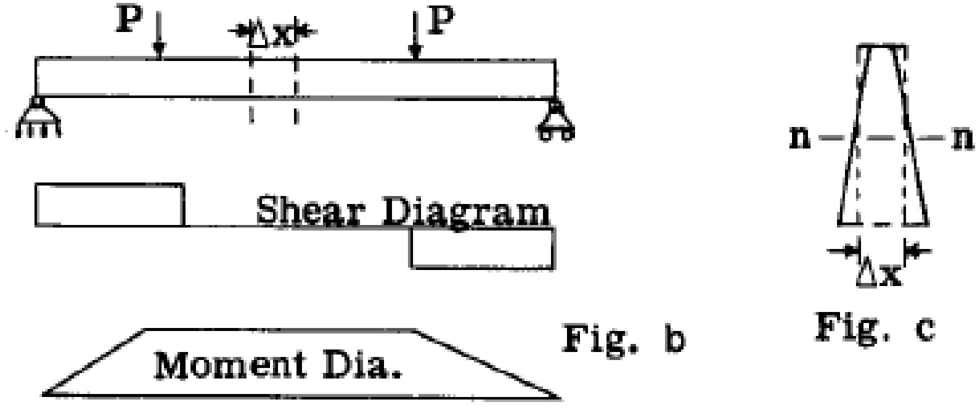
- E is the elastic modulus
- I is the second moment of area
- q distributed load

Bending stress

- Plane sections remains planes after bending, but they rotate w.r.t each other.

- After applying stresses, the top fiber are shortened, and the bottom fibers are elongated.

- At certain plane on the cross-section, the fiber suffer no deformation and no stresses, and this location is referred to as the *neutral axis*.



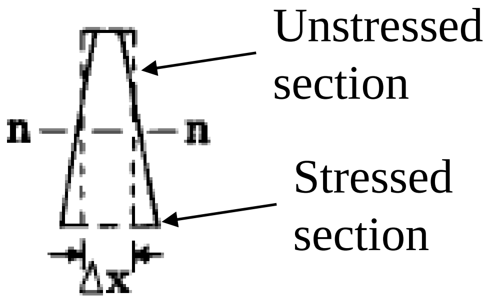


Fig. c

Location of Neutral axis

The neutral axis passes through the centroid of the cross-sectional area when the material follows Hooke's law and there is no axial force acting on the cross section.

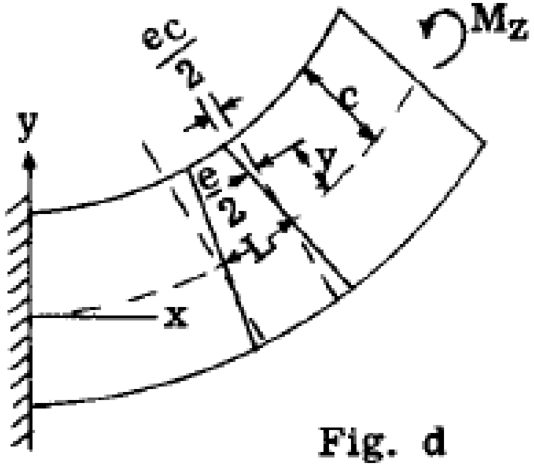
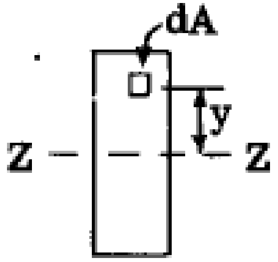


Fig. d



Beam Section

Equation of bending stresses

Assumptions:

- ➢Straight cantilever beam with constant cross-section.
- ➢The beam subject to pure bending such that no torsion moments applied.

It is required to determine the *neutral axis* direction and the bending stress at any point within the cross-section.

