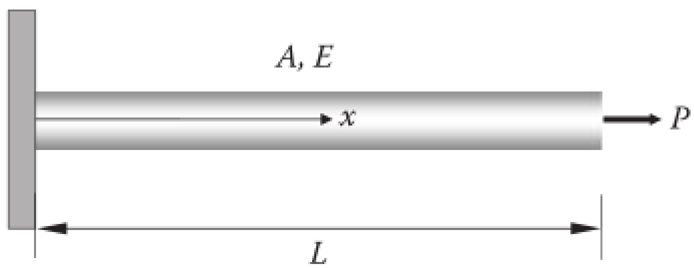
Strain–displacement relation:

$$\varepsilon(x) = \frac{du(x)}{dx}$$

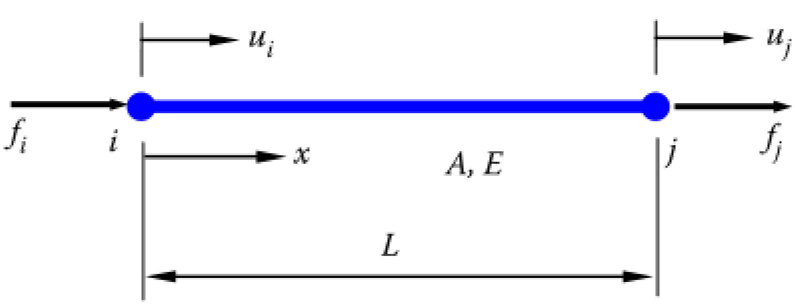


Stress–strain relation:

$$\sigma(x) = E\varepsilon(x)$$

Equilibrium equation:

$$\frac{d\sigma(x)}{dx} + f(x) = 0$$



Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University

2.4.2 Stiffness Matrix: Energy Approach

We derive the same stiffness matrix for the bar using a formal approach which can be applied to many other more complicated situations.

First, we define two *linear shape functions* as follows (Figure 2.5):

$$N_i(\xi) = 1 - \xi, \quad N_j(\xi) = \xi \tag{2.10}$$

where

$$\xi = \frac{x}{L}, \quad 0 \le \xi \le 1 \tag{2.11}$$

From Equation 2.4, we can write the displacement as

$$u(x) = u(\xi) = N_i(\xi)u_i + N_j(\xi)u_j$$

or

$$u = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{Nu} \tag{2.12}$$

Strain is given by Equations 2.1 and 2.12 as

$$\varepsilon = \frac{du}{dx} = \left[\frac{d}{dx}\mathbf{N}\right]\mathbf{u} = \mathbf{Bu} \tag{2.13}$$

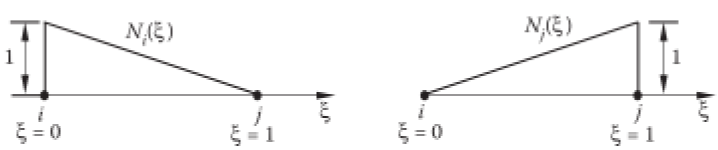


FIGURE 2.5
The shape functions for a bar element.

where **B** is the element *strain–displacement matrix*, which is

$$\mathbf{B} = \frac{d}{dx}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} = \frac{d}{d\xi}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \cdot \frac{d\xi}{dx}$$

that is,

$$\mathbf{B} = \begin{bmatrix} -1/L & 1/L \end{bmatrix} \tag{2.14}$$

Stress can be written as

$$\sigma = E\varepsilon = E\mathbf{Bu} \tag{2.15}$$

Consider the *strain energy* stored in the bar

$$\begin{aligned} \mathrm{U} &= \frac{1}{2}\int_V \sigma^{\mathrm{T}}\varepsilon \, dV = \frac{1}{2}\int_V (\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}E\mathbf{Bu})\,dV \\ &= \frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V (\mathbf{B}^{\mathrm{T}}E\mathbf{B})\,dV\right]\mathbf{u} \end{aligned} \tag{2.16}$$

where Equations 2.13 and 2.15 have been used.

