

System Dynamics

Modeling in frequency domain

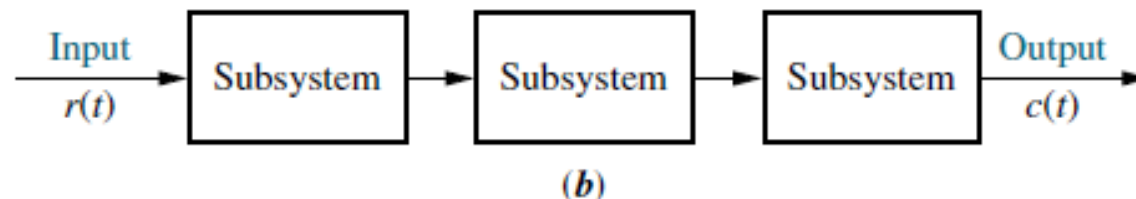
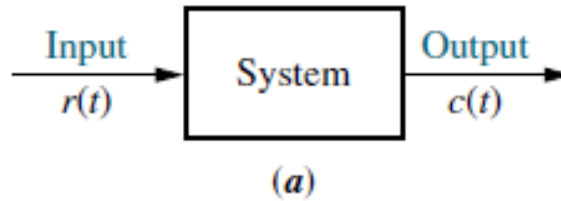
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System Modeling

- We aim to develop mathematical models from schematics of physical systems.
- The mathematical model is obtained by applying the fundamental physical laws of science and engineering.
- A system mathematical model is usually a differential equation that relates the system input to the system output.



Transfer function

The transfer function is a function that algebraically relates a system's output to its input.

This function will allow separation of the input, system, and output into three separate and distinct parts.

It allows us to algebraically combine mathematical representations of subsystems to yield a total system representation.

- Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Transfer function in frequency domain

D.E that mathematically represent the system

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t)$$

Solve the D.E. in Frequency domain using Laplace Transform.

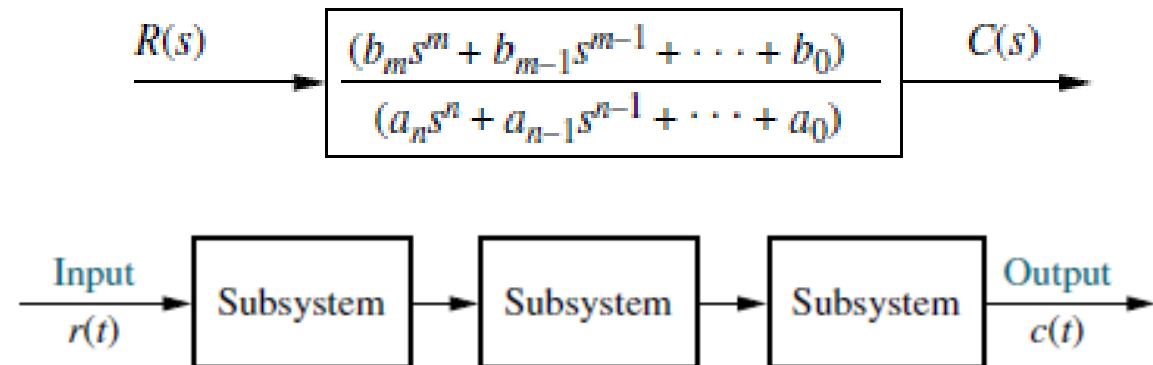
$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0) R(s)$$

Construct the T.F.

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$

$$C(s) = R(s) G(s)$$

The main T.F. function can be divided into subsystems using Partial Fraction.



Laplace Transform

- Operational method that is used to solve linear differential equations.
- It transforms functions such as the exponential functions into algebraic functions of *complex variable s* .
- Operations such as integration or differentiation can be replaced by algebraic equations in the complex plane.
- It allows the use of graphical techniques for predicting the system performance without actually solving the system differential equation.

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$$

$$s = \sigma + \omega j$$

Complex variables

Complex variable

$$s = \sigma + \omega j$$

Complex function

$$F(s) = F_x + F_y j$$

Magnitude

$$|F(s)| = \sqrt{F_x^2 + F_y^2}$$

Angle

$$\theta = \tan^{-1} \left(\frac{F_y}{F_x} \right)$$

Complex conjugate

$$\overline{F(s)} = F_x - F_y j$$

Euler's theorem

The power series expansion of $\cos \theta$ and $\sin \theta$ take the form

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

Then

$$\cos \theta + j \sin \theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \dots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

So we can see that

$$\cos \theta + j \sin \theta = e^{j\theta} \quad \text{Euler's theorem}$$

Euler's theorem

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$\cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta})$$

$$\sin \theta = \frac{1}{2j} (e^{j\theta} - e^{-j\theta})$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\cos \theta + j \sin \theta = e^{j\theta}$$

Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$$

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

\mathcal{L} = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^{\infty} e^{-st} dt$

$F(s)$ = Laplace transform of $f(t)$

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds, \quad \text{for } t > 0$$

Laplace transform theorem

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

Laplace transform formulas

	$f(t)$	$F(s)$			
1	Unit impulse $\delta(t)$	1			
2	Unit step $1(t)$	$\frac{1}{s}$	10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	t	$\frac{1}{s^2}$	11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$	12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$	13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
6	e^{-at}	$\frac{1}{s + a}$	14	$\frac{1}{a} (1 - e^{-at})$	$\frac{1}{s(s + a)}$
7	te^{-at}	$\frac{1}{(s + a)^2}$	15	$\frac{1}{b - a} (e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s + a)^n}$	16	$\frac{1}{b - a} (be^{-bt} - ae^{-at})$	$\frac{s}{(s + a)(s + b)}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s + a)^{n+1}}$	17	$\frac{1}{ab} \left[1 + \frac{1}{a - b} (be^{-at} - ae^{-bt}) \right]$	$\frac{1}{s(s + a)(s + b)}$

Laplace transform formulas

18	$\frac{1}{a^2} (1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$			
19	$\frac{1}{a^2} (at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$	25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$	26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$	27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
22	$\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_n \sqrt{1-\xi^2} t$	$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$	28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
23	$-\frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$	$\frac{s}{s^2 + 2\xi\omega_n s + \omega_n^2}$	29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
24	$1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$	$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$	30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
			31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Examples

Exponential function. Consider the exponential function

$$\begin{aligned} f(t) &= 0, & \text{for } t < 0 \\ &= Ae^{-at}, & \text{for } t \geq 0 \end{aligned}$$

where A and a are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathcal{L}[Ae^{-at}] = \int_0^{\infty} Ae^{-at}e^{-st} dt = A \int_0^{\infty} e^{-(a+s)t} dt = \frac{A}{s+a}$$

Using the Laplace theorem

$$\mathcal{L}[A] = \frac{A}{s} \Rightarrow \mathcal{L}[Ae^{-at}] = F(s+a) = \frac{A}{s+a}$$

$$\left\| \begin{array}{c|c} e^{-at} & \frac{1}{s+a} \end{array} \right\|$$

Examples

Sinusoidal function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0, & \text{for } t < 0 \\ &= A \sin \omega t, & \text{for } t \geq 0 \end{aligned}$$

where A and ω are constants, is obtained as follows. Referring to Equation (2-3), $\sin \omega t$ can be written

$$\sin \omega t = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

Hence

$$\begin{aligned} \mathcal{L}[A \sin \omega t] &= \frac{A}{2j} \int_0^{\infty} (e^{j\omega t} - e^{-j\omega t}) e^{-st} dt \\ &= \frac{A}{2j} \frac{1}{s - j\omega} - \frac{A}{2j} \frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2} \end{aligned}$$

Similarly, the Laplace transform of $A \cos \omega t$ can be derived as follows:

$$\mathcal{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}$$

$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

Inverse Laplace transform

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)] \\ &= f_1(t) + f_2(t) + \cdots + f_n(t)\end{aligned}$$

The function is partitioned using *partial fraction*

Partial fraction

F(s) has real and distinct poles only

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}, \quad \text{for } m < n$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \quad (2-14)$$

where a_k ($k = 1, 2, \dots, n$) are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (2-14) by $(s + p_k)$ and letting $s = -p_k$, which gives

$$\begin{aligned} \left[(s + p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1} (s + p_k) + \frac{a_2}{s + p_2} (s + p_k) \right. \\ &\quad \left. + \cdots + \frac{a_k}{s + p_k} (s + p_k) + \cdots + \frac{a_n}{s + p_n} (s + p_k) \right]_{s=-p_k} \\ &= a_k \end{aligned}$$

Partial fraction

$F(s)$ has real and distinct poles only

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}$$

where a_1 and a_2 are found by using Equation (2-15):

$$a_1 = \left[(s + 1) \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-1} = \left[\frac{s + 3}{s + 2} \right]_{s=-1} = 2$$
$$a_2 = \left[(s + 2) \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-2} = \left[\frac{s + 3}{s + 1} \right]_{s=-2} = -1$$

Thus

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s + 1}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s + 2}\right] \\ &= 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0 \end{aligned}$$

$$\left. \begin{array}{c} e^{-at} \\ \hline \frac{1}{s + a} \end{array} \right\|$$

Partial fraction - Example

F(s) has real and distinct poles only

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

Case 2. Roots of the Denominator of $F(s)$ Are Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

put $s = -1$ which gives $K_1 = 2$, and letting $s = -2$,
then $K_2 = -2$

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3$$

K_3 can be found by differentiating the
previous equation w.r.t. s to isolate K_3 and put $s = -2$.
Hence $K_3 = -2$

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

K_1 is found in the usual way to be $3/5$.
 K_2 and K_3 can be found by first multiplying the shown equation by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions.

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5} \Rightarrow F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{(s+1) + (1/2)(2)}{(s+1)^2 + 2^2} \Rightarrow f(t) = \frac{3}{5} - \frac{3}{5} e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right)$$

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2s + K_3}{s^2 + 2s + 5}$$

Balancing coefficients, $\left(K_2 + \frac{3}{5}\right) = 0$, and $\left(K_3 + \frac{6}{5}\right) = 0$
then, $K_2 = -\frac{3}{5}$, and $K_3 = -\frac{6}{5}$

Transfer function in frequency domain

D.E that mathematically represent the system

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t)$$

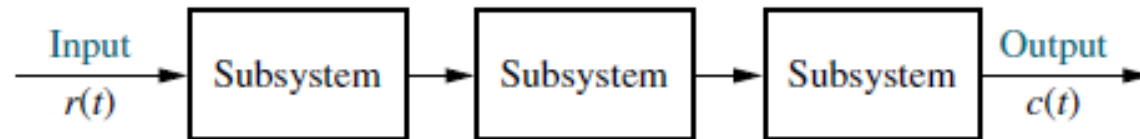
Solve the D.E. in Frequency domain using Laplace Transform.

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0) R(s)$$

Construct the T.F.

The main T.F. function can be divided into subsystems using Partial Fraction.

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$



System transfer function-Example

Given the system D.E (mathematical model), find the system T.F.?

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution:

$$sC(s) + 2C(s) = R(s)$$

The T.F. takes the form

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Assume that $r(t)$ is defined as step input $u(t)$, find the system response $c(t)$?

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

From the Laplace transform table (the differentiation theory)

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$

$$r(t) = u(t), R(s) = 1/s,$$

System transfer function-Example

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

Using partial fraction

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Using L.T. Table

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

From the Laplace transform table

Unit step $1(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

System Dynamics and control- Remember

- Our objective is to understand the system dynamics behavior.
- To do so, we desire to model the system using the proper mathematical model.
- Then, we want to solve this mathematical model to obtain the system response.
- Finally, we wish to study the system response (performance).
- But, we can not model, solve, or analyze a system without deep understanding of it physics 😊 .



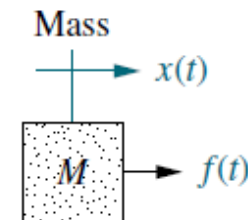
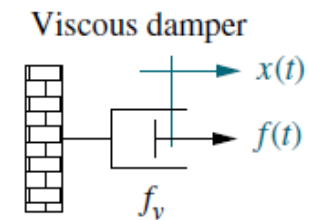
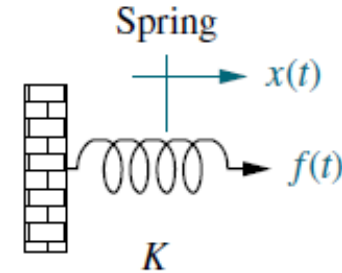
A - Translational Mechanical Systems



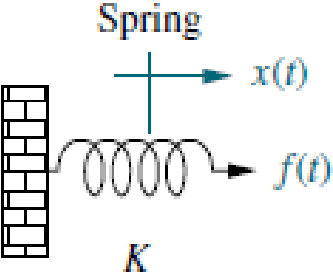
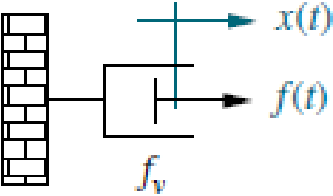
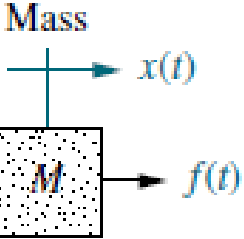
Translational Mechanical Systems

- Mechanical systems have three passive, linear components.
- Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy.