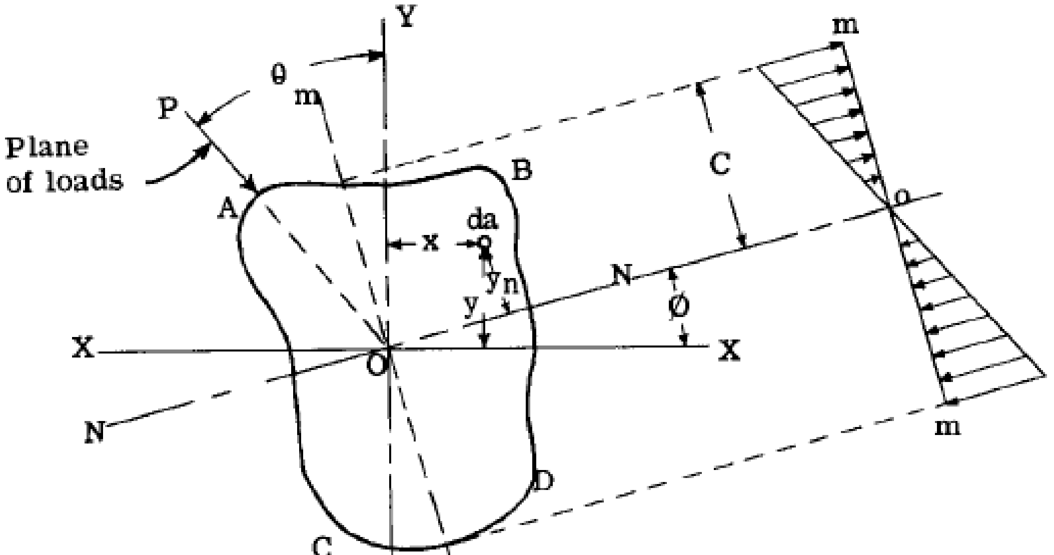


Fig. A13.1

Fig. A13.2

Equation of bending stresses

Let σ represent unit bending stress at any point a distance y_n from the neutral axis. Then the stress σ on da is

$$\sigma = ky_n \quad \text{- - - - - - - - - - - - - - - - -} \quad (1)$$

where k is a constant. Since the position of the neutral axis is unknown, y_n will be expressed for convenience in terms of rectangular coordinates with respect to the axes X-X and Y-Y.

Thus, $y_n = (y - x \tan \emptyset) \cos \emptyset$ - - - - - -

$$= y \cos \emptyset - x \sin \emptyset \quad \text{- - - - - - - - - - - - -} \quad (2)$$

Then Eq. (1) becomes

$$\sigma = k\,(y \cos \emptyset - x \sin \emptyset) \quad \text{- - - - - - - - - -} \quad (3)$$

