# Dynamics of Structures
# SDOF Systems

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# Free Vibration

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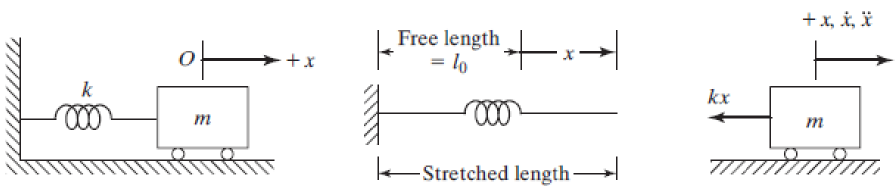
# Introduction

- A system is said to undergo *free vibration* when it oscillates only under an initial disturbance with no external forces.
- Figure below shows a spring-mass system that represents the simplest possible vibratory system.
- It is called a *single-degree-of-freedom system*, since one coordinate ($x$) is sufficient to specify the position of the mass at any time.
- There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be free vibration.

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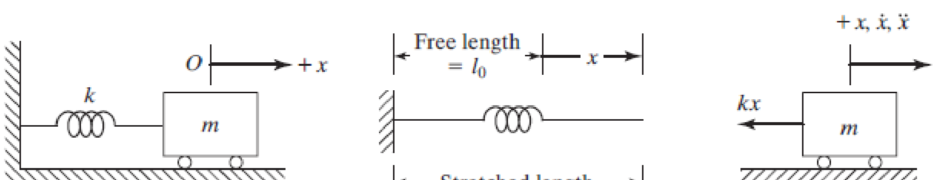
# Introduction

- Since there is no element that causes dissipation of energy during the motion of the mass, the *amplitude of motion remains constant with time*. So, it is an *undamped* system.
- In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (such as air).
- Such vibrations are said to be *damped*.

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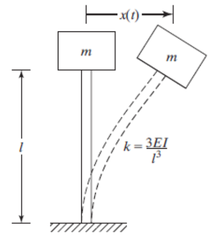
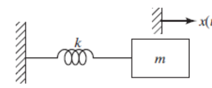
# Introduction

- Several mechanical and structural systems can be idealized as single-degree-of-freedom systems.
- In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass.
- Similarly, the elasticity of the system, which may be distributed throughout the system, can also be idealized by a *single spring.*




(a) Idealization of the tall structure

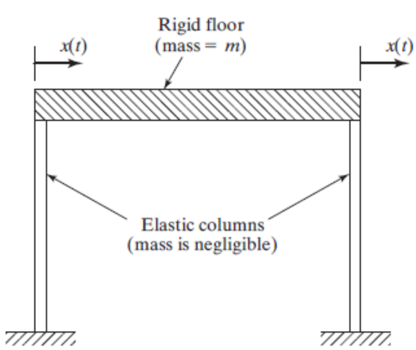



(b) Equivalent spring-mass system

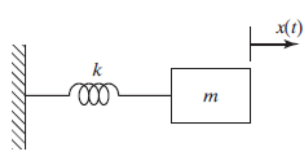


# Introduction

Another example




(a) Building frame




(b) Equivalent spring-mass system

# Undamped Systems – Newton's second law

if a mass $m$ is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction, Newton's second law of motion gives

$$\vec{F}(t) = \frac{d}{dt}\left(m\frac{d\vec{x}(t)}{dt}\right)$$

If the mass $m$ is constant, this equation reduces to

$$\vec{F}(t) = m\frac{d^2\vec{x}(t)}{dt^2} = m\ddot{\vec{x}}$$

where

$$\ddot{\vec{x}} = \frac{d^2\vec{x}(t)}{dt^2}$$

In the form

Resultant force on the mass = mass × acceleration

For a rigid body undergoing rotational motion, Newton's law gives

$$\vec{M}(t) = J\ddot{\vec{\theta}}$$

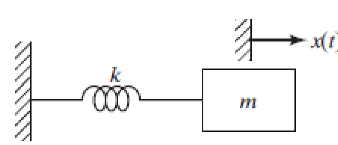


# Undamped Systems

Then, the equation of motion has the form

$$F(t) = -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$




(b) Equivalent spring-mass system

# Undamped systems – Conservation of Energy

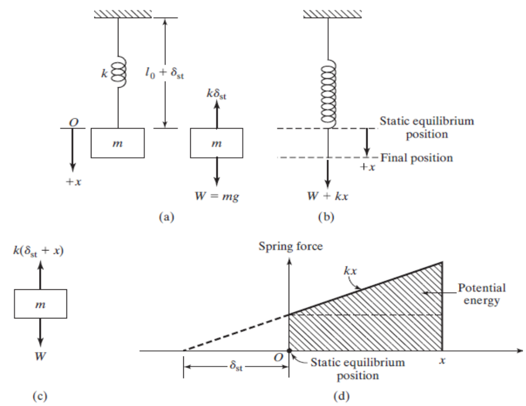
- A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members.
- If no work is done on a conservative system by external forces (other than gravity or other potential forces), then the total energy of the system remains constant.

$$T + U = \text{constant} \qquad \frac{d}{dt}(T + U) = 0$$

$$T = \tfrac{1}{2}m\dot{x}^2 \qquad U = \tfrac{1}{2}kx^2$$

$$m\ddot{x} + kx = 0$$

Represents a 2nd order homogenous differential equation with constant coefficients.

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# 2nd order homogenous DE with constant coeffecients

The general solution for this type of equations has the form,

$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

Where $m_1$ and $m_2$ are the roots of the corresponding algebraic equation.

Example :- $y'' - 5y' + 6y = 0$

The corresponding algebraic eqn :- $m^2 - 5m + 6 = 0$

$\Rightarrow m = 3, 2$

$\Rightarrow$ The general sol. $y = C_1 e^{2x} + C_2 e^{3x}$

$C_1, C_2$ can be obtained from the BC's.

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# Undamped Systems – Solution

Another method

$$x(t) = Ce^{st} \tag{2.11}$$

where $C$ and $s$ are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since $C$ cannot be zero, we have

$$ms^2 + k = 0 \tag{2.12}$$

and hence

$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n \tag{2.13}$$

where $i = (-1)^{1/2}$ and

$$\omega_n = \left(\frac{k}{m}\right)^{1/2}$$

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# Undamped Systems

Which has the general solution

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

where $C_1$ and $C_2$ are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$\begin{aligned} x(t = 0) &= A_1 = x_0 \\ \dot{x}(t = 0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \tag{2.17}$$

Hence $A_1 = x_0$ and $A_2 = \dot{x}_0/\omega_n$. Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \tag{2.18}$$

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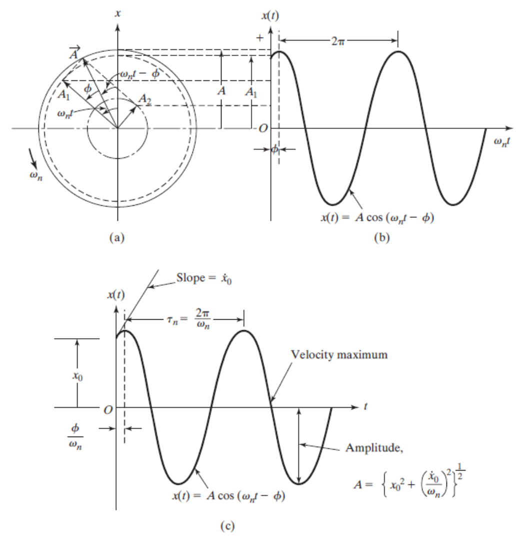
# Undamped Systems




$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

This equation represents a harmonic function in time that can be expressed in different forms

$$x(t) = A \cos(\omega_n t - \phi)$$

$$A = (A_1^2 + A_2^2)^{1/2} = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude}$$

$$\phi = \tan^{-1}\left( \frac{A_2}{A_1} \right) = \tan^{-1}\left( \frac{\dot{x}_0}{x_0 \omega_n} \right) = \text{phase angle}$$

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$A_0 = A = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left( \frac{x_0 \omega_n}{\dot{x}_0} \right)$$

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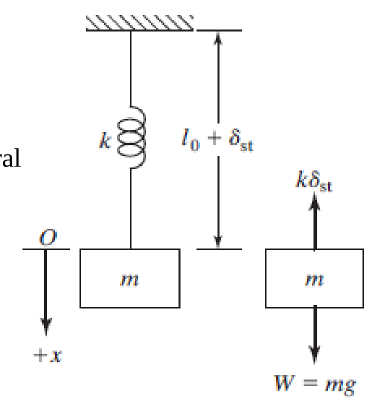
# Comments




- If the spring-mass system is in a vertical position, as shown in Fig, the natural frequency can be expressed as

$$\omega_n = \left( \frac{k}{m} \right)^{1/2}$$

- The spring constant $k$ can be expressed in terms of the mass $m$ as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \qquad \omega_n = \left( \frac{g}{\delta_{st}} \right)^{1/2}$$

where $\delta_{st}$ is the static deflection (i.e. the elongation due to the weight $W$ of the mass $m$).