K , f_v , and M are called spring constant, coefficient of viscous friction, and mass, respectively.



Translational Mechanical Systems

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
 <p>Spring</p> <p>$x(t)$</p> <p>$f(t)$</p> <p>K</p>	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
 <p>Viscous damper</p> <p>$x(t)$</p> <p>$f(t)$</p> <p>f_v</p>	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
 <p>Mass</p> <p>$x(t)$</p> <p>$f(t)$</p> <p>M</p>	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	$M s^2$

$F(t)$ [N], $x(t)$ [m], $v(t)$ [m/s], K [N/m], f_v [N.s/m], M [kg]

Mechanical system equation of motion

The mechanical system requires one differential equation, called *the equation of motion*, to describe it. That can be constructed by the following steps:

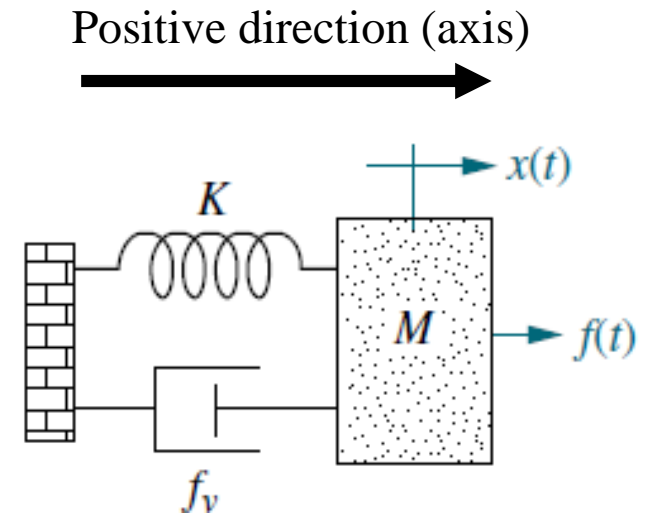
1. begin by assuming a positive direction of motion, for example, to the right.
2. Then, draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.
3. Next use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.
4. Finally, take the Laplace transform of the differential equation, separate the variables, and derive the transfer function.

Mechanical system equation of motion

Example

Find the transfer function (T.F.) of the Mass, spring, and damper system shown in the figure.

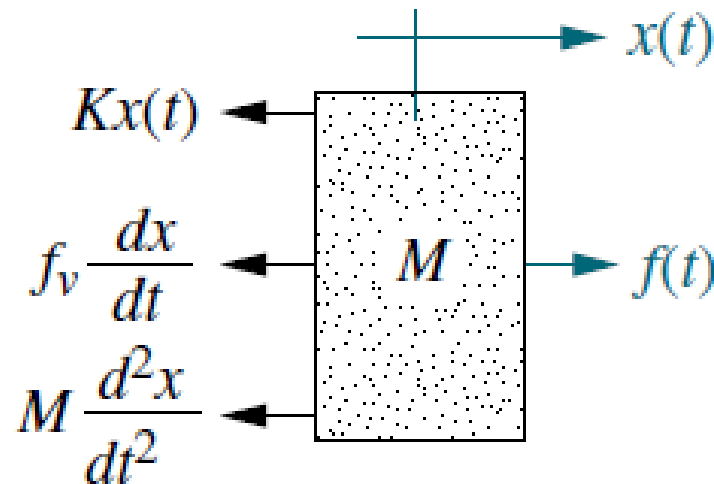
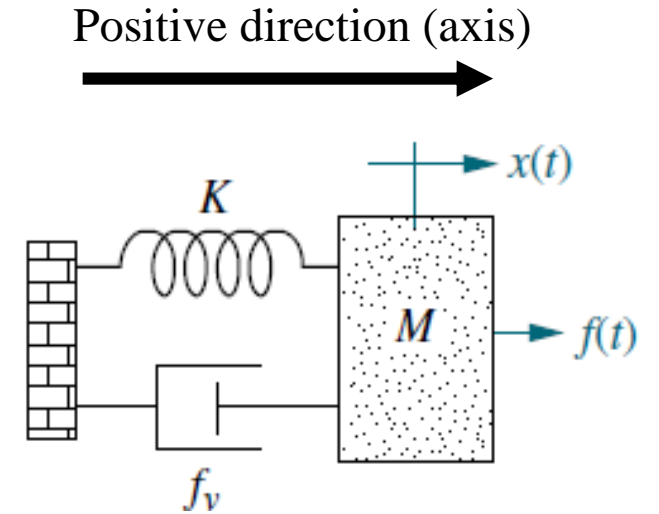
1. begin by assuming a positive direction of motion, for example, to the right.



Mechanical system equation of motion

Example

2. Then, draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.



From the element table

$$f(t) = Kx(t)$$

$$f(t) = f_v \frac{dx(t)}{dt}$$

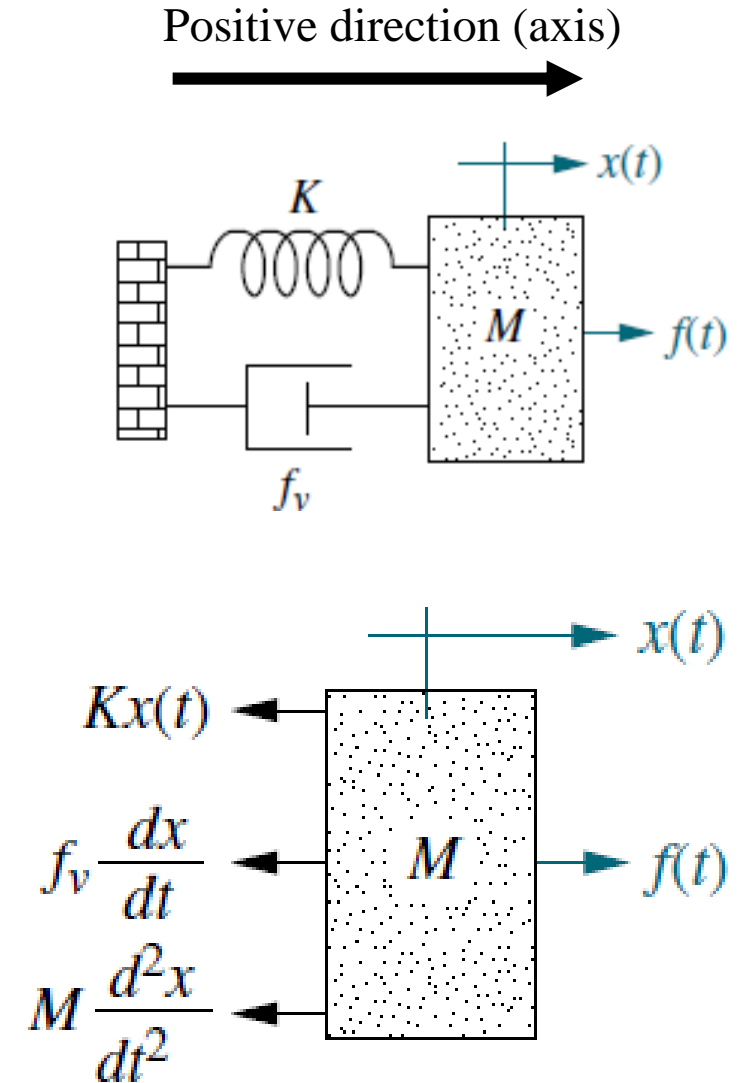
$$f(t) = M \frac{d^2x(t)}{dt^2}$$

Mechanical system equation of motion

Example

3. Next use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.

$$M \frac{d^2 x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t)$$



Mechanical system equation of motion

Example

4. Finally, take the Laplace Transform of the differential equation, separate the variables, and derive the transfer function.

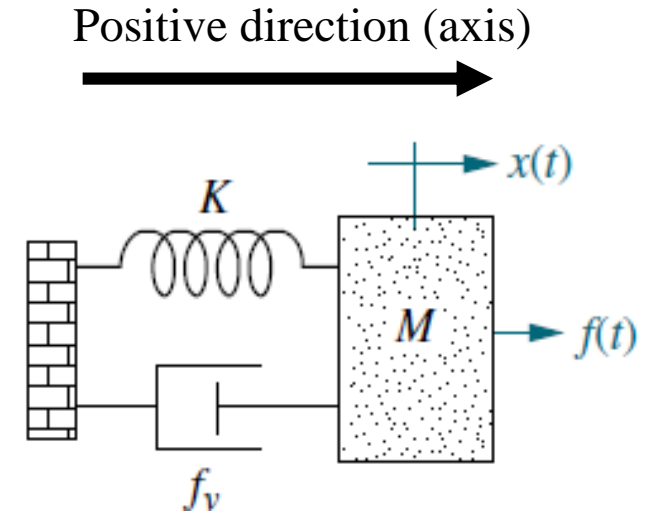
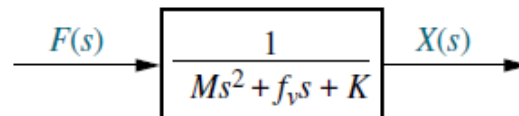
Assume zero initial conditions,

$$M \frac{d^2 x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t)$$

$$Ms^2 X(s) + f_v s X(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

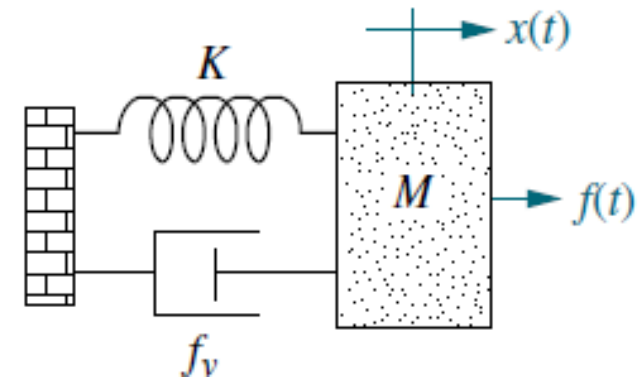


From the Laplace Transform table
(differentiation theorem)

$$\begin{aligned} \mathcal{L} \left[\frac{df}{dt} \right] &= sF(s) - f(0-) \\ \mathcal{L} \left[\frac{d^2 f}{dt^2} \right] &= s^2 F(s) - sf(0-) - f'(0-) \end{aligned}$$

Single degree of freedom system

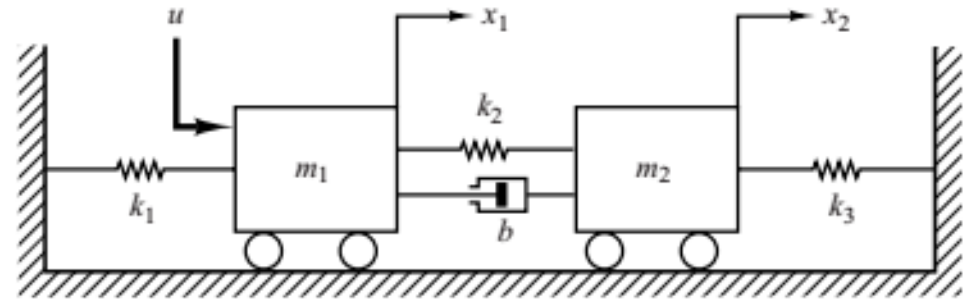
- The system that we solved early has one degree of freedom. i.e. the system has one independent motion.
- We called this system “Single degree of freedom system”.
- Subsequently, there is one equation of motion for this system.



Number of equations of motion = number of degrees of freedom

Multi-degree of freedom system

- In multi-degree of freedom systems: we have more than one degree of freedom.
- And we have more than one equation of motion.
- In order to solve such a problem, we draw a free-body diagram for each point of motion and then use superposition.