The *potential* of the external forces is written as (this is by definition, and remember the negative sign)

$$\Omega = -f_i u_i - f_j u_j = -\mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{2.17}$$

The total potential of the system is

$$\Pi = \mathrm{U} + \Omega$$

which yields by using Equations 2.16 and 2.17

$$\Pi = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V (\mathbf{B}^{\mathrm{T}}E\mathbf{B})\,dV\right]\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{2.18}$$

Setting $d\Pi = 0$ by the principle of minimum potential energy, we obtain (verify this)

$$\left[\int_V \left(\mathbf{B}^{\mathrm{T}}E\mathbf{B}\right)dV\right]\mathbf{u} = \mathbf{f}$$

or

$$\mathbf{ku} = \mathbf{f} \tag{2.19}$$

Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University

where

$$\mathbf{k} = \int_V (\mathbf{B}^T E \mathbf{B})\, dV \qquad (2.20)$$

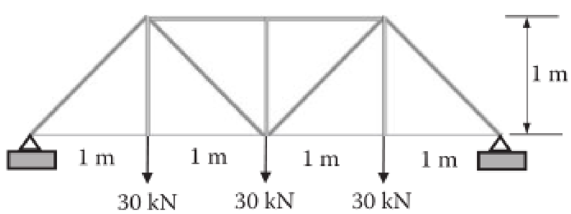
is the *element stiffness matrix*.

Equation 2.20 is a general result which can be used for the construction of other types of elements.

Now, we evaluate Equation 2.20 for the bar element by using Equation 2.14

$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \begin{bmatrix} -1/L & 1/L \end{bmatrix} A dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Problem Description: Truss bridges can span long distances and support heavy weights without intermediate supports. They are economical to construct and are available in a wide variety of styles. Consider the following planar truss, constructed of wooden timbers, which can be used in parallel to form bridges. Determine the deflections at each joint of the truss under the given loading conditions.



Material: Douglas fir

E = 13.1 GPa

ν = 0.29

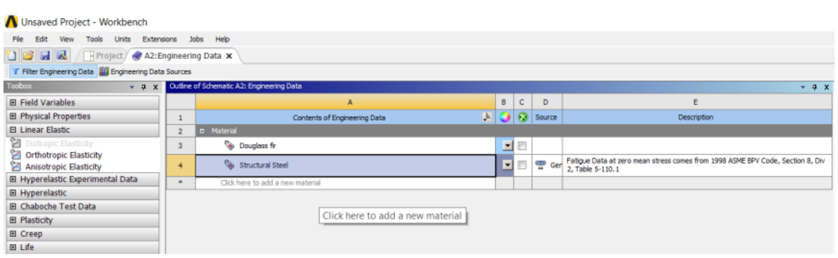
Member cross section:

Height = 6 cm

Width = 6 cm

Step 1: Define the material properties

1. Double click on Engineering Data
2. Click to define material and name it Douglas fir



3. Click on linear elastic tab and select isotropic elasticity; then define the Young's Modulus and Poisson's ratio.

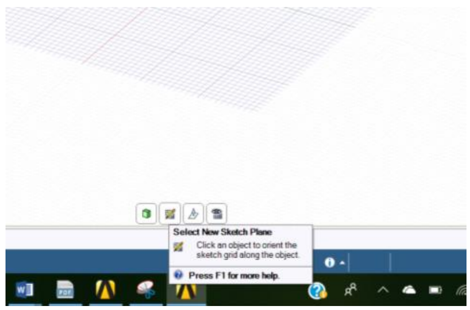
Then return to the workbench

Properties of Outline Row 4: Structural Steel

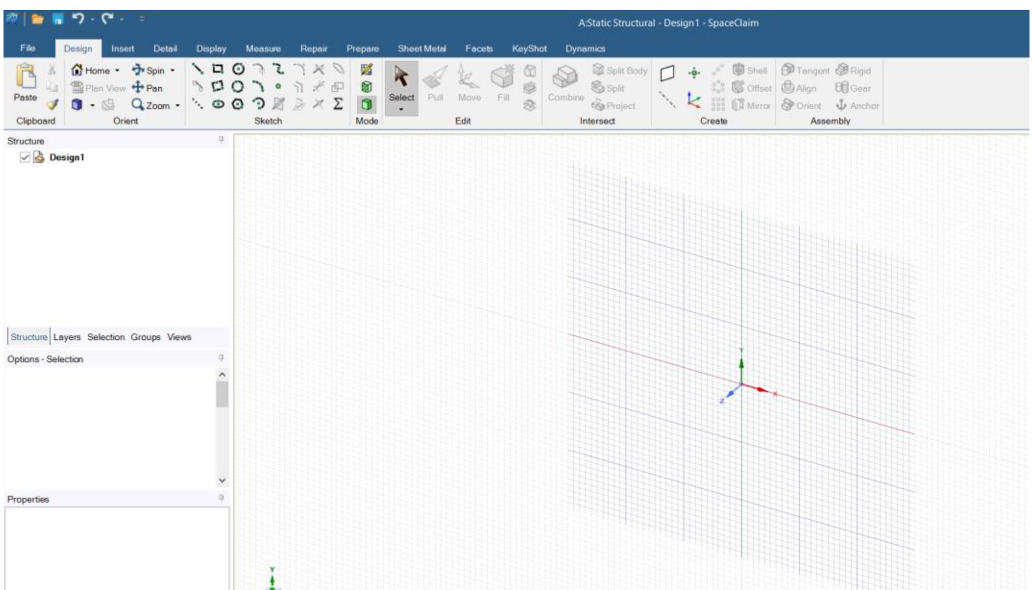
	A	B	C	D	E
1	Property	Value	Unit		
2	Material Field Variables	Table			
3	Density	7850	kg m^-3		
4	Isotropic Secant Coefficient of Thermal Expansion				
6	Isotropic Elasticity				
7	Derive from	Young's Modulus and Poisson...			
8	Young's Modulus	2E+11	Pa		
9	Poisson's Ratio	0.3			