Hence the natural frequency in cycles per second and the natural period are given by

$$f_n = \frac{1}{2\pi} \left( \frac{g}{\delta_{st}} \right)^{1/2}$$

$$\tau_n = \frac{1}{f_n} = 2\pi \left( \frac{\delta_{st}}{g} \right)^{1/2}$$

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection (i.e. we don t need to know the spring stiffness $k$ and the mass $m$).

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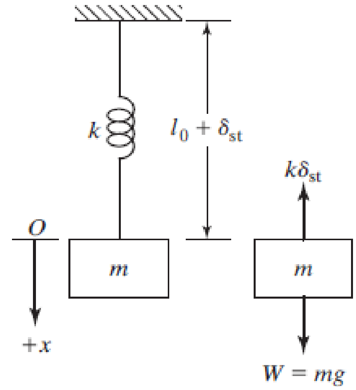
# Comments

The mass velocity and acceleration can be expressed as ,

$$\dot{x}(t) = \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right)$$

$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi)$$




$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$$

Thus, the velocity leads the displacement by $\frac{\pi}{2}$ and the acceleration leads the displacement by $\pi$.

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right)$$

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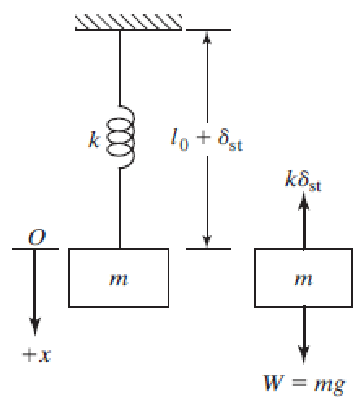
# Comments

If the initial displacement is zero,

$$x(t) = \frac{\dot{x}_0}{\omega_n}\cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$




If the initial velocity is zero, however, the solution becomes

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$x(t) = x_0 \cos \omega_n t$$

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right)$$

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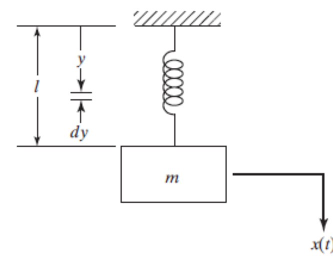
# Effect of spring mass

- In general, we neglect the spring mass in the vibration problem, because the spring mass is relatively small than the structure mass.
- One can study the effect of spring mass by assuming the mass-spring system shown in figure with spring constant $k$ and spring mass $m_s$.
- Then, we can add the spring kinetic energy (K.E.) to the K.E. of the mass.
- Since the displacement and velocity of the lower end of the spring are $x, \dot{x}$, then the displacement and velocity at distance $y$ from the support are $\left(y\frac{x}{l}\right), and \left(y\frac{\dot{x}}{l}\right)$.
- Then, the K.E. of a spring of length $dy$.




$$dT_s = \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y\dot{x}}{l}\right)^2$$

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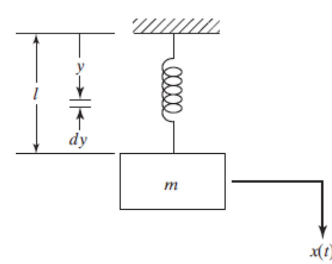
# Effect of spring mass

The total K.E. has the form,

$$T = \text{kinetic energy of mass } (T_m) + \text{kinetic energy of spring } (T_s)$$
$$= \frac{1}{2}m\dot{x}^2 + \int_{y=0}^{l}\frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y^2\dot{x}^2}{l^2}\right)$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$




The total potential energy of the system is given by

$$U = \tfrac{1}{2}kx^2$$

By assuming a harmonic motion

$$x(t) = X\cos\omega_n t$$

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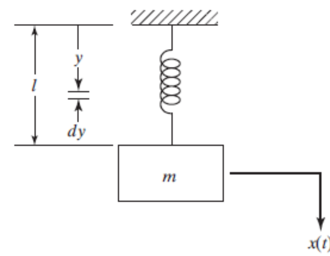
# Effect of spring mass

where $X$ is the maximum displacement of the mass and $\omega_n$ is the natural frequency, the maximum kinetic and potential energies can be expressed as

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)X^2\omega_n^2$$

$$U_{\max} = \frac{1}{2}kX^2$$




By equating $T_{\max}$ and $U_{\max}$, we obtain the expression for the natural frequency:

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}}\right)^{1/2}$$

Thus the effect of the mass of the spring can be accounted for by adding one-third of its mass to the main mass

Since the spring mass is relatively small than the structure mass, we usually neglects the spring weight.



# Example 1 – Harmonic Response of a Water Tank

The column of the water tank shown in Fig. 2.10(a) is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs $6 \times 10^5$ lb when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as $4 \times 10^6$ psi, determine the following:

**a.** the natural frequency and the natural time period of transverse vibration of the water tank.
**b.** the vibration response of the water tank due to an initial transverse displacement of 10 in.
**c.** the maximum values of the velocity and acceleration experienced by the water tank.

**Solution:** Assuming that the water tank is a point mass, the column has a uniform cross section, and the mass of the column is negligible, the system can be modeled as a cantilever beam with a concentrated load (weight) at the free end as shown in Fig. 2.10(b).

**a.** The transverse deflection of the beam, $\delta$, due to a load $P$ is given by $\frac{Pl^3}{3EI}$, where $l$ is the length, $E$ is the Young's modulus, and $I$ is the area moment of inertia of the beam's cross section. The stiffness of the beam (column of the tank) is given by

$$k = \frac{P}{\delta} = \frac{3EI}{l^3}$$

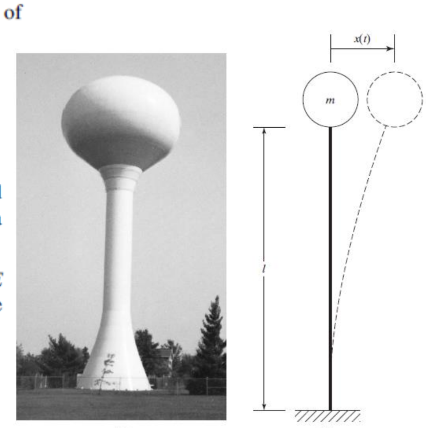



**FIGURE 2.10** Elevated tank. (Photo courtesy of West Lafayette Water Company.)

# Example 1 – Harmonic Response of a Water Tank

In the present case, $l = 3600$ in., $E = 4 \times 10^6$ psi,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4) = 600.9554 \times 10^4 \text{ in.}^4$$

and hence

$$k = \frac{3(4 \times 10^6)(600.9554 \times 10^4)}{3600^3} = 1545.6672 \text{ lb/in.}$$

The natural frequency of the water tank in the transverse direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672 \times 386.4}{6 \times 10^5}} = 0.9977 \text{ rad/sec}$$

The natural time period of transverse vibration of the tank is given by

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{0.9977} = 6.2977 \text{ sec}$$

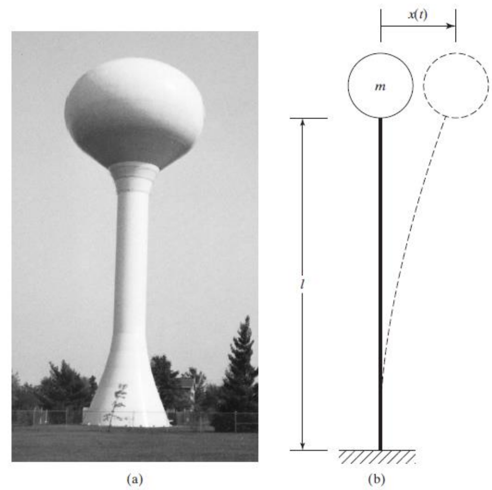



**FIGURE 2.10** Elevated tank. (Photo courtesy of West Lafayette Water Company.)



# Example 1 – Harmonic Response of a Water Tank

**b.** Using the initial displacement of $x_0 = 10$ in. and the initial velocity of the water tank $(\dot{x}_0)$ as zero, the harmonic response of the water tank can be expressed, using Eq. (2.23), as

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where the amplitude of transverse displacement $(A_0)$ is given by

$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} = x_0 = 10 \text{ in.}$$

and the phase angle $(\phi_0)$ by

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{0}\right) = \frac{\pi}{2}$$