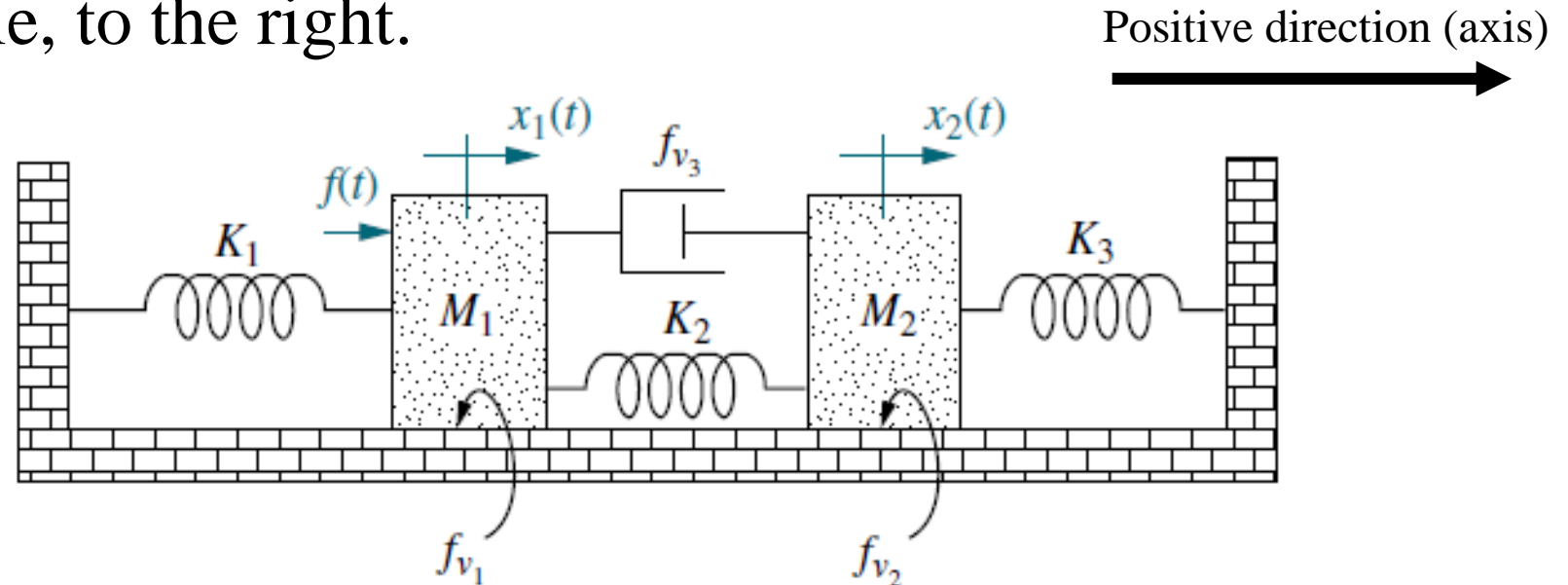


Mechanical system equation of motion

Example-Two-degree-of-freedom system

Find the transfer function (T.F.) of the Mass, spring, and damper system shown in the figure.

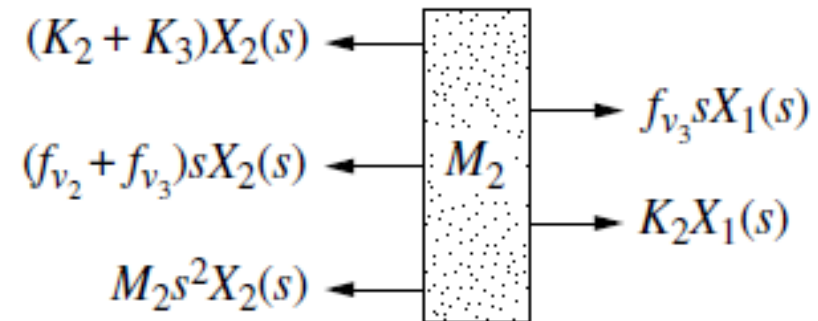
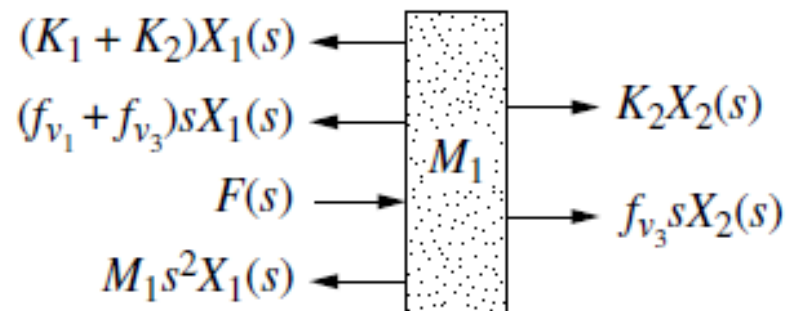
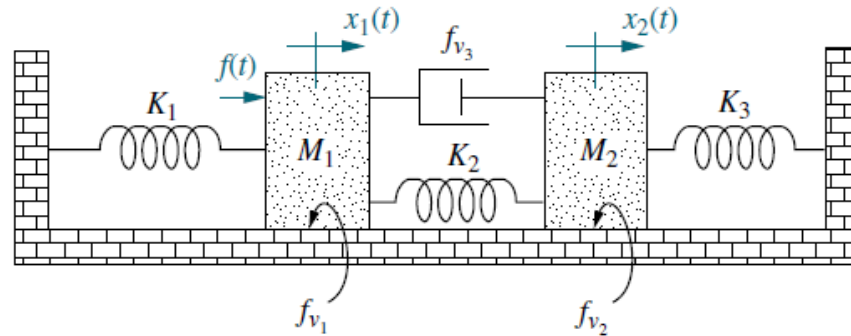
1. begin by assuming a positive direction of motion, for example, to the right.



Mechanical system equation of motion

Example-Two-degree-of-freedom system

2. Then, draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.



Mechanical system equation of motion

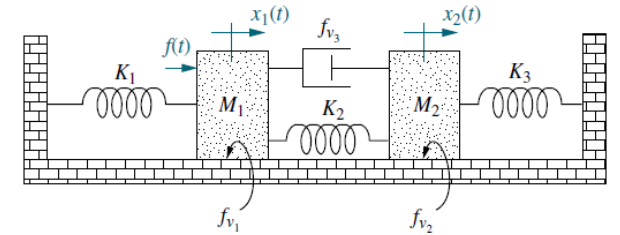
Example-Two-degree-of-freedom system

3. Next use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.

4. Finally, take the Laplace Transform of the differential equation, separate the variables, and arrive at the transfer function.

Assume zero initial conditions,

$$M \frac{d^2 x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t)$$



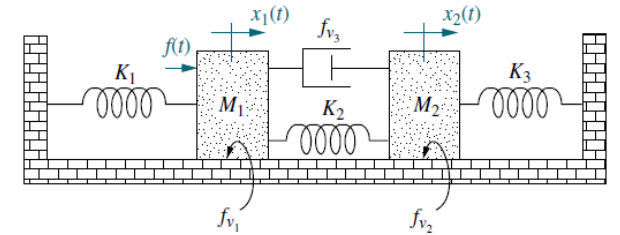
Mechanical system equation of motion

Example-Two-degree-of-freedom system

3. Next use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.

4. Finally, take the Laplace Transform of the differential equation, separate the variables, and arrive at the transfer function.

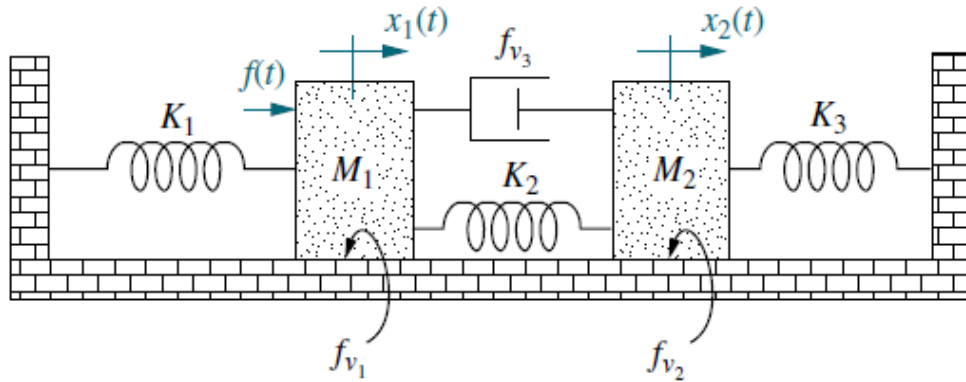
Assume zero initial conditions,



$$\begin{aligned} [M_1 s^2 (f_{v1} + f_{v3}) s + (K_1 + K_2)] X_1(s) - (f_{v3} s + K_2) X_2(s) &= F(s) \\ -(f_{v3} s + K_2) X_1(s) + [M_2 s^2 + (f_{v2} + f_{v3}) s + (K_2 + K_3)] X_2(s) &= 0 \end{aligned}$$

Mechanical system equation of motion

Example-Two-degree-of-freedom system



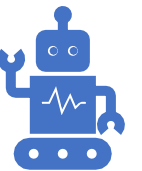
$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta}$$

$$\Delta = \begin{vmatrix} [M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

$\xrightarrow{F(s)}$

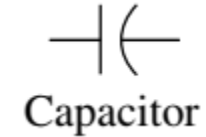
 $\frac{(f_{v_3}s + K_2)}{\Delta}$
 $\xrightarrow{X_2(s)}$

B – Electrical Systems



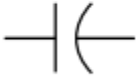


Electrical systems

Equivalent circuits for the electric networks consist of three passive linear components: resistors, capacitors, and inductors.



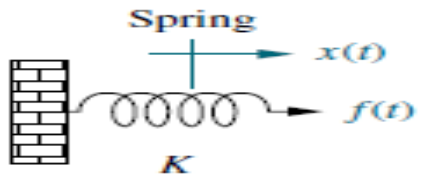
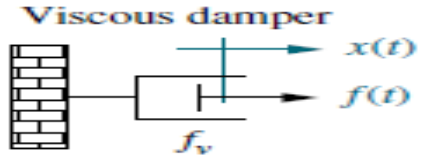
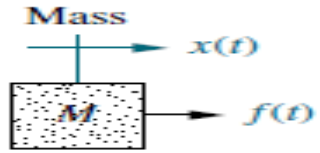
Electrical systems

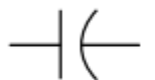


Equivalent circuits for the electric networks consist of three passive linear components: resistors, capacitors, and inductors.

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C} \int_0^1 i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^1 v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical versus mechanical systems

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	Ms^2

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical systems

Transfer functions can be obtained using Kirchhoff's voltage law and summing voltages around loops or meshes.

We call this method loop or mesh analysis and demonstrate it in the following example.

- Kirchhoff's current law: "The sum of the currents at any junction must equal zero"
- Kirchhoff's voltage law: "The sum of the potential differences across all elements around any closed circuit loop must be zero"

Example

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

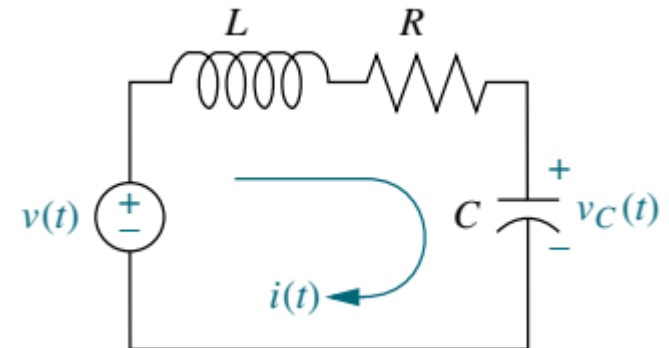
$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = v(t)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$, yields

$$L \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = v(t)$$

From the voltage-charge relationship for a capacitor

$$q(t) = Cv_C(t)$$



Example

Substituting in the equation

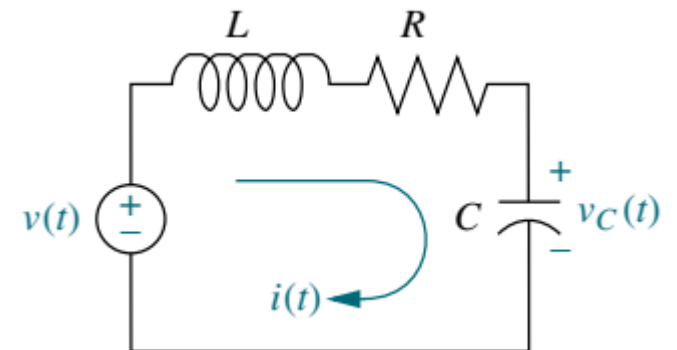
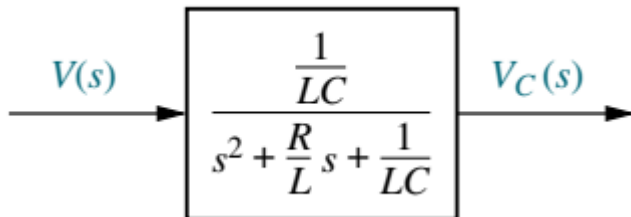
$$L \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = v(t)$$

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

Solving for the transfer function

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



Electric systems

Let us now develop a technique for simplifying the solution for future problems. take the Laplace transform of the equations in the voltage-current column, assuming zero initial conditions.