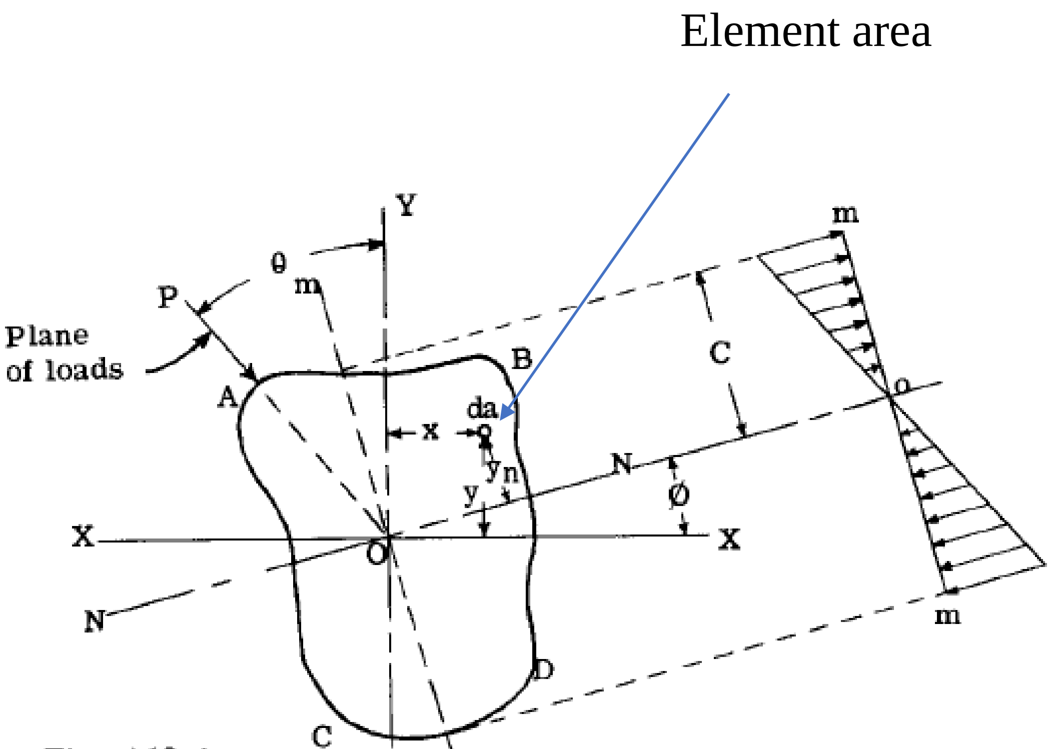


Fig. A13.1

Fig. A13.2

Equation of bending stresses

Let M represent the bending moment in the plane of the loads; then the moment about axis X-X and Y-Y is $M_x = M \cos \theta$ and $M_y = M \sin \theta$. The moment of the stresses on the beam section about axis X-X is $\int \sigma \, da \, y$. Hence, taking moments about axis X-X, we obtain for equilibrium,

$$M \cos \theta = \int \sigma \, da \, y$$

$$= \int k \, (\cos \phi \, y^2 da - \sin \phi \, xyda)$$

$$= k \cos \phi \int y^2 da - k \sin \phi \int xyda \quad - - - - \quad (4)$$

In similar manner, taking moments about the Y-Y axis

$$M \sin \theta = \int \sigma \, da \, x$$

whence

$$M \sin \theta = - k \sin \phi \int x^2 da + k \cos \phi \int xyda \quad (4a)$$

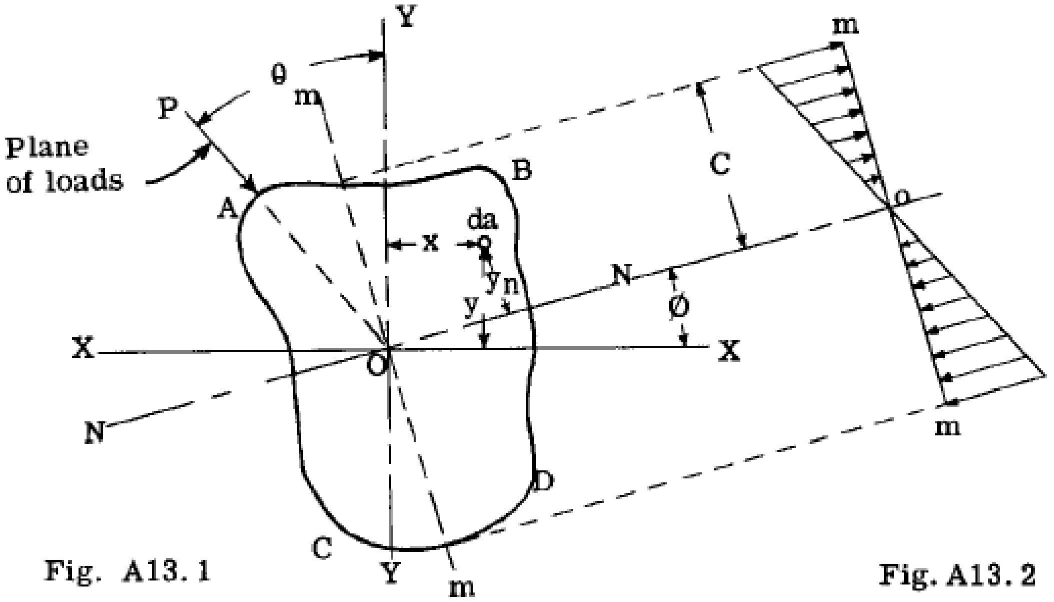


Fig. A13.1

Fig. A13.2

Equation of bending stresses

$$M \cos \theta = \int \sigma \, da \, y$$

$$= \int k (\cos \phi \, y^2 da - \sin \phi \, xyda)$$

$$= k \cos \phi \int y^2 da - k \sin \phi \int xyda \quad ---- \quad (4)$$

The fiber stresses can be found without resort to principal axes or to the neutral axis.