Thus

$$x(t) = 10 \sin\left(0.9977t + \frac{\pi}{2}\right) = 10 \cos 0.9977t \text{ in.} \quad \text{(E.1)}$$

**c.** The velocity of the water tank can be found by differentiating Eq. (E.1) as

$$\dot{x}(t) = 10(0.9977)\cos\left(0.9977t + \frac{\pi}{2}\right) \quad \text{(E.2)}$$

and hence

$$\dot{x}_{max} = A_0\omega_n = 10(0.9977) = 9.977 \text{ in./sec}$$

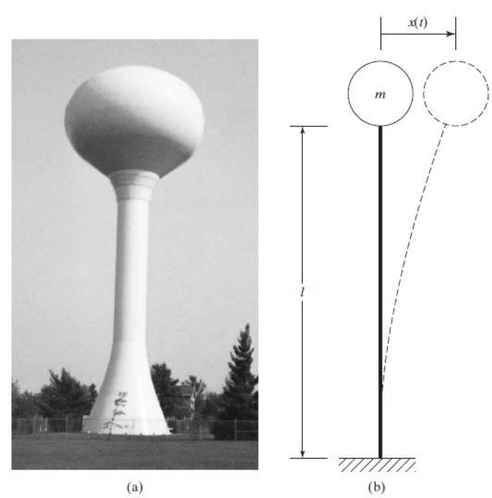



**FIGURE 2.10** Elevated tank. (Photo courtesy of West Lafayette Water Company.)

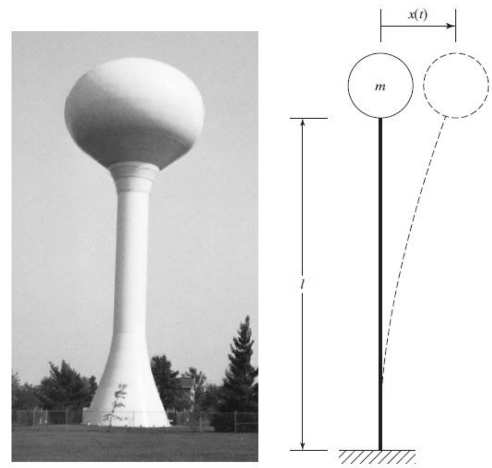
# Example 1 – Harmonic Response of a Water Tank

The acceleration of the water tank can be determined by differentiating Eq. (E.2) as

$$\ddot{x}(t) = -10(0.9977)^2 \sin\left(0.9977t + \frac{\pi}{2}\right)$$

and hence the maximum value of acceleration is given by

$$\ddot{x}_{\max} = A_0(\omega_n)^2 = 10(0.9977)^2 = 9.9540 \text{ in./sec}^2$$



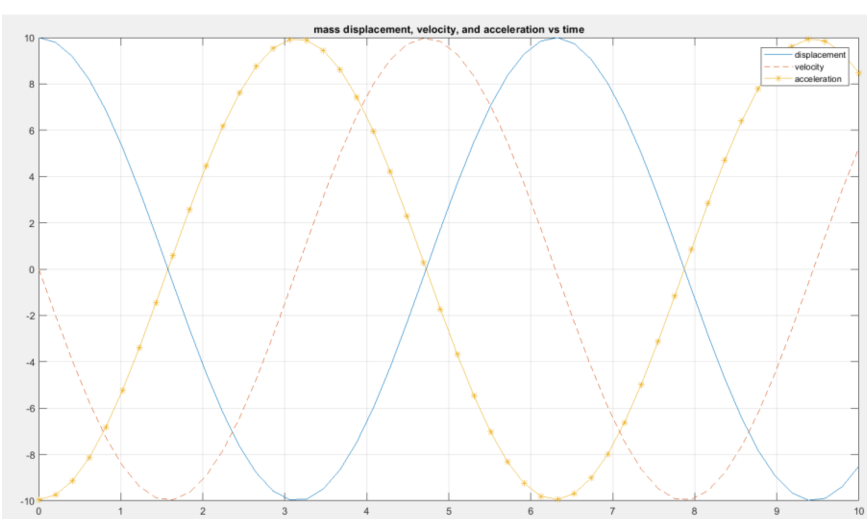

(a) (b)

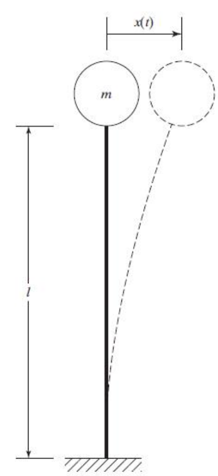
**FIGURE 2.10** Elevated tank. (Photo courtesy of West Lafayette Water Company.)



# Example 1 – Harmonic Response of a Water Tank

**System response**



# Different interpretation of the response equation

The general solution takes the form

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

By using the exponential definition $e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$

The solution will take the form

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

A1 and A2 are obtained from the initial conditions.

$$x(t) = A \cos (\omega_n t - \phi)$$

$$A = (A_1^2 + A_2^2)^{1/2} = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude}$$

$$\phi = \tan^{-1}\left( \frac{A_2}{A_1} \right) = \tan^{-1}\left( \frac{\dot{x}_0}{x_0 \omega_n} \right) = \text{phase angle}$$

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

$$A_0 = A = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left( \frac{x_0 \omega_n}{\dot{x}_0} \right)$$

$x_0$ *and* $\dot{x}_0$ are the initial conditions for the displacement and velocity

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# Example 1 – column mass effect on the natural frequencies

- To include the column mass, we find the equivalent mass of the column at the free end using the *equivalence of kinetic energy* and use a single degree-of-freedom model to find the natural frequency of vibration.
- The column of the tank is considered as a cantilever beam fixed at one end (ground) and carrying a mass *M* (water tank) at the other end.
- The static deflection of a cantilever beam under a concentrated end load is given by

$$y(x) = \frac{Px^2}{6EI}(3l - x) = \frac{y_{max} x^2}{2l^3}(3l - x)$$
$$= \frac{y_{max}}{2l^3}(3x^2 l - x^3)$$

The maximum kinetic energy of the beam itself ($T_{max}$) is given by

$$T_{max} = \frac{1}{2}\int_0^l \frac{m}{l}\{\dot{y}(x)\}^2 \, dx$$

$$T_{max} = \int_0^L \tfrac{1}{2} m \dot{y}^2$$

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# Example 1 – column mass effect on the natural frequencies

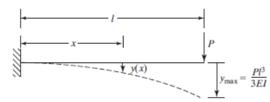
where $m$ is the total mass and $(m/l)$ is the mass per unit length of the beam. Then the velocity takes the form,

$$\dot{y}(x) = \frac{\dot{y}_{max}}{2l^3}(3x^2 l - x^3)$$

And

$$T_{max} = \frac{m}{2l}\left(\frac{\dot{y}_{max}}{2l^3}\right)^2 \int_0^l (3x^2 l - x^3)^2 dx$$

$$= \frac{1}{2}\frac{m}{l}\frac{\dot{y}_{max}^2}{4l^6}\left(\frac{33}{35}l^7\right) = \frac{1}{2}\left(\frac{33}{140}m\right)\dot{y}_{max}^2$$




$$y(x) = \frac{Px^2}{6EI}(3l - x) = \frac{y_{max}x^2}{2l^3}(3l - x)$$

$$= \frac{y_{max}}{2l^3}(3x^2 l - x^3)$$

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# Example 1 – column mass effect on the natural frequencies