For the capacitor,

$$V(s) = \frac{1}{Cs} I(s)$$

For the resistor,

$$V(s) = RI(s)$$

For the inductor,

$$V(s) = LsI(s)$$

Now define the following transfer function:

$$\frac{V(s)}{I(s)} = Z(s)$$

Electric systems

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current.

But, unlike resistance, this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation.

We call these particular transfer functions impedance.

For the capacitor,

$$V(s) = \frac{1}{Cs} I(s)$$

For the resistor,

$$V(s) = RI(s)$$

For the inductor,

$$V(s) = LsI(s)$$

Now define the following transfer function:

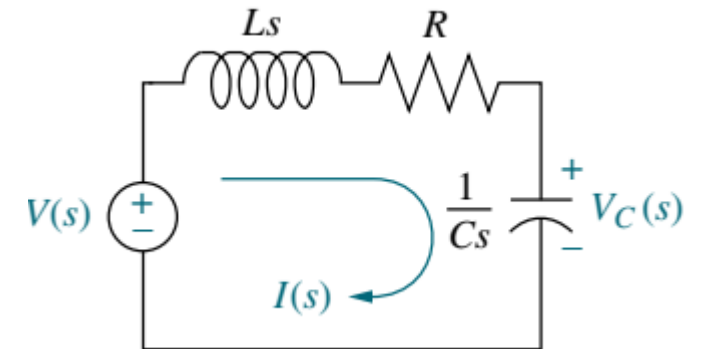
$$\frac{V(s)}{I(s)} = Z(s)$$

Electric Systems

Let us now demonstrate how the concept of impedance simplifies the solution for the transfer function. The Laplace transform of equation in the previous example, assuming zero initial conditions, is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s)$$

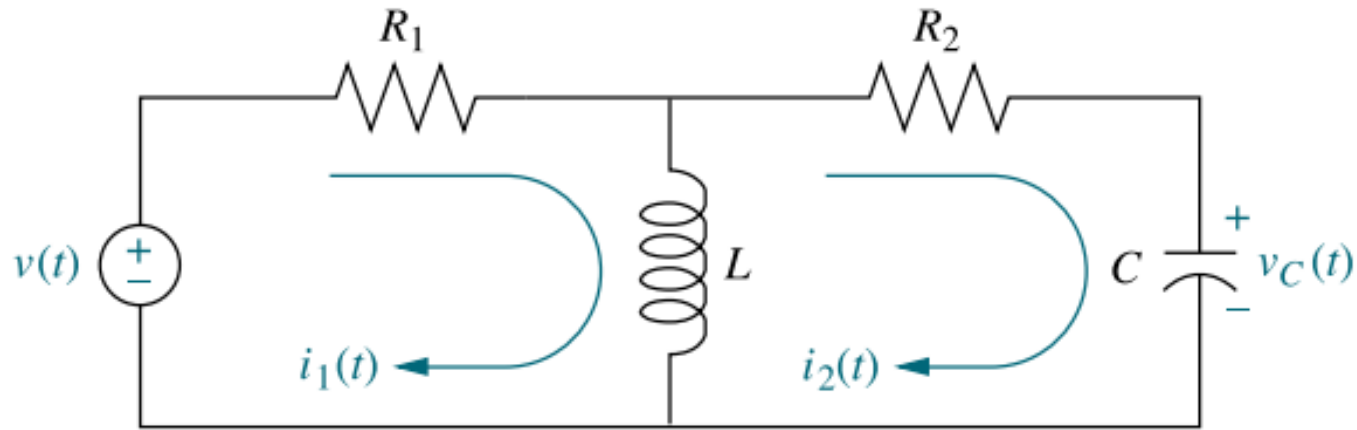
$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}]$$



Example

Given the network in the following Figure, find the transfer function,

$$I_2(s)/V(s).$$



Example

The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.

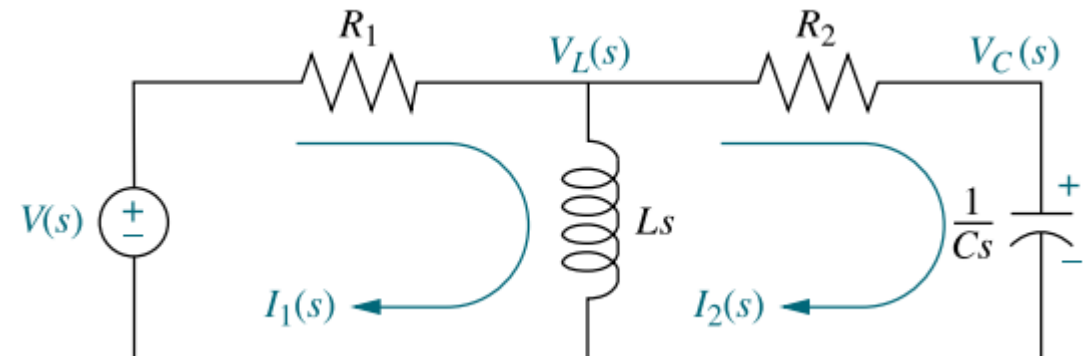
Around Mesh 1

$$R_1 I_1(s) + Ls I_1(s) - Ls I_2(s) = V(s)$$

Around Mesh 2,

$$Ls I_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - Ls I_1(s) = 0$$

$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{array} \right] I_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{array} \right] I_2(s) &= \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{array} \right] \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{array} \right] I_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{array} \right] I_2(s) &= \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{array} \right] \end{aligned}$$



Example

By combine the two terms and solving the two equations together to obtain the T.F.

$$\begin{aligned}(R_1 + Ls)I_1(s) - LsI_2(s) &= V(s) \\ -LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) &= 0\end{aligned}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

C – Rotational Mechanical Systems



Rotational Mechanical System

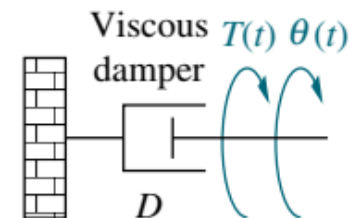
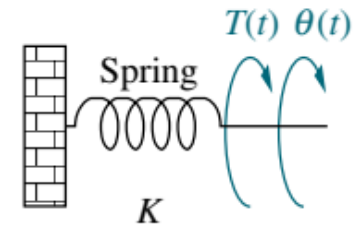
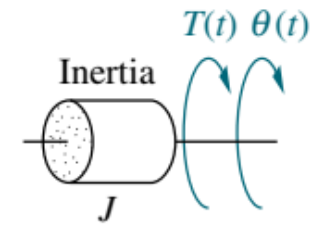
Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement.

The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation.

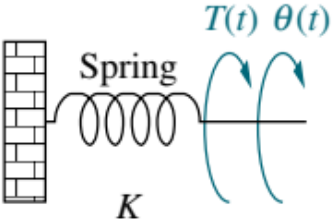
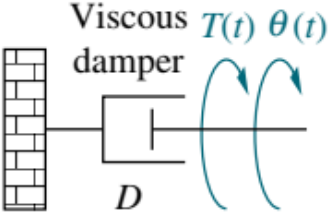
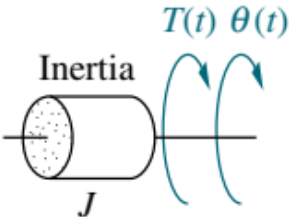
Rotational Mechanical System

Rotational mechanical systems are based on three passive components:

- Inertia
- Spring, and
- Damper

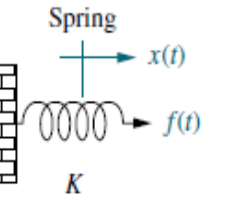
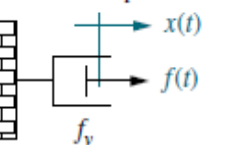
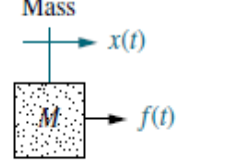


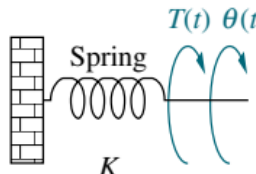
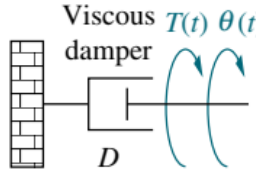
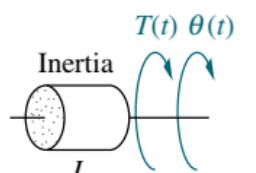
Rotational Mechanical System

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m²(kilograms-meters² – newton-meters-seconds²/radian).

Translational and rotational mechanical systems

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	Ms^2

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2 \theta(t)}{dt^2}$	Js^2

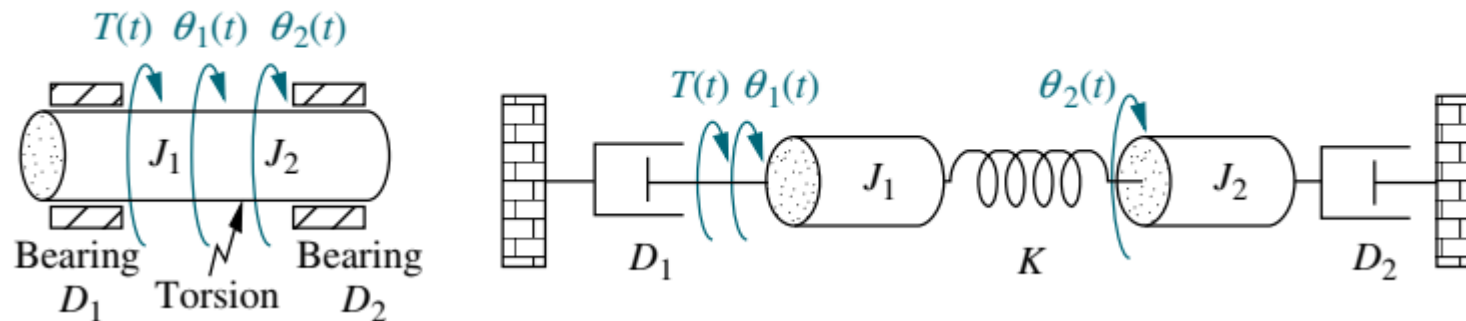
Rotational Mechanical System

Notice that the symbols for the components look the same as translational symbols, but they are undergoing rotation and not translation.

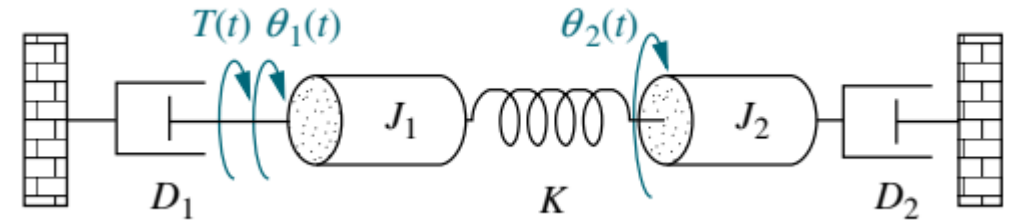
Also notice that the term associated with the mass is replaced by inertia. The values of K , D , and J are called spring constant, coefficient of viscous friction, and moment of inertia, respectively.

Example - 1

- Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in the Figure.
- The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.



Example - 1



First, obtain the schematic from the physical system. Even though torsion occurs throughout the rod in Figure 2.22(a),⁹ we approximate the system by assuming that the torsion acts like a spring concentrated at one particular point in the rod, with an inertia J_1 to the left and an inertia J_2 to the right.¹⁰ We also assume that the damping inside the flexible shaft is negligible. The schematic is shown in Figure 2.22(b). There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system.

Next, draw a free-body diagram.