Equation (4) can be written:

$$M_x = k \cos \phi \, I_x - k \sin \phi \, I_{xy} \quad ------ \quad (11)$$

where $I_x = \int y^2 da$ and $I_{xy} = \int xyda$, and $M_x = M \cos \theta$.

In like manner,

$$M_y = - k \sin \phi \, I_y + k \cos \phi \, I_{xy} \quad ----- \quad (12)$$

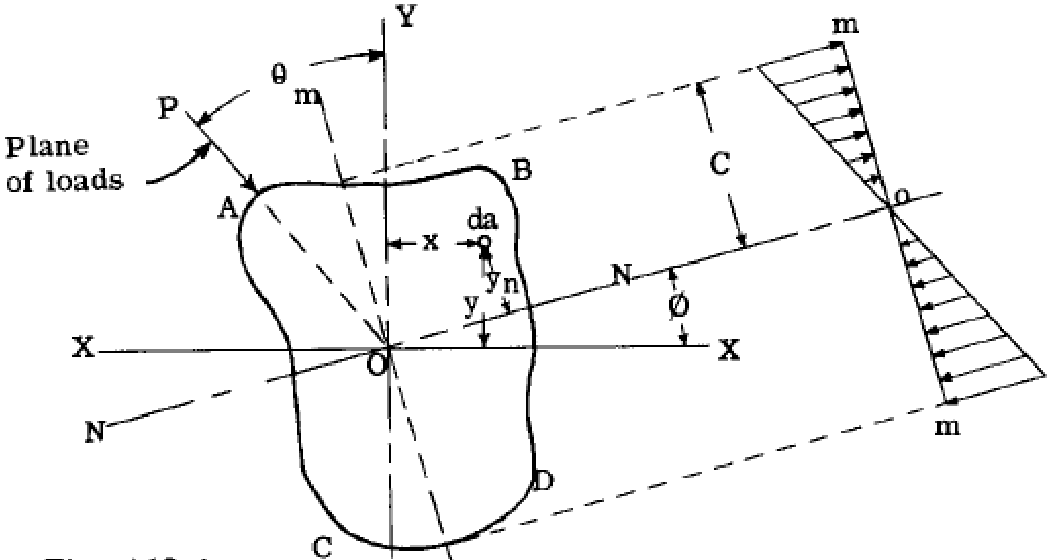


Fig. A13.1

Fig. A13.2

Equation of bending stresses

Solving equations (11) and (12) for sin Ø and cos Ø and substituting their values in equation (3), we obtain the following expression for the fiber stress σ_b: -

$$\sigma_b = -\frac{(M_y I_x - M_x I_{xy})}{I_x I_y - I_{xy}^2} x - \frac{(M_x I_y - M_y I_{xy})y}{I_x I_y - I_{xy}^2} \quad - (13)$$