If $m_{eq}$ denotes the equivalent mass of the cantilever (water tank) at the free end, its maximum kinetic energy can be expressed as

$$T_{max} = \frac{1}{2}m_{eq}\dot{y}_{max}^2$$

By substitute in the K.E. equation,

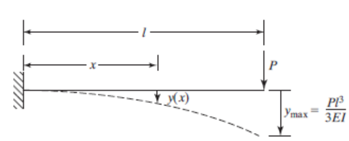
$$m_{eq} = \frac{33}{140}m$$

Thus the total effective mass acting at the end of the cantilever beam is given by

$$M_{eff} = M + m_{eq}$$

where $M$ is the mass of the water tank. The natural frequency of transverse vibration of the water tank is given by

$$\omega_n = \sqrt{\frac{k}{M_{eff}}} = \sqrt{\frac{k}{M + \frac{33}{140}m}}$$

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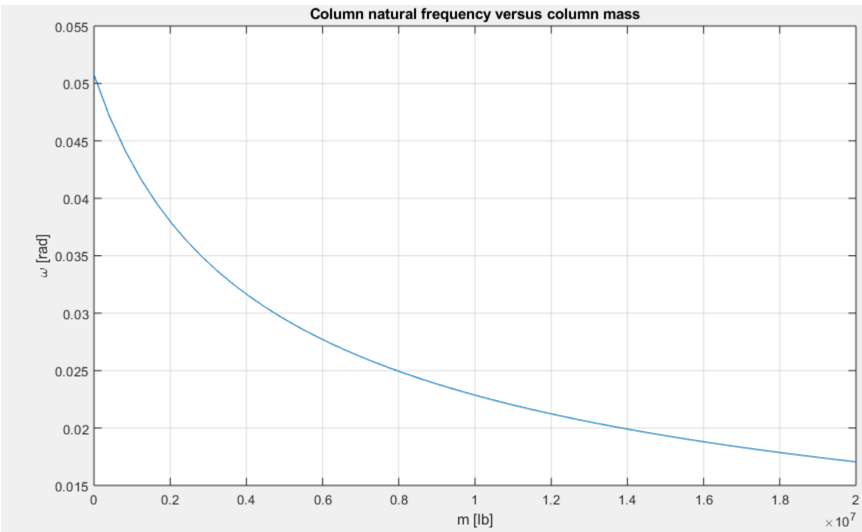
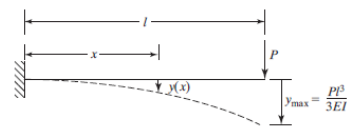
# Example 1 – column mass effect on the natural frequencies




$$\omega_n = \sqrt{\frac{k}{M_{\text{eff}}}} = \sqrt{\frac{k}{M + \frac{33}{140}m}}$$

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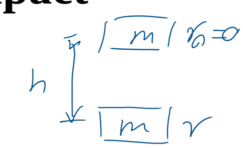
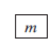
# Example 2 – Free-vibration response due to impact




A cantilever beam carries a mass $M$ at the free end as shown in Fig. 2.11(a). A mass $m$ falls from a height $h$ onto the mass $M$ and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.

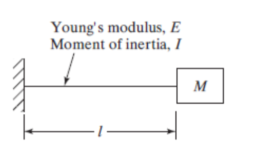



then $a = v\frac{dv}{dx} = \text{const} = g$

$\Rightarrow \int_{x_0}^{x} g\,dx = \int_{v_0=0}^{v} v\,dv$

$\Rightarrow g(x - x_0) = \frac{1}{2}v^2$

$\Rightarrow v^2 = 2gh$




$k = \frac{3EI}{l^3}$

The mass $m$ applies an initial velocity $v$ to the mass $M$.

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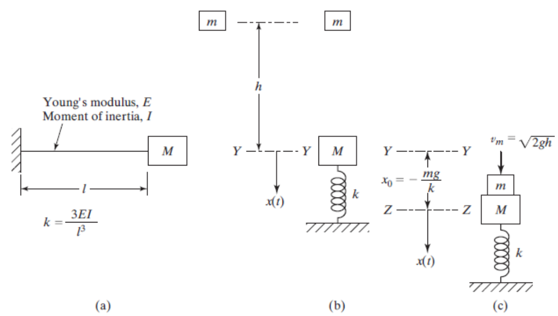
# Example 2 – Free-vibration response due to impact

**Solution:** When the mass $m$ falls through a height $h$, it will strike the mass $M$ with a velocity of $v_m = \sqrt{2gh}$, where $g$ is the acceleration due to gravity. Since the mass $m$ adheres to $M$ without rebounding, the velocity of the combined mass $(M + m)$ immediately after the impact $(\dot{x}_0)$ can be found using the principle of conservation of momentum:

$$m v_m = (M + m)\dot{x}_0$$




YY = static equilibrium position of $M$
ZZ = static equilibrium position of $M + m$



# Example 2 – Free-vibration response due to impact

or

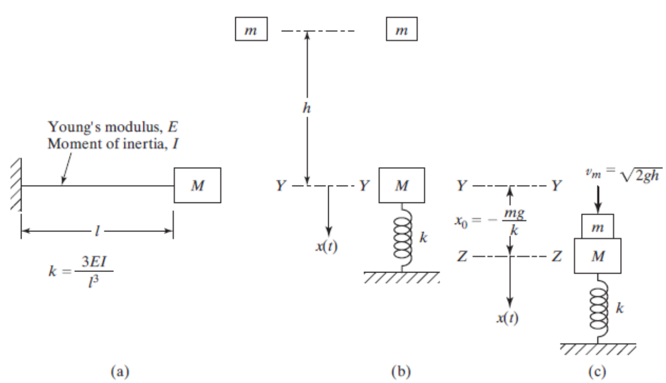
$$\dot{x}_0 = \left(\frac{m}{M + m}\right) v_m = \left(\frac{m}{M + m}\right)\sqrt{2gh} \qquad \text{(E.1)}$$

The static equilibrium position of the beam with the new mass $(M + m)$ is located at a distance of $\frac{mg}{k}$ below the static equilibrium position of the original mass $(M)$ as shown in Fig. 2.11(c). Here $k$ denotes the stiffness of the cantilever beam, given by

$$k = \frac{3EI}{l^3}$$

Since free vibration of the beam with the new mass $(M + m)$ occurs about its own static equilibrium position, the initial conditions of the problem can be stated as

$$x_0 = -\frac{mg}{k}, \qquad \dot{x}_0 = \left(\frac{m}{M + m}\right)\sqrt{2gh} \qquad \text{(E.2)}$$




YY = static equilibrium position of $M$
ZZ = static equilibrium position of $M + m$

# Example 2 – Free-vibration response due to impact

Thus the resulting free transverse vibration of the beam can be expressed as (see Eq. (2.21)):

$$x(t) = A\cos(\omega_n t - \phi)$$

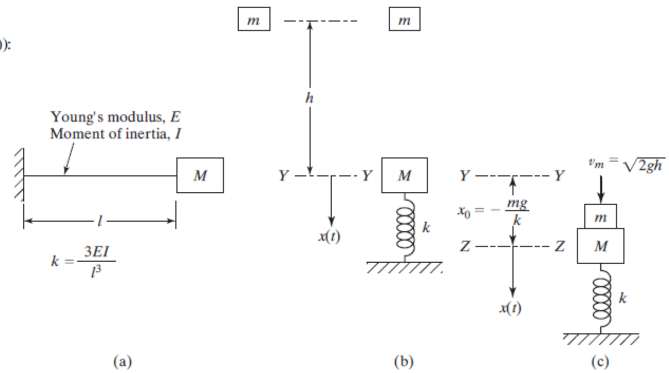
where

$$A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$$

$$\phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

$$\omega_n = \sqrt{\frac{k}{M+m}} = \sqrt{\frac{3EI}{l^3(M+m)}}$$

with $x_0$ and $\dot{x}_0$ given by Eq. (E.2).




$YY$ = static equilibrium position of $M$
$ZZ$ = static equilibrium position of $M + m$

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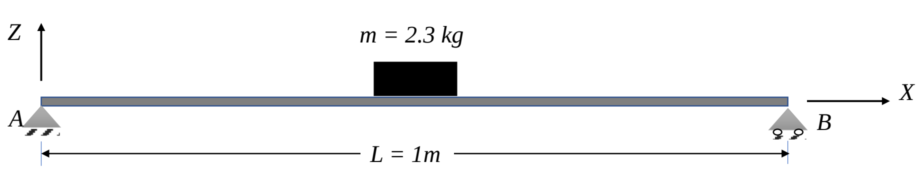
# Example 3 – Young's Modulus from Natural Frequency Measurement

A simply supported beam of square cross section 5 mm × 5 mm and length 1 *m*, carrying a mass of 2.3 kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.

**Solution:** By neglecting the self weight of the beam, the natural frequency of transverse vibration of the beam can be expressed as

$$\omega_n = \sqrt{\frac{k}{m}} \tag{E.1}$$

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# Example 3 – Young s Modulus from Natural Frequency Measurement

where

$$k = \frac{192EI}{l^3} \tag{E.2}$$

where $E$ is the Young's modulus, $l$ is the length, and $I$ is the area moment of inertia of the beam:

$$I = \frac{1}{12}(5 \times 10^{-3})(5 \times 10^{-3})^3 = 0.5208 \times 10^{-10} \text{ m}^4$$

Since $m = 2.3$ kg, $l = 1.0$ m, and $\omega_n = 30.0$ rad/s, Eqs. (E.1) and (E.2) yield

$$k = \frac{192EI}{l^3} = m\omega_n^2$$

or

$$E = \frac{m\omega_n^2 l^3}{192I} = \frac{2.3(30.0)^2(1.0)^3}{192(0.5208 \times 10^{-10})} = 207.0132 \times 10^9 \text{ N/m}^2$$

This indicates that the material of the beam is probably carbon steel.