Example - 1

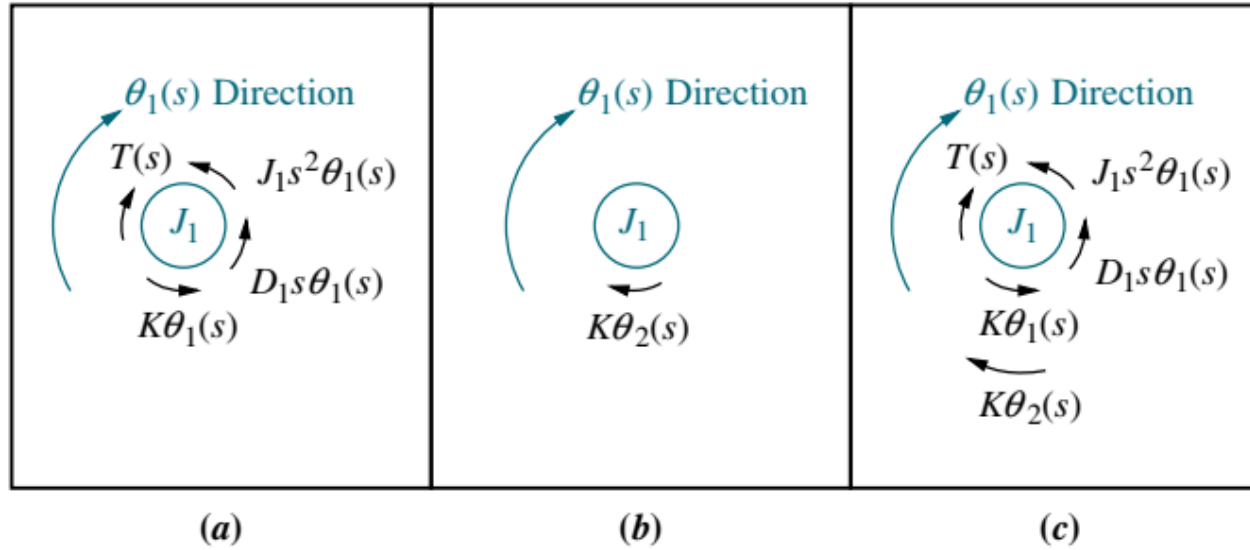
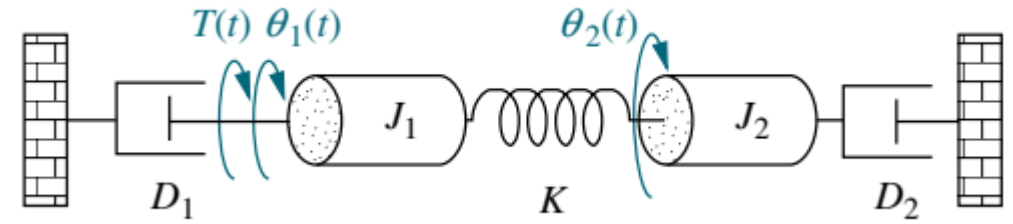
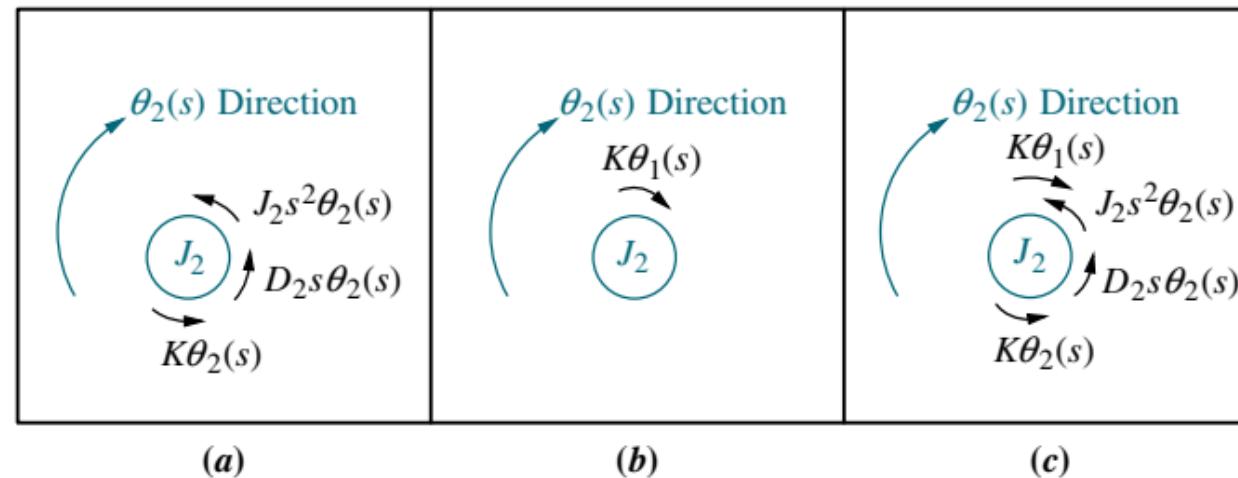
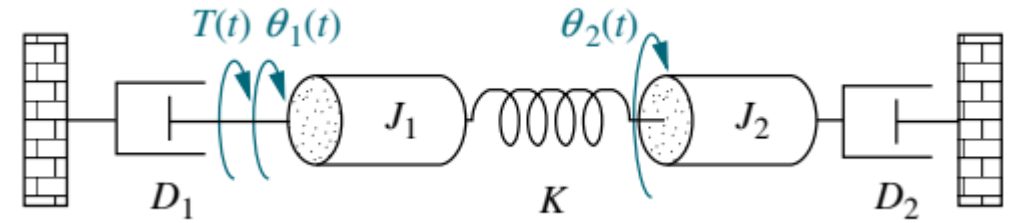


FIGURE 2.23 a. Torques on J_1 due only to the motion of J_1 ; b. torques on J_1 due only to the motion of J_2 ; c. final free-body diagram for J_1

FIGURE 2.24 a. Torques on J_2 due only to the motion of J_2 ; b. torques on J_2 due only to the motion of J_1 ; c. final free-body diagram for J_2



Example - 1



Summing torques respectively from Figures 2.23(c) and 2.24(c) we obtain the equations of motion,

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s) \quad (2.127a)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0 \quad (2.127b)$$

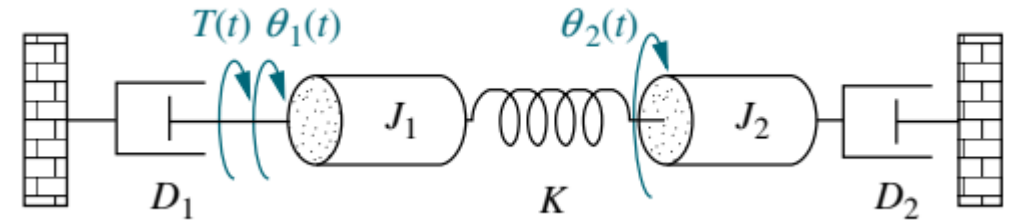
from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \quad (2.128)$$

as shown in Figure 2.22(c), where

$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

Example - 1



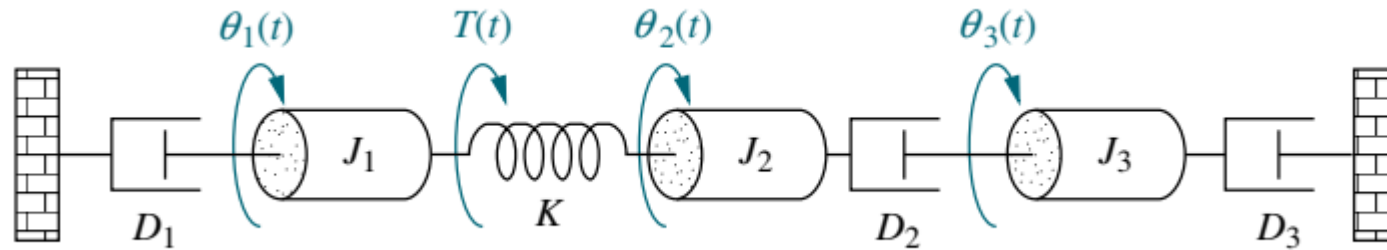
Notice that Eq. (2.127) have that now well-known form

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix} \quad (2.129a)$$

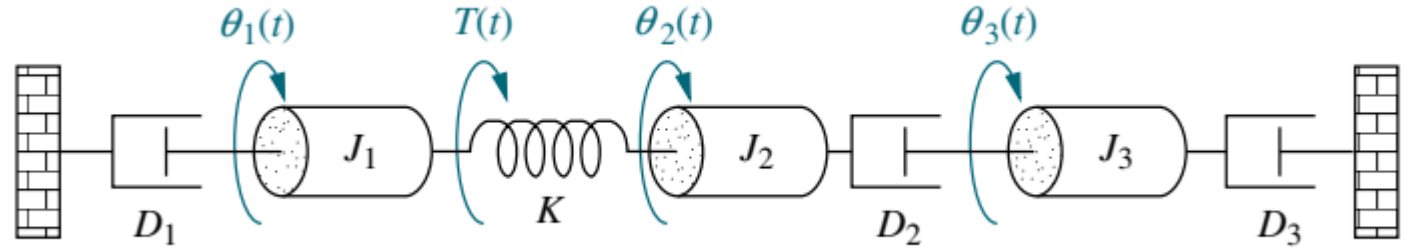
$$- \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix} \quad (2.129b)$$

Example - 2

Write, but do not solve, the Laplace transform of the equations of motion for the system shown in the Figure.



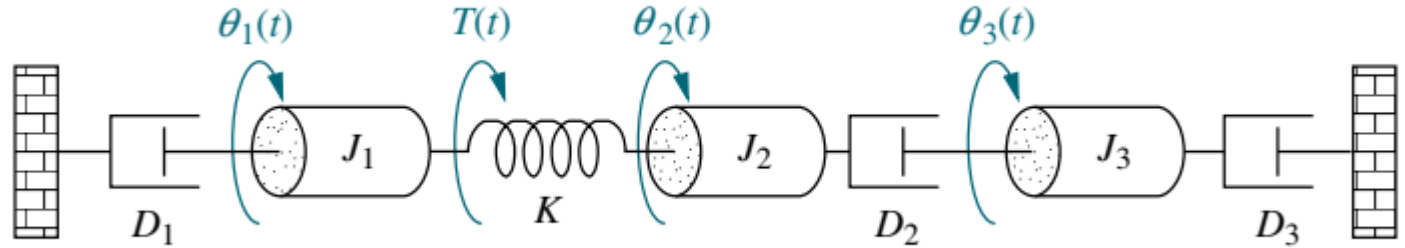
Example - 2



The equations will take on the following form, similar to electrical mesh equations:

$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{array} \right] \theta_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_2(s) \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_3 \end{array} \right] \theta_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{array} \right] \end{aligned}$$

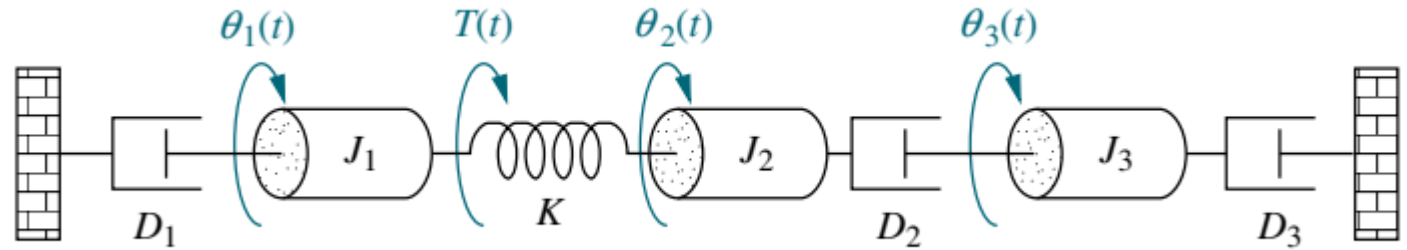
Example - 2



The equations will take on the following form, similar to electrical mesh equations:

$$-\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{array} \right] \theta_2(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_2 \text{ and } \theta_3 \end{array} \right] \theta_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{array} \right]$$

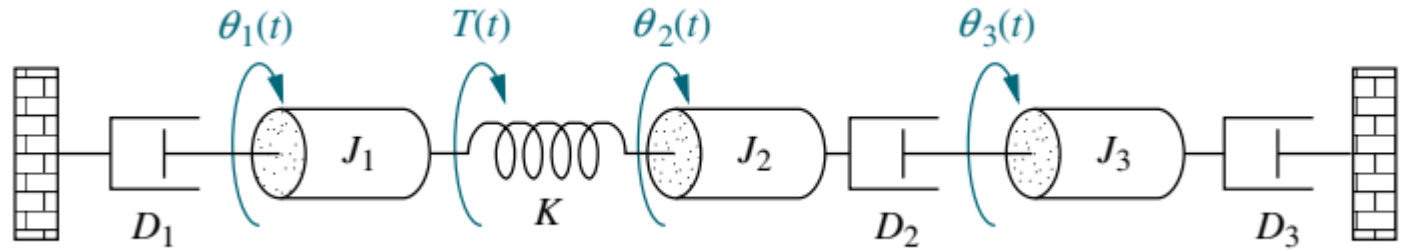
Example - 2



The equations will take on the following form, similar to electrical mesh equations:

$$\begin{aligned} - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_3 \end{array} \right] \theta_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_2 \text{ and } \theta_3 \end{array} \right] \theta_2(s) \\ + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_3 \end{array} \right] \theta_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_3 \end{array} \right] \end{aligned}$$