For simplification, let

$$K_1 = I_{xy}/(I_x I_y - I^2{}_{xy})$$

$$K_2 = I_y/(I_x I_y - I^2{}_{xy})$$

$$K_3 = I_x/(I_x I_y - I^2{}_{xy})$$

Substituting these values in Equation (13): -

$$\sigma_b = -(K_3 M_y - K_1 M_x)\, x - (K_2 M_x - K_1 M_y) y \quad - (14)$$

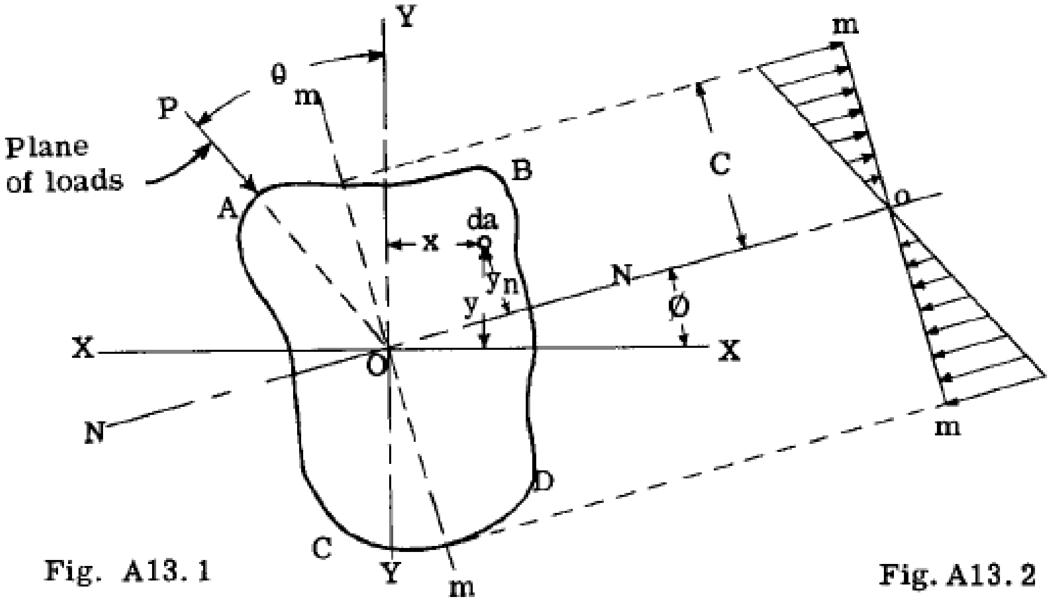


Fig. A13.1

Fig. A13.2

Equation of bending stresses

The bending stresses about symmetric *x* and *y* axes (Principal axes),

$$\sigma_b = -\frac{M_x y}{I_x} - \frac{M_y x}{I_y}$$

If the section is symmetric and $M_x = 0$, then

$$\sigma_b = -\frac{M_y x}{I_y}$$

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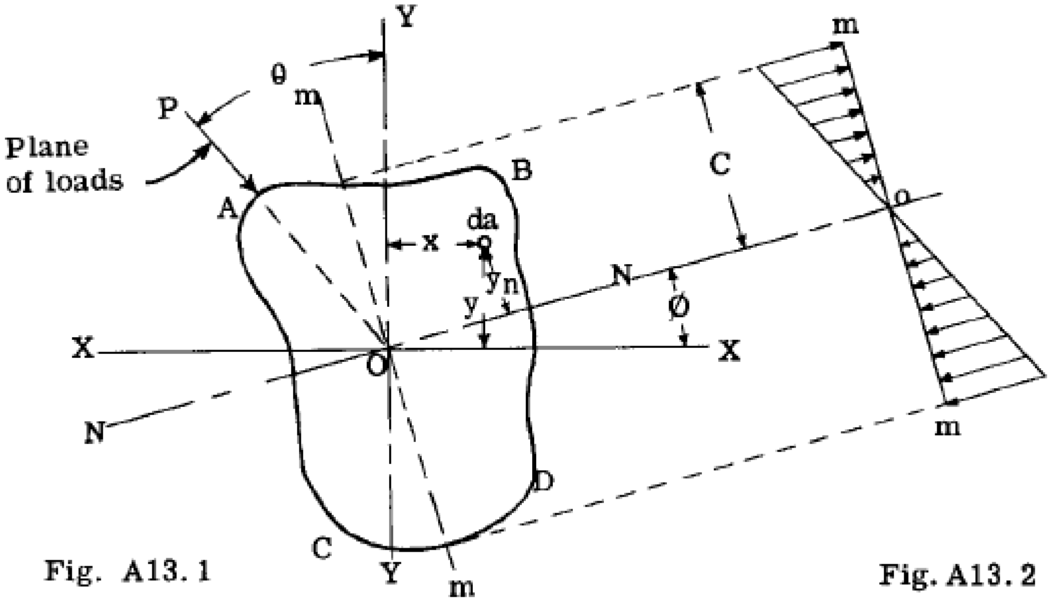