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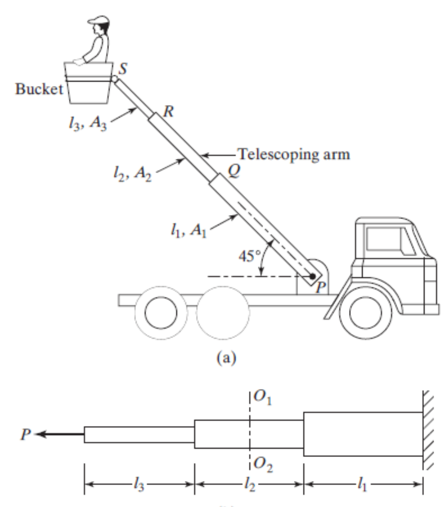
# Example 4 – Natural Frequency of Cockpit of a Firetruck

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.12(a). The cockpit, along with the fireman, weighs 2000 N. Find the cockpit's natural frequency of vibration in the vertical direction.

*Data:* Young's modulus of the material: $E = 2.1 \times 10^{11}$ N/m$^2$; lengths: $l_1 = l_2 = l_3 = 3$ m; cross-sectional areas: $A_1 = 20$ cm$^2$, $A_2 = 10$ cm$^2$, $A_3 = 5$ cm$^2$.

**Solution:** To determine the system's natural frequency of vibration, we find the equivalent stiffness of the boom in the vertical direction and use a single-degree-of-freedom idealization. For this we assume that the mass of the telescoping boom is negligible and the telescoping boom can deform only in the axial direction (with no bending). Since the force induced at any cross section $O_1O_2$ is equal to the axial load applied at the end of the boom, as shown in Fig. 2.12(b), the axial stiffness of the boom ($k_b$) is given by

$$\frac{1}{k_b} = \frac{1}{k_{b_1}} + \frac{1}{k_{b_2}} + \frac{1}{k_{b_3}} \tag{E.1}$$

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# Example 4 – Natural Frequency of Cockpit of a Firetruck

where $k_{b_i}$ denotes the axial stiffness of the $i$th segment of the boom:

$$k_{b_i} = \frac{A_i E_i}{l_i}; \qquad i = 1, 2, 3 \tag{E.2}$$

From the known data ($l_1 = l_2 = l_3 = 3$ m, $A_1 = 20$ cm$^2$, $A_2 = 10$ cm$^2$; $A_3 = 5$ cm$^2$, $E_1 = E_2 = E_3 = 2.1 \times 10^{11}$ N/m$^2$),

$$k_{b_1} = \frac{(20 \times 10^{-4})(2.1 \times 10^{11})}{3} = 14 \times 10^7 \text{ N/m}$$

$$k_{b_2} = \frac{(10 \times 10^{-4})(2.1 \times 10^{11})}{3} = 7 \times 10^7 \text{ N/m}$$

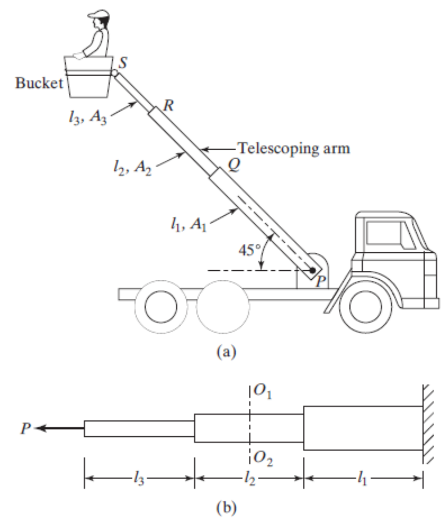
$$k_{b_3} = \frac{(5 \times 10^{-4})(2.1 \times 10^{11})}{3} = 3.5 \times 10^7 \text{ N/m}$$

Thus Eq. (E.1) gives

$$\frac{1}{k_b} = \frac{1}{14 \times 10^7} + \frac{1}{7 \times 10^7} + \frac{1}{3.5 \times 10^7} = \frac{1}{2 \times 10^7}$$

or

$$k_b = 2 \times 10^7 \text{ N/m}$$

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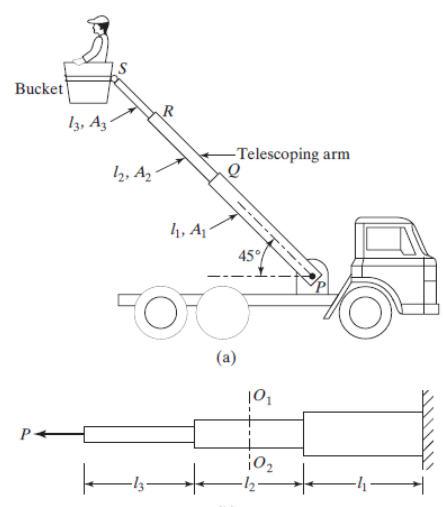
# Example 4 – Natural Frequency of Cockpit of a Firetruck

The stiffness of the telescoping boom in the vertical direction, $k$, can be determined as

$$k = k_b \cos^2 45° = 10^7 \text{ N/m}$$

The natural frequency of vibration of the cockpit in the vertical direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(10^7)(9.81)}{2000}} = 221.4723 \text{ rad/s}$$

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# Free Vibration with Viscous Damping

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## Equation of motion

As stated in Section 1.9, the viscous damping force $F$ is proportional to the velocity $\dot{x}$ or $v$ and can be expressed as

$$F = -c\dot{x} \tag{2.58}$$

where $c$ is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single-degree-of-freedom system with a viscous damper is shown in Fig. 2.21. If $x$ is measured from the equilibrium position of the mass $m$, the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

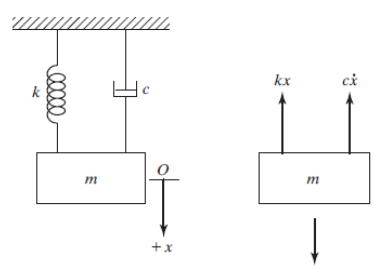
$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.59}$$

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \tag{2.60}$$

where $C$ and $s$ are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \tag{2.61}$$




System

Free-body diagram

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# Solution

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.62}$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \tag{2.63}$$

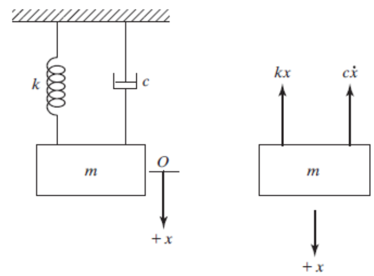
Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$
$$= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} \tag{2.64}$$

where $C_1$ and $C_2$ are arbitrary constants to be determined from the initial conditions of the system.

**Critical Damping Constant and the Damping Ratio.** The critical damping $c_c$ is defined as the value of the damping constant $c$ for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$




System Free-body diagram

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# Solution

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \tag{2.65}$$

For any damped system, the damping ratio $\zeta$ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \tag{2.66}$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n \tag{2.67}$$

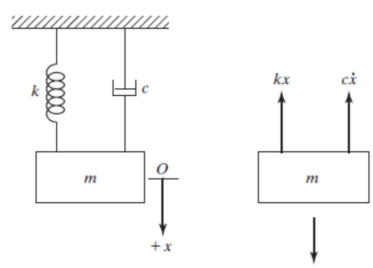
and hence

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \tag{2.68}$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t} \tag{2.69}$$

The nature of the roots $s_1$ and $s_2$ and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.