Example - 2



The equations will take on the following form,

$$\begin{array}{rcl}
 (J_1 s^2 + D_1 s + K)\theta_1(s) & -K\theta_2(s) & -0\theta_3(s) = T(s) \\
 -K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) & & -D_2 s\theta_3(s) = 0 \\
 -0\theta_1(s) & -D_2 s\theta_2(s) + (J_3 s^2 + D_3 s + D_2 s)\theta_3(s) & = 0
 \end{array}$$

D – Systems with Gears



Systems with gears

- Gears provide mechanical advantage to rotational systems.
- Gears can provide more torque and less speed.
- On the straightaway, you can shift to obtain more speed and less torque.
- Thus, gears allow you to match the drive system and the load—a trade-off between speed and torque

Systems with gears

The linearized interaction between two gears is depicted in Figure 2.27. An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$. Let us now find the relationship between the rotation of Gear 1, $\theta_1(t)$, and Gear 2, $\theta_2(t)$.

$$r_1\theta_1 = r_2\theta_2$$

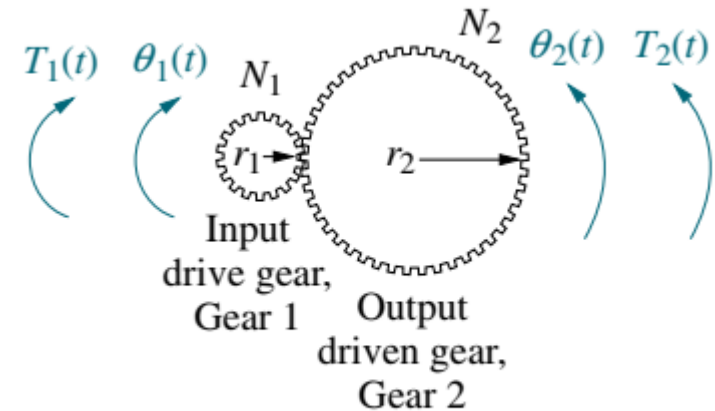
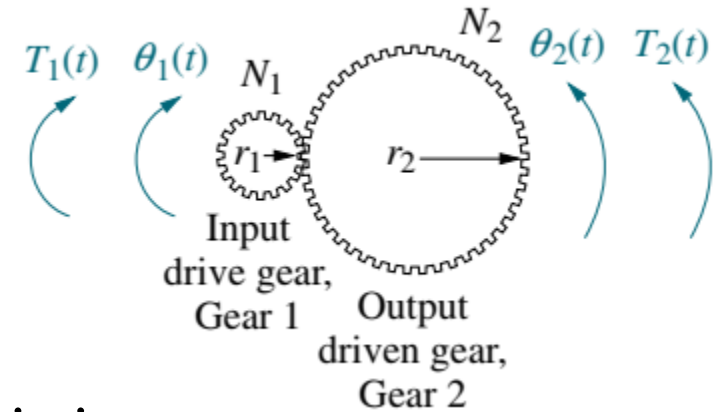


FIGURE 2.27 A gear system

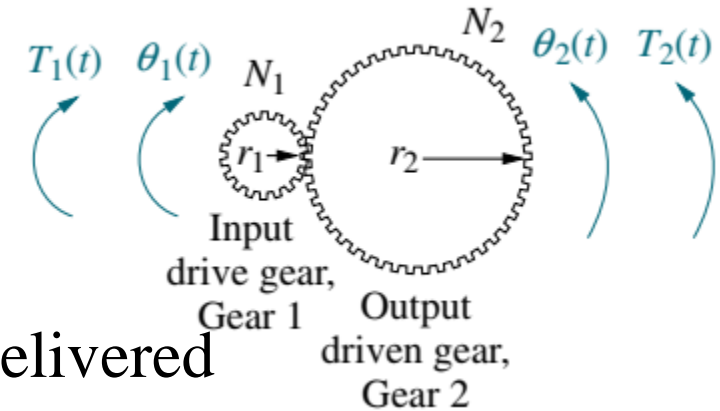
Systems with gears

since the ratio of the number of teeth along the circumference is in the same proportion as the ratio of the radii. We conclude that the ratio of the angular displacement of the gears is inversely proportional to the ratio of the number of teeth.

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$



Systems with gears



What is the relationship between the input torque, T_1 , and the delivered torque, T_2 ? If we assume the gears are lossless, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2.

Since the translational energy of force times displacement becomes the rotational energy of torque times angular displacement,

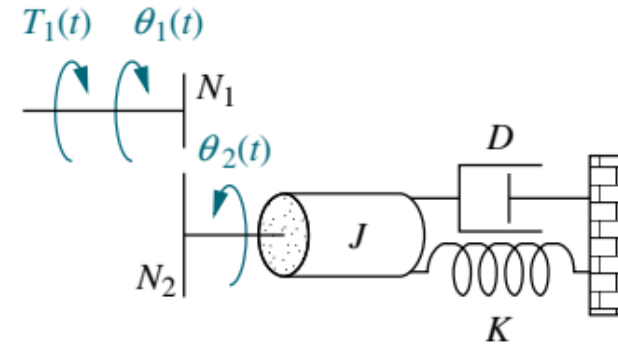
$$T_1\theta_1 = T_2\theta_2$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

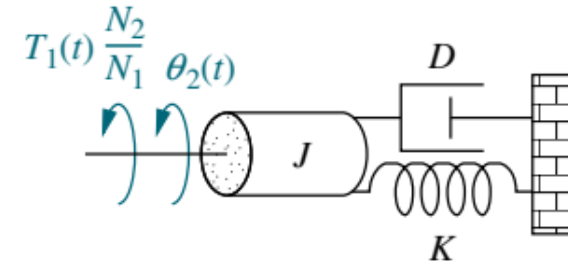
Thus, the torques are directly proportional to the ratio of the number of teeth.

Gears mechanical impedance

- Figure a shows gears driving a rotational inertia, spring, and viscous damper.

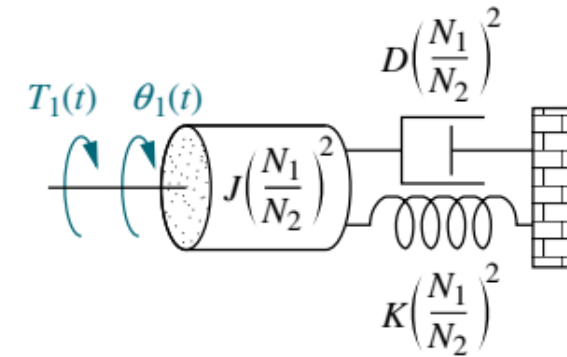


(a)



(b)

- For clarity, the gears are shown by an end-on view. We want to represent Figure (a) as an equivalent system at θ_1 without the gears.



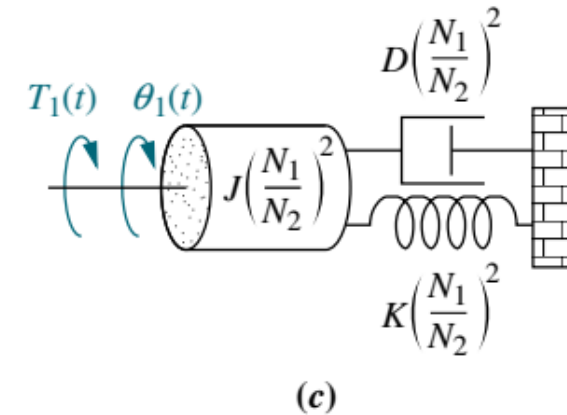
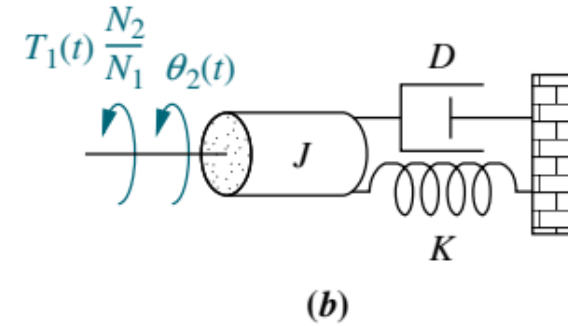
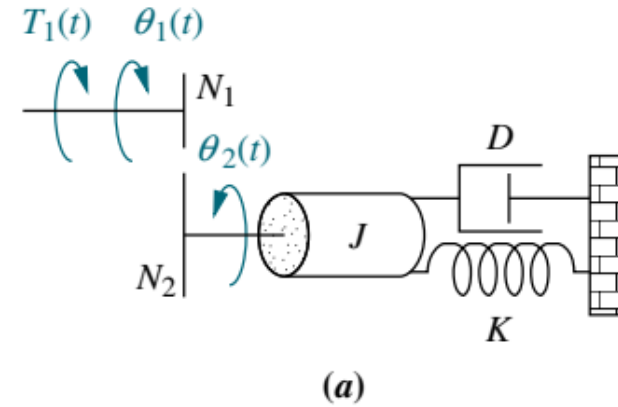
(c)

- In other words, can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?

Gears mechanical impedance

From Figure (b), T_1 can be reflected to the output by multiplying by N_2/N_1 . The result is shown in Figure (b), from which we write the equation of motion as

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s) \frac{N_2}{N_1}$$



Gears mechanical impedance

Now convert $\theta_2(s)$ into an equivalent $\theta_1(s)$, so that the previous equation

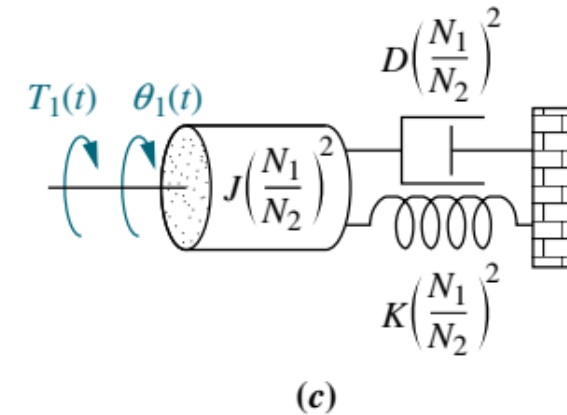
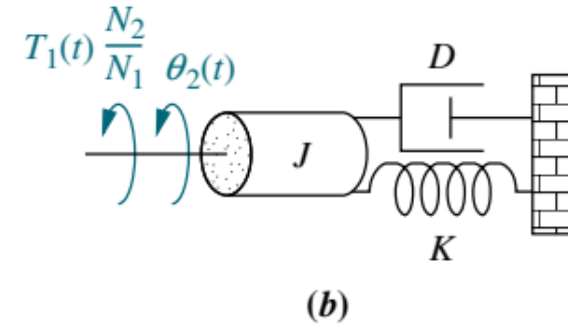
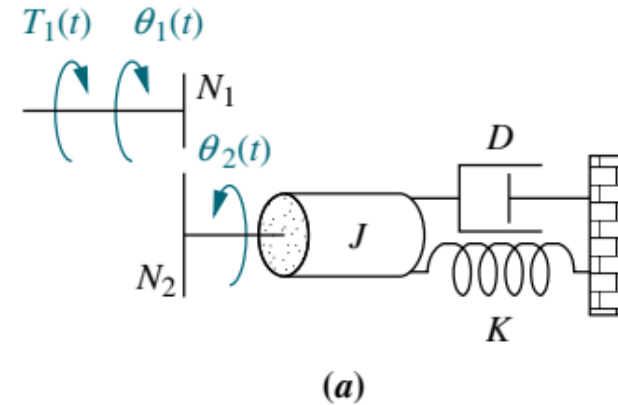
$$(Js^2 + Ds + K)\theta_2(s) = T_1(s) \frac{N_2}{N_1}$$

will look as if it were written at the input. Using Figure (a), we get

$$(Js^2 + Ds + K) \frac{N_1}{N_2} \theta_1(s) = T_1(s) \frac{N_2}{N_1}$$

After simplification,

$$\left[J \left(\frac{N_1}{N_2} \right)^2 s^2 + D \left(\frac{N_1}{N_2} \right)^2 s + K \left(\frac{N_1}{N_2} \right)^2 \right] \theta_1(s) = T_1(s)$$