Step 2: Create geometry using SpaceClaim

1. Click on select new sketch plane

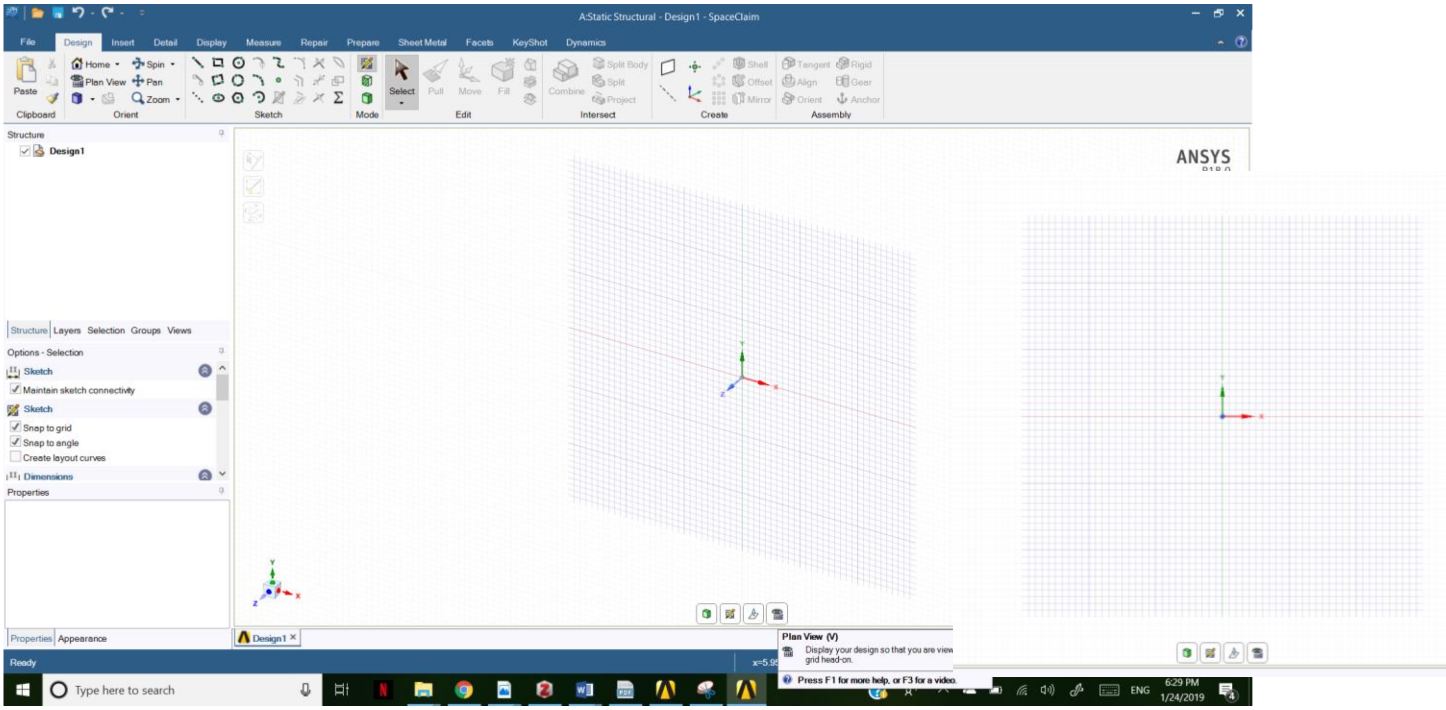


2. Select the x-y plane

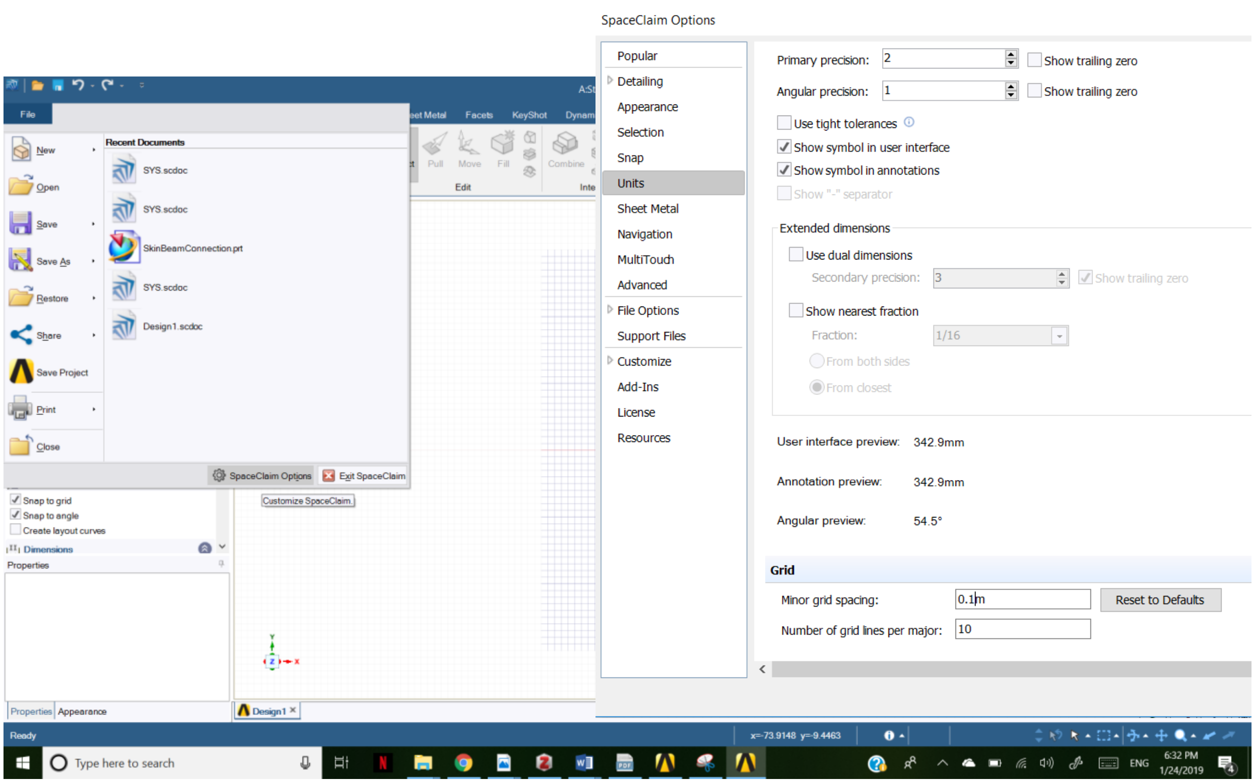


3. Click on plane view

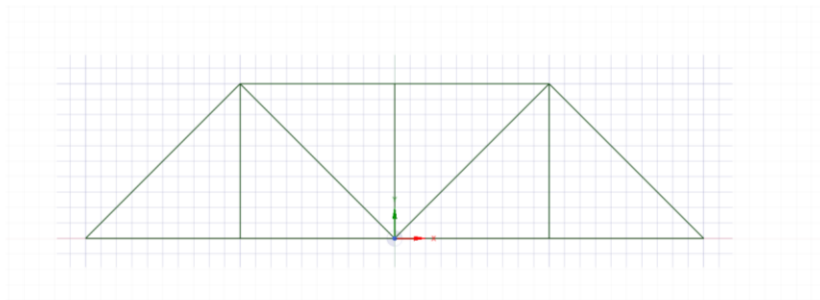
Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University