System Free-body diagram

$$ms^2 + cs + k = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

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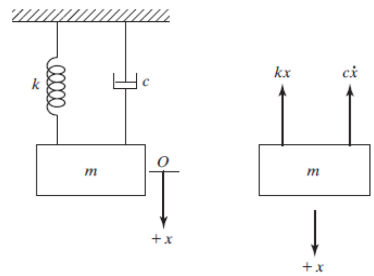
# Solution

**Case 1.** *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots $s_1$ and $s_2$ can be expressed as

$$s_1 = (-\zeta + i\sqrt{1-\zeta^2})\omega_n$$
$$s_2 = (-\zeta - i\sqrt{1-\zeta^2})\omega_n$$

and the solution, Eq. (2.69), can be written in different forms:

$$\begin{aligned} x(t) &= C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} \\ &= e^{-\zeta\omega_n t}\left\{ C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right\} \\ &= e^{-\zeta\omega_n t}\left\{ (C_1 + C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\omega_n t \right\} \\ &= e^{-\zeta\omega_n t}\left\{ C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t \right\} \end{aligned}$$




System    Free-body diagram

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# Solution

$$\begin{aligned} &= X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\omega_n t + \phi_0\right) \\ &= X e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\omega_n t - \phi\right) \end{aligned} \quad (2.70)$$

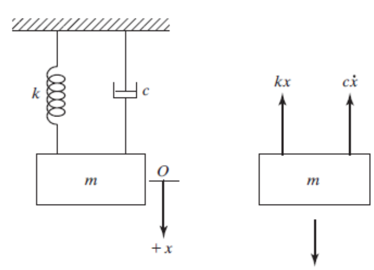
where $(C_1', C_2')$, $(X, \phi)$, and $(X_0, \phi_0)$ are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, $C_1'$ and $C_2'$ can be found:

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \quad (2.71)$$

and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{ x_0\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \quad (2.72)$$




System    Free-body diagram

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# Solution

The constants ($X$, $\phi$) and ($X_0$, $\phi_0$) can be expressed as

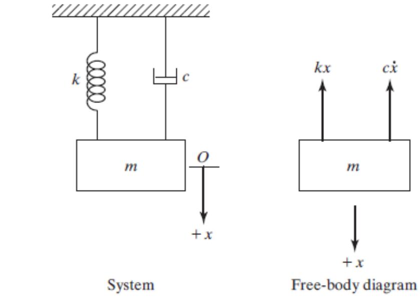
$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.73}$$

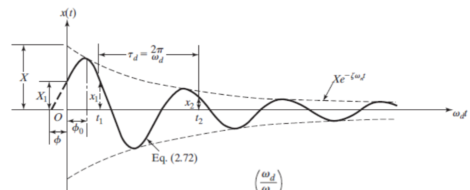
$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right) \tag{2.74}$$

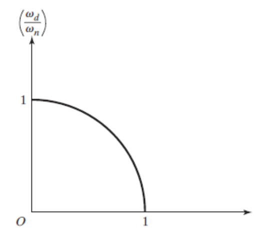
$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right) \tag{2.75}$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$, but because of the factor $e^{-\zeta\omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.76}$$




$\omega_d$ is the frequency of damped vibration. Then the response can take the form

$$x(t) = e^{-\xi\omega_n t}\left\{x_0\cos(\omega_d t) + \left(\frac{\dot{x}_0}{\omega_d} + \frac{\xi}{\sqrt{1-\xi^2}}x_0\right)\sin(\omega_d t)\right\}$$

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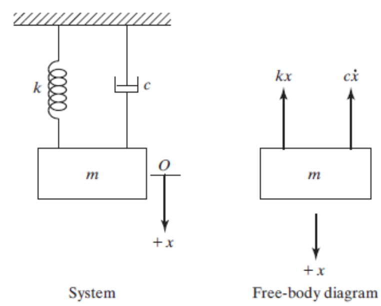
# Solution

**Case 2.** *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots $s_1$ and $s_2$ in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \tag{2.77}$$

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6][1]

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$



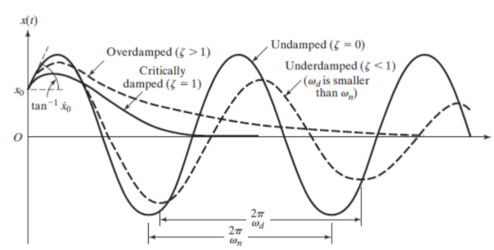

The application of the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ for this case gives

$$C_1 = x_0$$
$$C_2 = \dot{x}_0 + \omega_n x_0 \tag{2.79}$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{2.80}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.




A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting. The property of critical damping is used in many practical applications. For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.

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# Solution

**Case 3.** *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots $s_1$ and $s_2$ are real and distinct and are given by

$$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0$$
$$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0$$

with $s_2 \ll s_1$. In this case, the solution, Eq. (2.69), can be expressed as

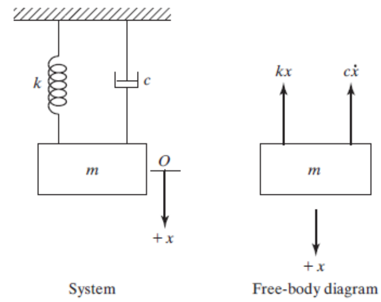
$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \tag{2.81}$$

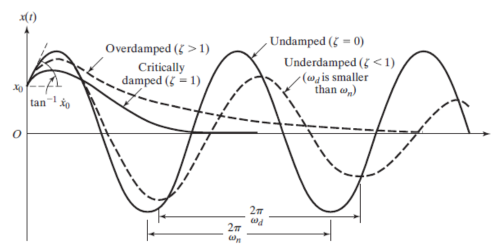
For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants $C_1$ and $C_2$ can be obtained:

$$C_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{2.82}$$

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots $s_1$ and $s_2$ are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

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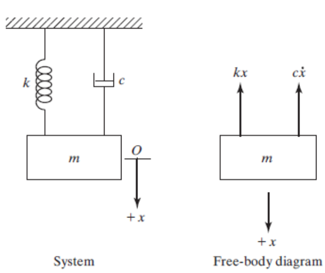
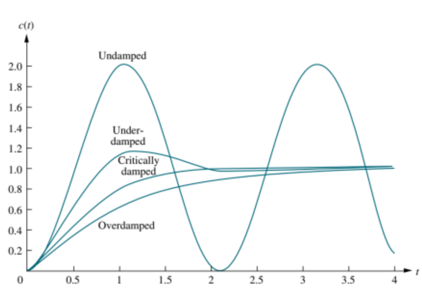
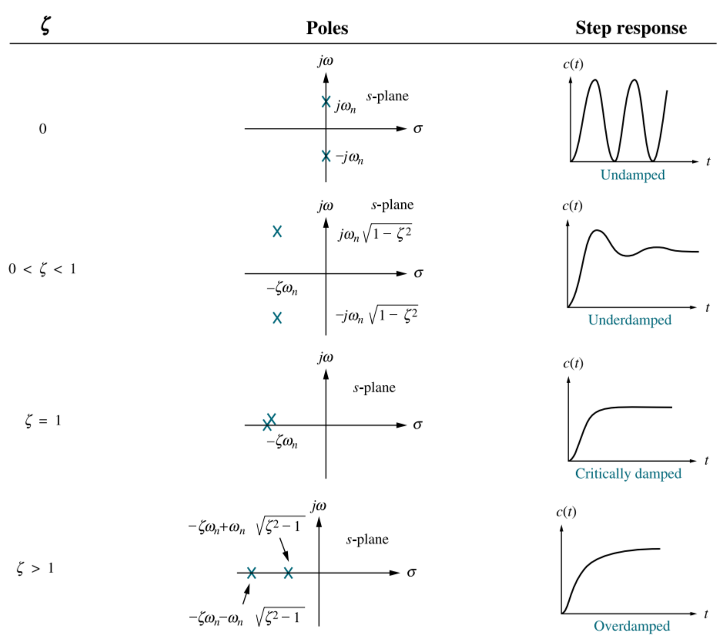
# Solution

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# Forced Vibration – Harmonic Excitation



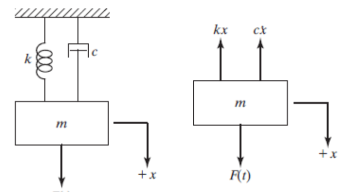
# Introduction

- A mechanical or structural system is said to undergo forced vibration whenever ***external energy*** is supplied to the system during vibration.
- External energy can be supplied through either an *applied force or an imposed displacement* excitation.
- The applied force or displacement excitation may be harmonic, or nonharmonic; periodic, or nonperiodic; or random.
- The response of a system to a *harmonic excitation* is called *harmonic response*.
- The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response*.

# Equation of motion

If a force $F(t)$ acts on a viscously damped spring-mass system as shown in Fig. the equation of motion can be obtained using Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$



(a) (b) Free-body diagram

Which represents a 2nd order, Nonhomogeneous O.D.E. with constant coefficients that can be solved using the *method of undetermined coefficients*.