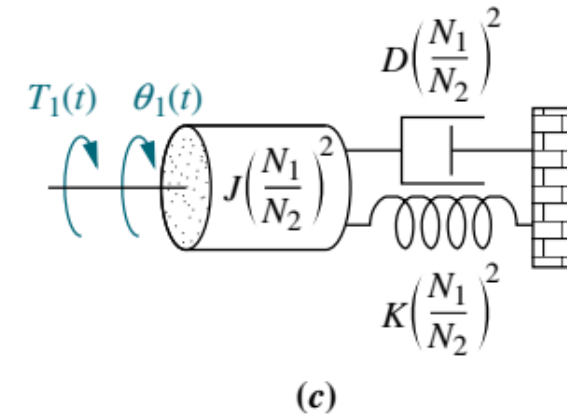
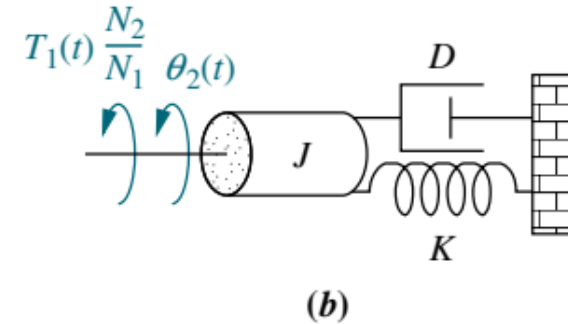
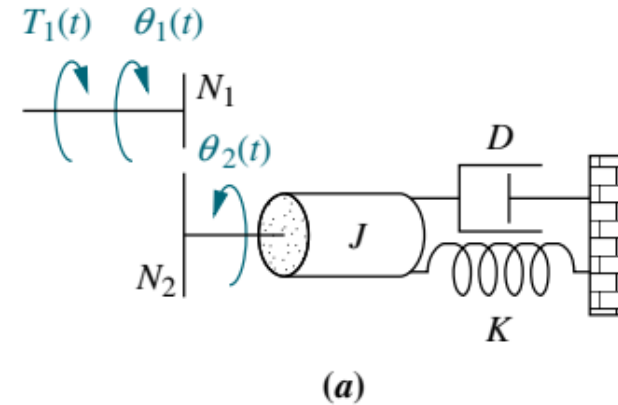
Gears mechanical impedance

The final equation suggests the equivalent system at the input and without gears shown in Figure (c). Thus, the load can be thought of as having been reflected from the output to the input.

$$\left[J \left(\frac{N_1}{N_2} \right)^2 s^2 + D \left(\frac{N_1}{N_2} \right)^2 s + K \left(\frac{N_1}{N_2} \right)^2 \right] \theta_1(s) = T_1(s)$$

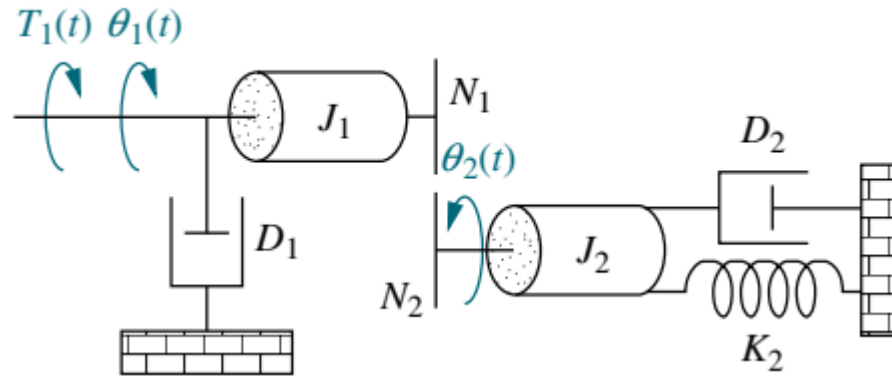
Generalizing the results, we can make the following statement: Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

$$\left(\frac{\text{Number of teeth of gear on destination shaft}}{\text{Number of teeth of gear on source shaft}} \right)^2$$



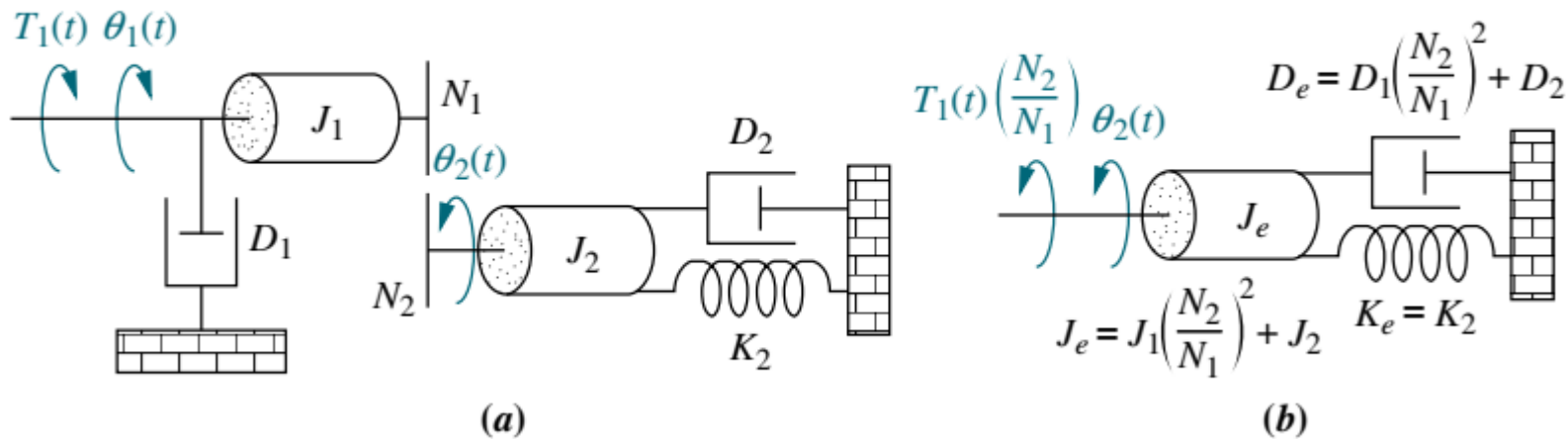
Gear example - 1

Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure (a).

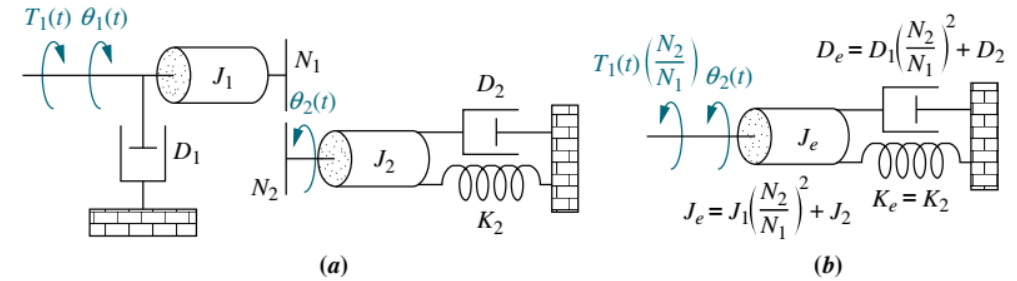


Gear example - 1

The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears. Thus, there is only one degree of freedom and hence one equation of motion.



Gear example - 1



Let us first reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output as shown in Figure (b), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by $\left(\frac{N_2}{N_1}\right)$. The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e) \theta_2(s) = T_1(s) \frac{N_2}{N_1}$$

$$J_e = J_1 \left(\frac{N_2}{N_1} \right)^2 + J_2; \quad D_e = D_1 \left(\frac{N_2}{N_1} \right)^2 + D_2; \quad K_e = K_2$$

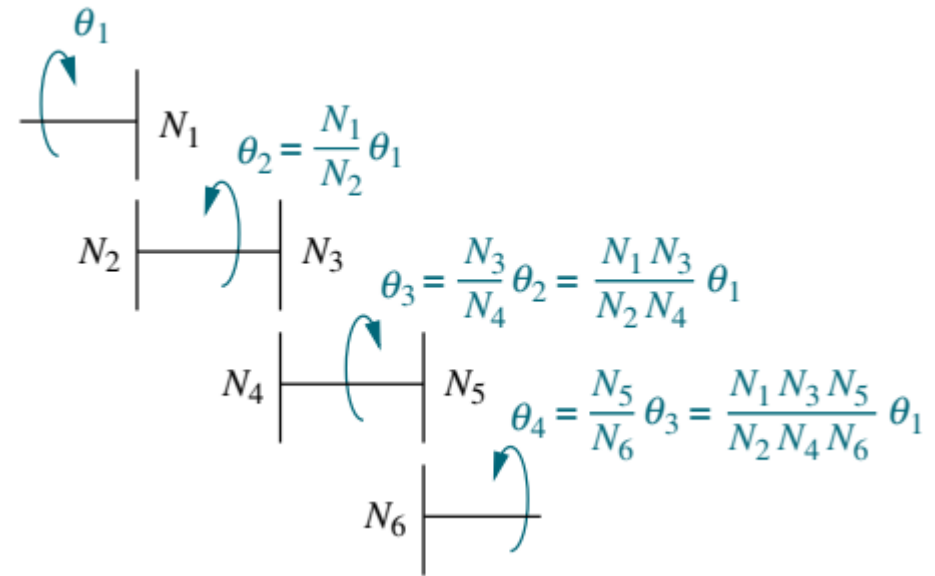
Solving for $\theta_2(s)/T_1(s)$, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$

Gear train

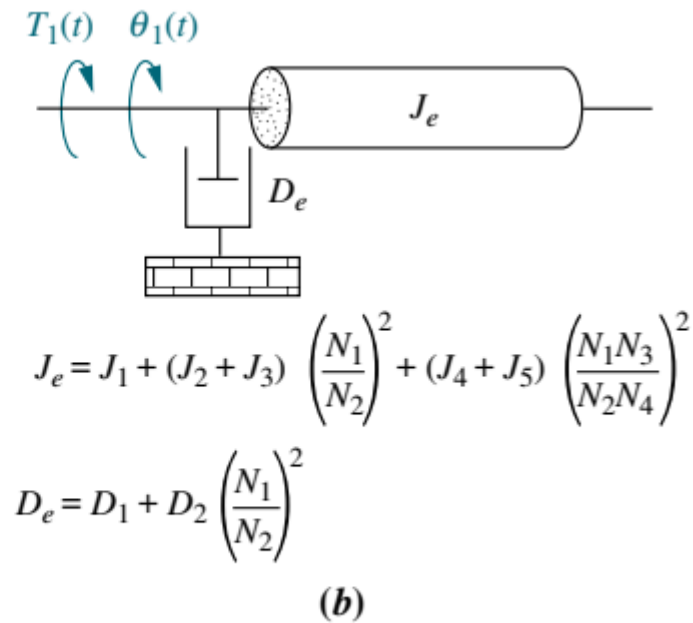
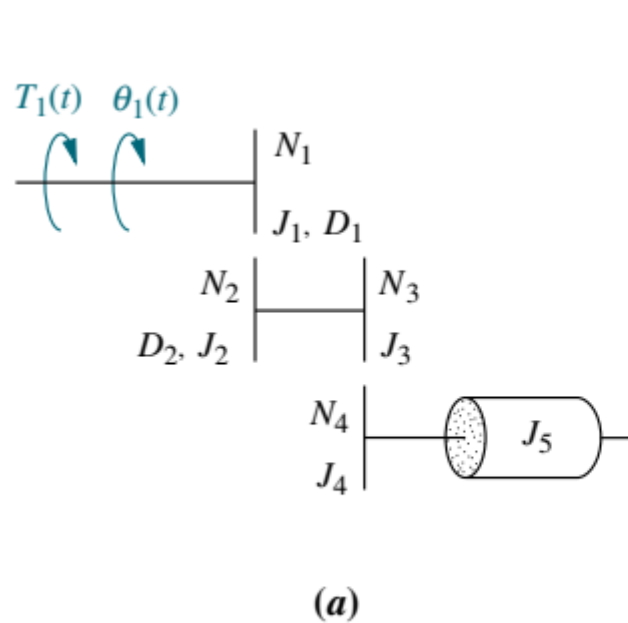
In order to eliminate gears with large radii, a gear train is used to implement large gear ratios by cascading smaller gear ratios.

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6} \theta_1$$

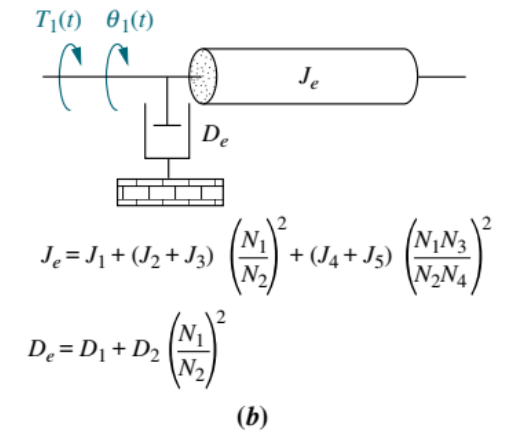
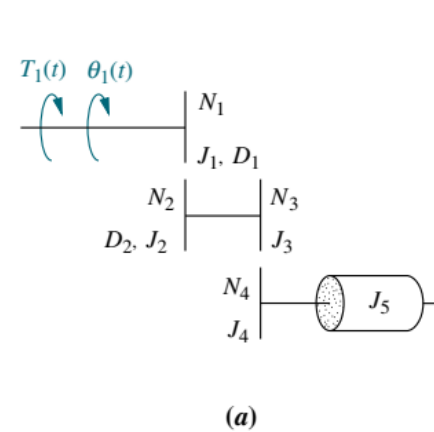


Gear example - 2

Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure (a).