Fig. A13.1 Fig. A13.2

Process for calculating the Neutral Axis position

1. Define the stress equation to equal to zero.

$$\sigma_b = -\frac{(M_y I_x - M_x I_{xy})}{I_x I_y - I_{xy}^2} x - \frac{(M_x I_y - M_y I_{xy}) y}{I_x I_y - I_{xy}^2} .$$

2. Then,

$$(K_3 M_y - K_1 M_x) x = -(K_2 M_x - K_2 M_y) y$$

Which applies to any point on the neutral axis.

3. The neutral axis equation is

$$\tan \phi = -\frac{(K_3 M_y - K_1 M_x)}{(K_2 M_x - K_1 M_y)} = \frac{y}{x}$$

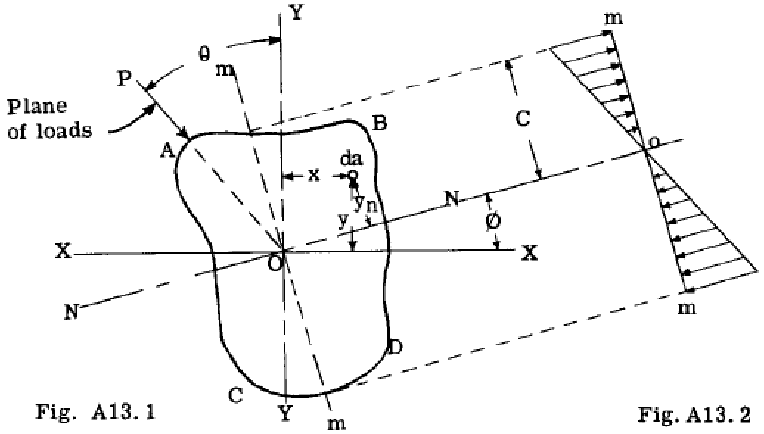

Fig. A13.1 Fig. A13.2

Neutral Axis and centroid

- ➢Neutral axis passes through the centroid.
- ➢Centroid depends only on the section geometry.

$$\bar{x} = \frac{\int x \, dA}{A} = \frac{\sum_{i=1}^{N} x_i A_i}{\sum_{i=1}^{N} A_i}$$

- ➢The neutral axis depends on the loading condition in addition to the section geometry

$$\tan \phi = -\frac{(K_3 M_y - K_1 M_x)}{(K_2 M_x - K_1 M_y)} = \frac{y}{x}$$

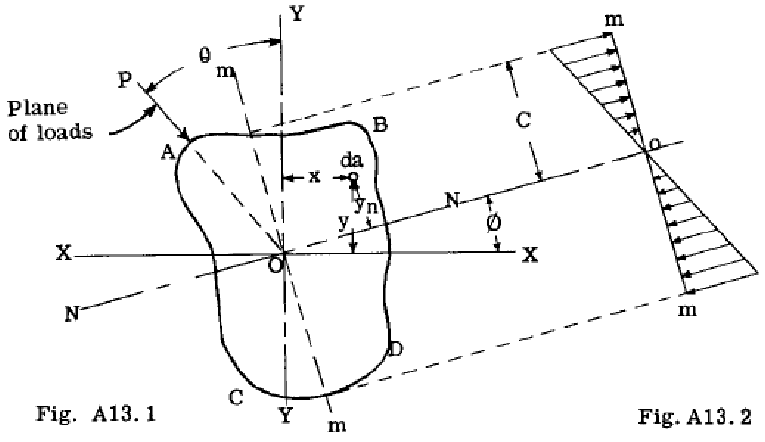

Fig. A13.1 Fig. A13.2

The centroid is important to determine the section moment of inertia and the neutral axis is important to determine the maximum stresses or the section stress distribution.