The solution has two parts: a. homogenous solution (transient response), and b. particular-integral solution (steady-state solution).

$$x(t) = x_h + x_p$$

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# Response of an Undamped System Under Harmonic Force

The homogeneous solution of this equation is given by

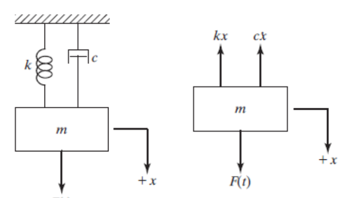
$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{3.4}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force $F(t)$ is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency $\omega$. Thus we assume a solution in the form

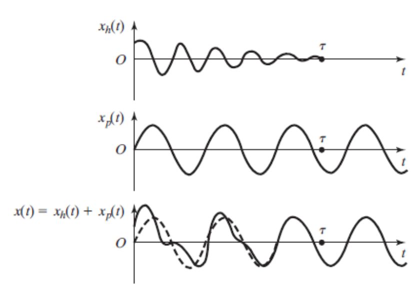
$$x_p(t) = X \cos \omega t \tag{3.5}$$

where $X$ is an constant that denotes the maximum amplitude of $x_p(t)$. By substituting Eq. (3.5) into Eq. (3.3) and solving for $X$, we obtain

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.6}$$



(a) (b) Free-body diagram

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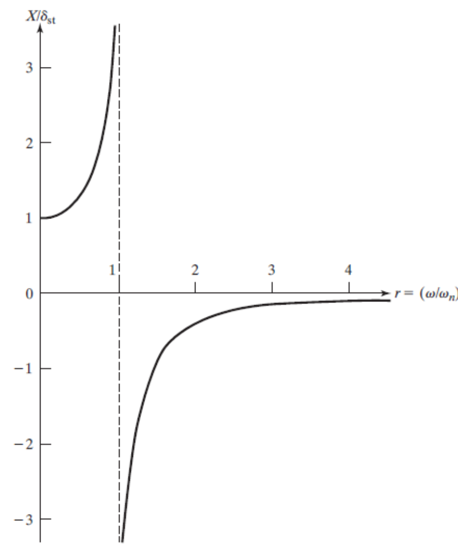
# Response of an Undamped System Under Harmonic Force

The maximum amplitude $X$ in Eq. (3.6) can be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$




This relation is called the *magnification factor*, *amplification factor, or amplitude ratio*. The variation of this ration with the frequency ration is shown in Figure.

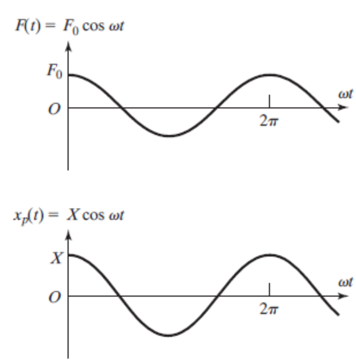
Based on the figure the system can be classified into three types.



# Response of an Undamped System Under Harmonic Force

**Case 1.** When $0 < \frac{\omega}{\omega_n} < 1$ the denominator in the equation is positive and the response is given by the equation,

$$x_p(t) = X \cos \omega t$$




The harmonic response of the system is said to be in phase with the external force as shown in Fig.

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

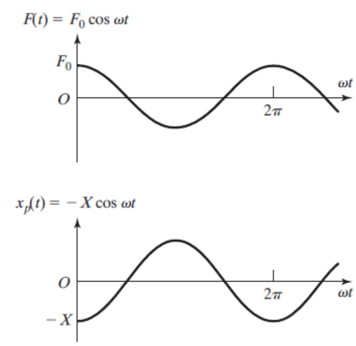
# Response of an Undamped System Under Harmonic Force

**Case 2.** When $\frac{\omega}{\omega_n} > 1$ the denominator in the equation is negative and the steady state solution can be expressed as,

$$x_p(t) = -X \cos \omega t$$

where the amplitude of motion $X$ is redefined to be a positive quantity as

$$X = \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$




The response is said to be $180^o$ out of phase with the external force. As $\frac{\omega}{\omega_n} \to \infty, X \to 0$. i.e. the system response to a force with very high frequency is close to zero.

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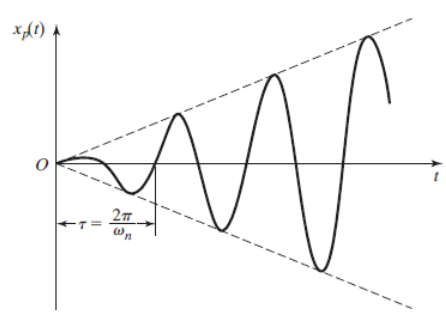
# Response of an Undamped System Under Harmonic Force

**Case 3.** When $\frac{\omega}{\omega_n} = 1$ the amplitude $X$ becomes infinite.

This condition, for which the forcing frequency is equal to the natural frequency of the system is called *resonance.* To find the response for this condition,

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st}\left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

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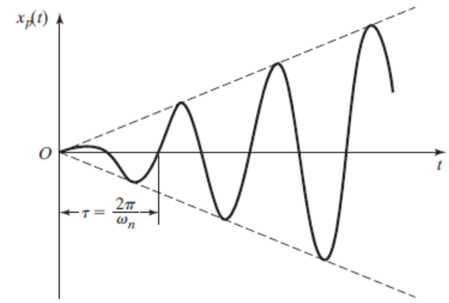
# Response of an Undamped System Under Harmonic Force

Since the last term of this equation takes an indefinite form for $\omega = \omega_n$, we apply L'Hospital's rule [3.1] to evaluate the limit of this term:

$$\lim_{\omega \to \omega_n} \left[ \frac{\cos \omega t - \cos \omega_n t}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \to \omega_n} \left[ \frac{\frac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\frac{d}{d\omega}\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \right]$$

$$= \lim_{\omega \to \omega_n} \left[ \frac{t \sin \omega t}{2 \frac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t$$




Thus the response of the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t$$

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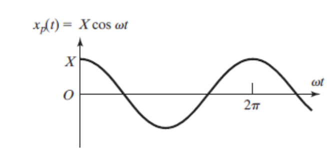
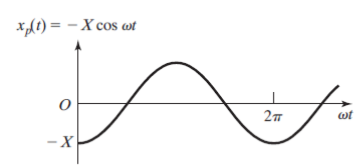
# Response of an Undamped System Under Harmonic Force

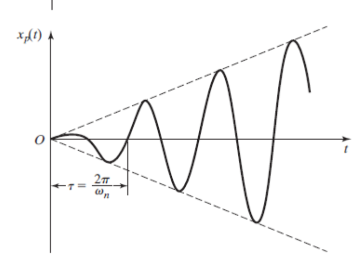
$$x(t) = A \cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \cos \omega t; \quad \text{for } \frac{\omega}{\omega_n} < 1$$




$$x(t) = A \cos(\omega_n t - \phi) - \frac{\delta_{st}}{-1 + \left( \frac{\omega}{\omega_n} \right)^2} \cos \omega t; \quad \text{for } \frac{\omega}{\omega_n} > 1$$




$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \quad \text{for } \frac{\omega}{\omega_n} = 1$$

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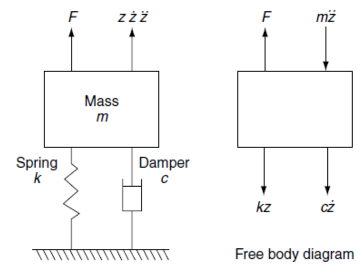
# Example 5 – Forced response

a. Derive an expression for the displacement response, z, when a step force of magnitude $P$ is applied to the single-DOF system shown in Fig.

Assume that the damping is less than critical, and that the initial conditions are $z = \dot{z} = 0$ at $t = 0$.

b. Plot the displacement, $z$, in non-dimensional form, as a multiple of the static displacement for the same load, $z_s = P/k$, with the non-dimensional viscous damping coefficient, ξ, equal to 0.1, and the undamped natural frequency, $\omega_n$, equal to 10 rad/s.






# Example 5 – Forced response

*Part (a):*
In this case the equation to be solved is

$$m\ddot{z} + c\dot{z} + kz = P$$

In the standard form

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = \frac{P}{m}$$

Since $\xi < 1$, the complementary solution will take the form

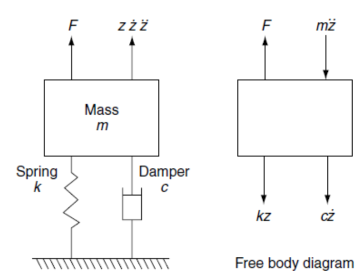
$$z = e^{-\xi\omega_n t}(A\cos(\omega_d t) + B\sin(\omega_d t))$$

For the particular-integral solution, assume z = C, and by substitute in the D.E.

$$C = \frac{P}{m\omega_n^2}$$

Then,

$$z = e^{-\xi\omega_n t}(A\cos(\omega_d t) + B\sin(\omega_d t)) + \frac{P}{m\omega_n^2}$$

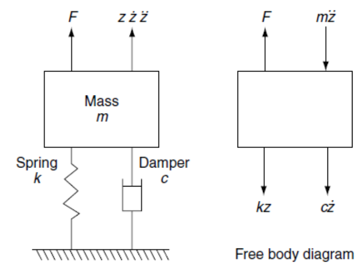


# Example 5 – Forced response

The constants A and B, can be obtained from the initial conditions $z(0) = 0, \dot{z}(0) = 0$.