Gear example - 2



SOLUTION: This system, which uses a gear train, does not have lossless gears. All of the gears have inertia, and for some shafts there is viscous friction. To solve the problem, we want to reflect all of the impedances to the input shaft, θ_1 . The gear ratio is not the same for all impedances. For example, D_2 is reflected only through one gear ratio as $D_2(N_1/N_2)^2$, whereas J_4 plus J_5 is reflected through two gear ratios as $(J_4 + J_5)[(N_3/N_4)(N_1/N_2)]^2$. The result of reflecting all impedances to θ_1 is shown in Figure 2.32(b), from which the equation of motion is

Gear example - 2

$$(J_e s^2 + D_e s) \theta_1(s) = T_1(s)$$

where

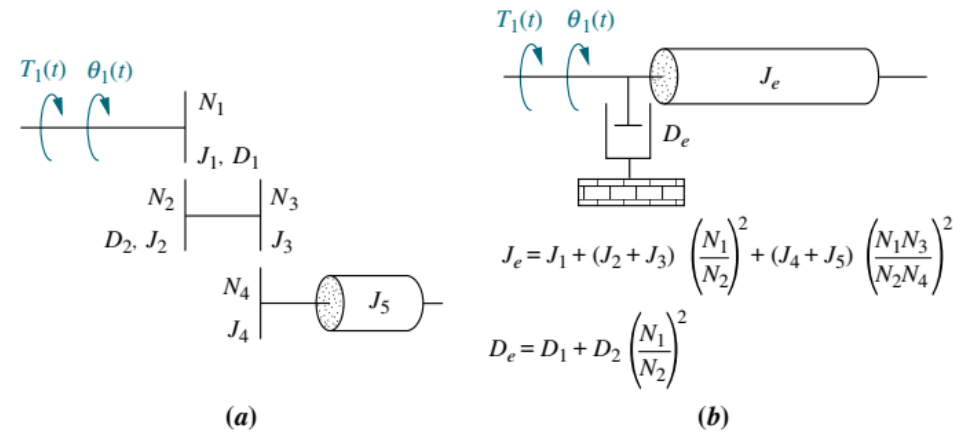
$$J_e = J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2} \right)^2 + (J_4 + J_5) \left(\frac{N_1 N_3}{N_2 N_4} \right)^2$$

and

$$D_e = D_1 + D_2 \left(\frac{N_1}{N_2} \right)^2$$

From Eq. (2.142), the transfer function is

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s}$$

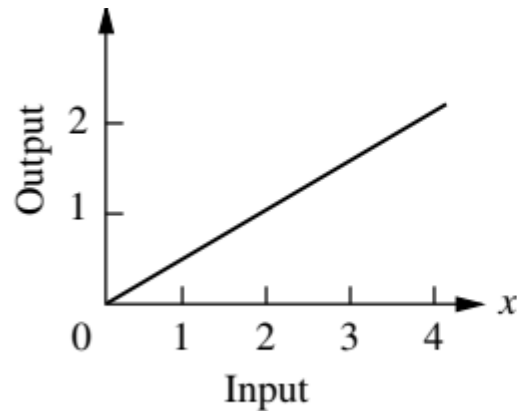


Linear and Nonlinear Systems

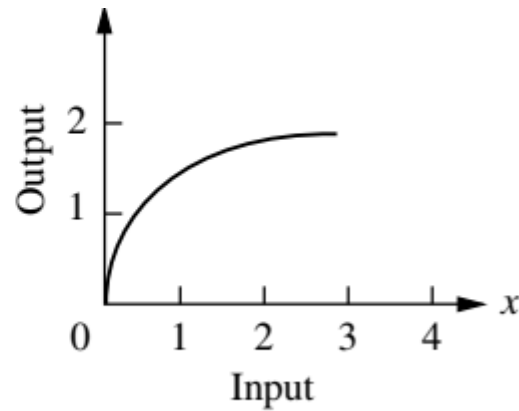


Nonlinearities

A linear system possesses two properties: superposition and homogeneity. The property of *superposition* means that the output response of a system to the sum of inputs is the sum of the responses to the individual inputs. Thus, if an input of $r_1(t)$ yields an output of $c_1(t)$ and an input of $r_2(t)$ yields an output of $c_2(t)$, then an input of $r_1(t) + r_2(t)$ yields an output of $c_1(t) + c_2(t)$. The property of *homogeneity* describes the response of the system to a multiplication of the input by a scalar. Specifically, in a linear system, the property of homogeneity is demonstrated if for an input of $r_1(t)$ that yields an output of $c_1(t)$, an input of $Ar_1(t)$ yields an output of $Ac_1(t)$; that is, multiplication of an input by a scalar yields a response that is multiplied by the same scalar.



(a)

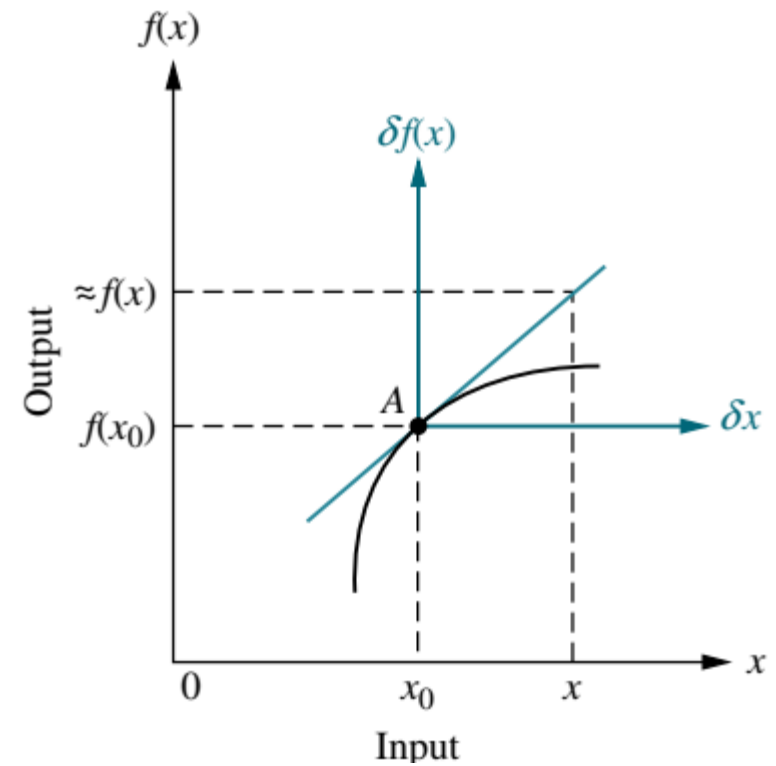


(b)

Linearization

The electrical and mechanical systems covered so far were assumed to be linear. However, if any nonlinear components are present, we must linearize the system before we can find the transfer function.

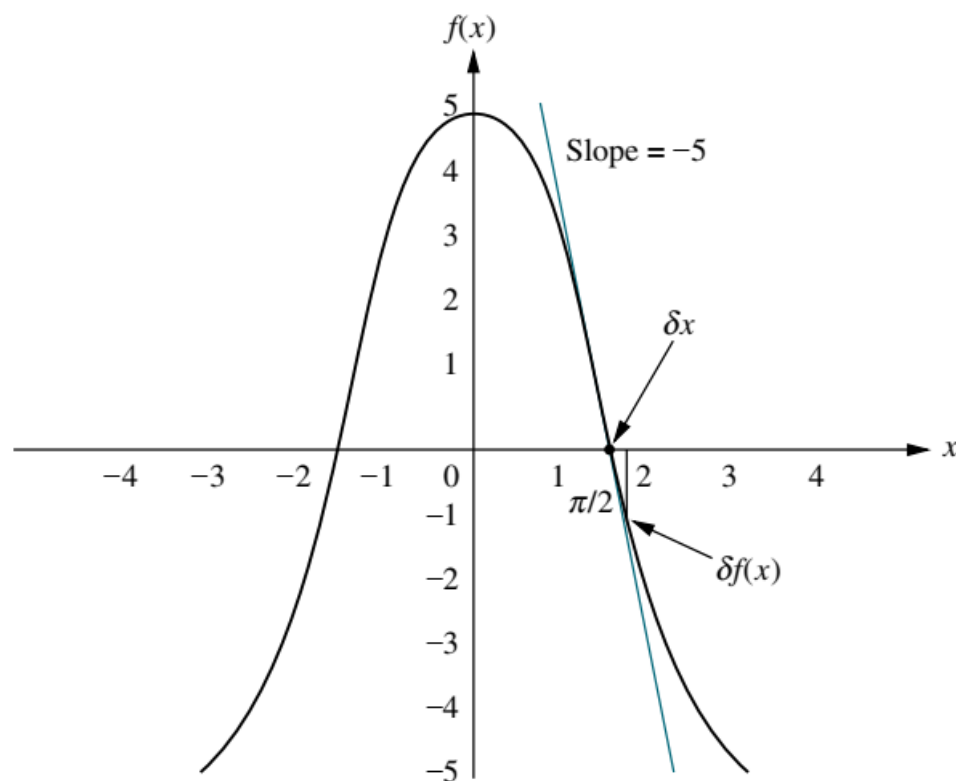
1. The first step is to recognize the nonlinear component and write the nonlinear differential equation.
2. We linearize it for small-signal inputs about the steady-state solution, this steady state solution called equilibrium and is selected as the second step in the linearization process.
3. Next, we linearize the nonlinear differential equation, and then we take the Laplace transform of the linearized differential equation.



We usually linearize the system about certain point.

Example

Linearize $f(x) = 5 \cos x$ about $x = \pi/2$.



SOLUTION: We first find that the derivative of $f(x)$ is $df/dx = (-5 \sin x)$. At $x = \pi/2$, the derivative is -5 . Also $f(x_0) = f(\pi/2) = 5 \cos(\pi/2) = 0$. Thus, from Eq. (2.180), the system can be represented as $f(x) = -5 \delta x$ for small excursions of x about $\pi/2$. The process is shown graphically in Figure 2.48, where the cosine curve does indeed look like a straight line of slope -5 near $\pi/2$.

Taylor series expansion

$$f(x) = f(x_0) + \left. \frac{df}{dx} \right|_{x=x_0} \frac{(x - x_0)}{1!} + \left. \frac{d^2f}{dx^2} \right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \dots$$

For small excursions of x from x_0 , we can neglect higher-order terms.

$$f(x) - f(x_0) \approx \left. \frac{df}{dx} \right|_{x=x_0} (x - x_0)$$

Example

Linearize the following for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0$$

SOLUTION: The presence of the term $\cos x$ makes this equation nonlinear. Since we want to linearize the equation about $x = \pi/4$, we let $x = \delta x + \pi/4$, where δx is the small excursion about $\pi/4$, and substitute x into Eq. (2.184):

$$\frac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} + 2\frac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} + \cos\left(\delta x + \frac{\pi}{4}\right) = 0$$

$$\frac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} = \frac{d^2\delta x}{dt^2}$$

$$\frac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} = \frac{d\delta x}{dt}$$

Example

Finally, the term $\cos(\delta x + (\pi/4))$ can be linearized with the truncated Taylor series. Substituting $f(x) = \cos(\delta x + (\pi/4))$, $f(x_0) = f(\pi/4) = \cos(\pi/4)$, and $(x - x_0) = \delta x$ into Eq. (2.182) yields

$$\cos\left(\delta x + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \left. \frac{d \cos x}{dx} \right|_{x=\frac{\pi}{4}} \delta x = -\sin\left(\frac{\pi}{4}\right) \delta x$$

Solving Eq. (2.188) for $\cos(\delta x + (\pi/4))$, we get

$$\cos\left(\delta x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \delta x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \delta x \quad (2.189)$$

Substituting Eqs. (2.186), (2.187), and (2.189) into Eq. (2.185) yields the following linearized differential equation:

$$\frac{d^2 \delta x}{dt^2} + 2 \frac{d \delta x}{dt} - \frac{\sqrt{2}}{2} \delta x = -\frac{\sqrt{2}}{2} \quad (2.190)$$