$$A = -\frac{P}{m\omega_n^2} = -\frac{P}{k}$$

$$B = -\frac{P\xi\omega_n}{m\omega_n^2\omega_d} = -\frac{P\xi}{k\sqrt{1-\xi^2}}$$




Then the general solution will be

$$z = \frac{P}{k}\left[1 - e^{-\xi\omega_n t}\left(\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right)\right]$$

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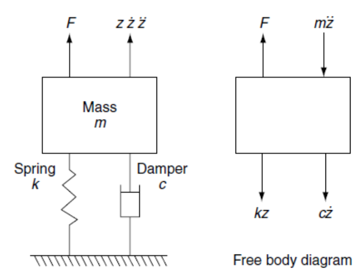
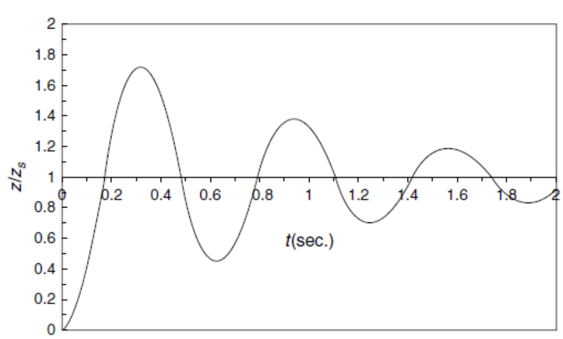
# Example 5 – Forced response

Part (b): divided by the static displacement $z_s = \frac{P}{k}$

$$\frac{z}{z_s} = \left[1 - e^{-\xi\omega_n t}\left(\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right)\right]$$




Which is plotted herein for $\xi = 0.1, \omega_n = 10$




It is seen that the displacement approaches twice the static value, before settling at the static value.

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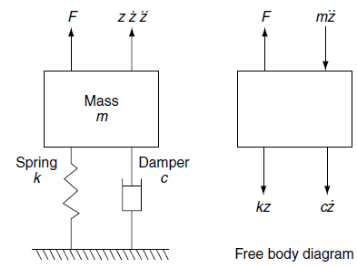
# Forced response – Exercise

Solve example 5 again with applied force $F = P\ sin(\omega t)$

In this case the particular integral solution will take the form

$$z = C\ \cos \omega t + D sin(\omega t)$$




And plot the magnitude of $z$ , $\frac{|z|}{z_s} = \sqrt{C^2 + D^2}$ as a function of the nondimensional frequency $\Omega = \frac{\omega}{\omega_n}$

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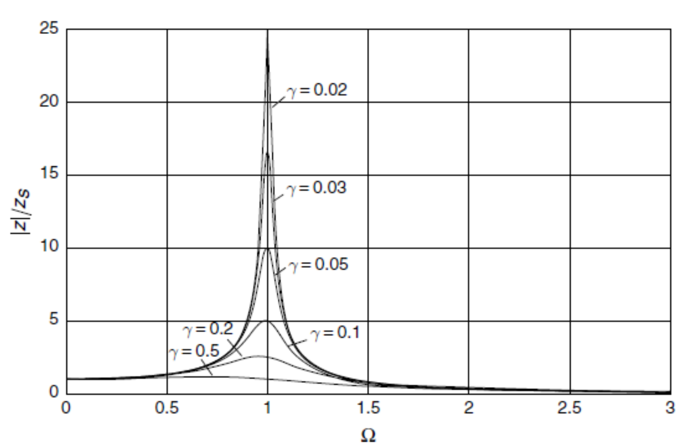
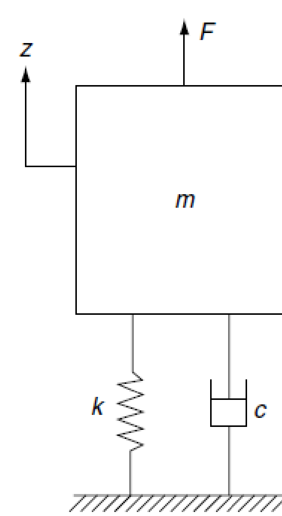
# Forced response – Exercise

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# Frequency Response Function (FRF)

The response (displacement, velocity or acceleration) divided by the input force.

Names of Frequency Response Functions

| Standard FRFs | | Inverse FRFs | |
|---|---|---|---|
| $\underline{z}/\underline{F}$ | Receptance, Admittance, Dynamic Compliance, or Dynamic Flexibility. | $\underline{F}/\underline{z}$ | Dynamic stiffness |
| $\underline{\dot{z}}/\underline{F}$ | Mobility | $\underline{F}/\underline{\dot{z}}$ | Impedance |
| $\underline{\ddot{z}}/\underline{F}$ | Inertance or Accelerance | $\underline{F}/\underline{\ddot{z}}$ | Apparent mass |

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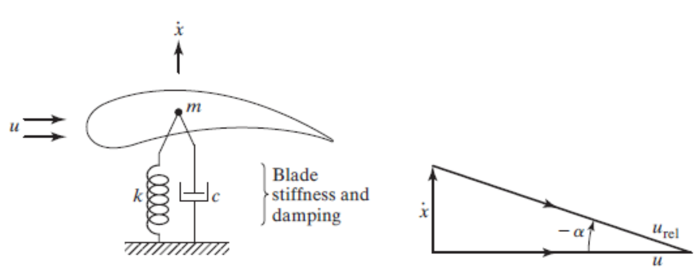
# Aerospace Application

Dynamic instability of an airfoil – Flutter

Find the value of free-stream velocity *u* at which the airfoil section (single-degree-of-freedom system) shown in Fig. becomes unstable.

*Approach:* Find the vertical force acting on the airfoil (or mass *m*) and obtain the condition that leads to zero damping.

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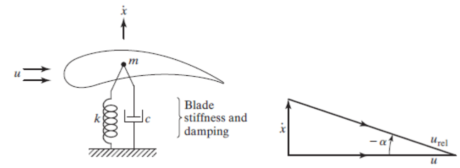
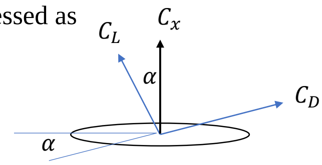
# Aerospace Application
## Dynamic instability of an airfoil




The vertical force acting on the airfoil (or mass $m$) due to fluid flow can be expressed as




$$F = \frac{1}{2}\rho u^2 \bar{c}\, C_x$$

The vertical force acting on the airfoil (or mass $m$) due to fluid flow can be expressed as

$$F = \frac{1}{2}\rho u^2 D C_x$$

where $\rho$ = density of the fluid, $u$ = free-stream velocity, $D$ = width of the cross section normal to the fluid flow direction, and $C_x$ = vertical force coefficient, which can be expressed as

$$C_x = C_L \cos\alpha + C_D \sin\alpha$$

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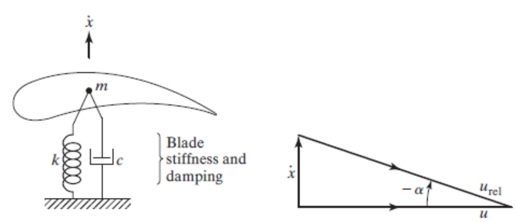
# Aerospace Application
## Dynamic instability of an airfoil

$$\alpha = -\tan^{-1}\left(\frac{\dot{x}}{u}\right)$$

For small angles of attack,

$$\alpha = -\frac{\dot{x}}{u}$$




$$C_x = C_L + C_D\alpha = C_L - C_D\frac{\dot{x}}{u}$$

Then the equation of motion takes the form

$$m\ddot{x} + c\dot{x} + kx = \frac{1}{2}\rho u^2 \bar{c}\left(C_L - C_D\frac{\dot{x}}{u}\right)$$

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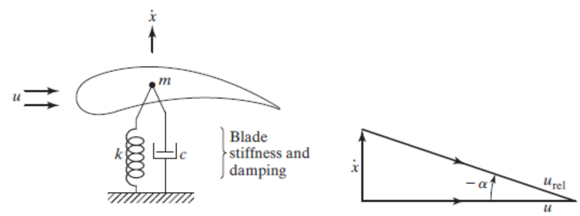
# Aerospace Application
## Dynamic instability of an airfoil

Which can be

$$m\ddot{x} + c\left(c + \frac{1}{2}\rho u^2 \bar{c}\frac{C_D}{u}\right)\dot{x} + kx = \frac{1}{2}\rho u^2 \bar{c}\, C_L$$

The wing (mass) will become unstable at zero damping,




$$c + \frac{1}{2}\rho u^2 \bar{c}\frac{C_D}{u} = 0 \;\Rightarrow u = \frac{-2c}{\rho \bar{c} C_D}$$

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