4. Click on file – spaceclaim options. In the options window select units and change the grid spacing as in the following figure

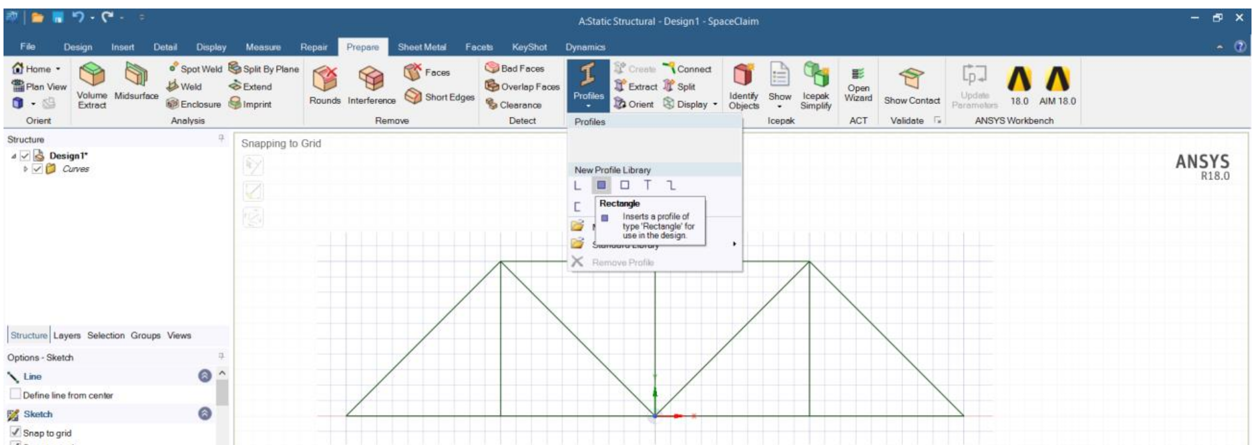


5. From the design tab – select line and sketch the following truss

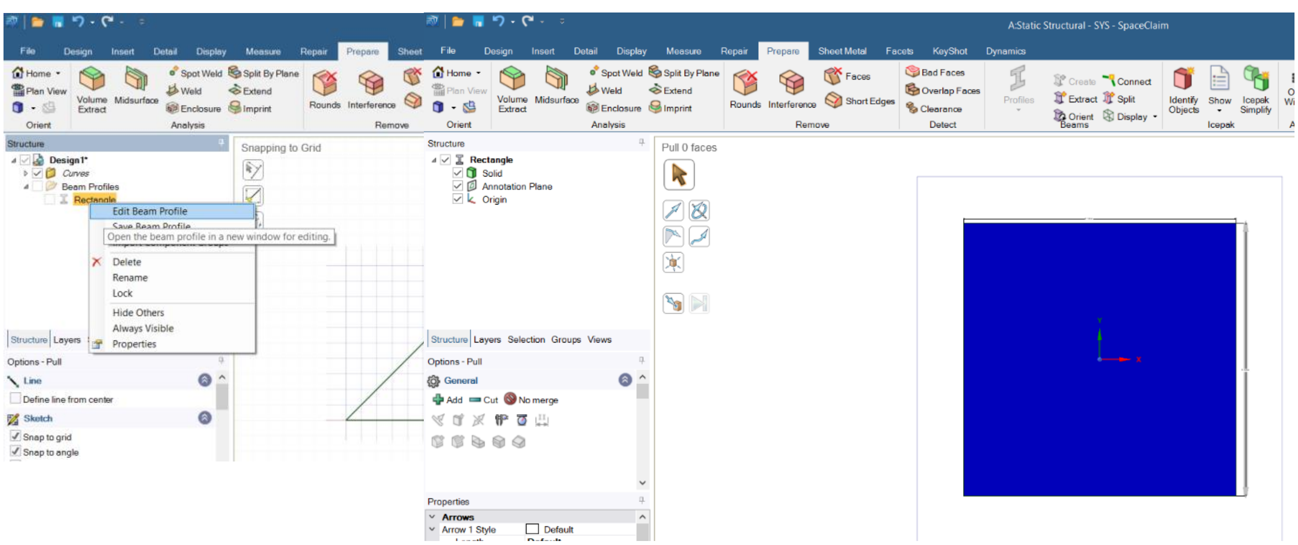


6. Define cross-section

In prepare tab – select your profile type

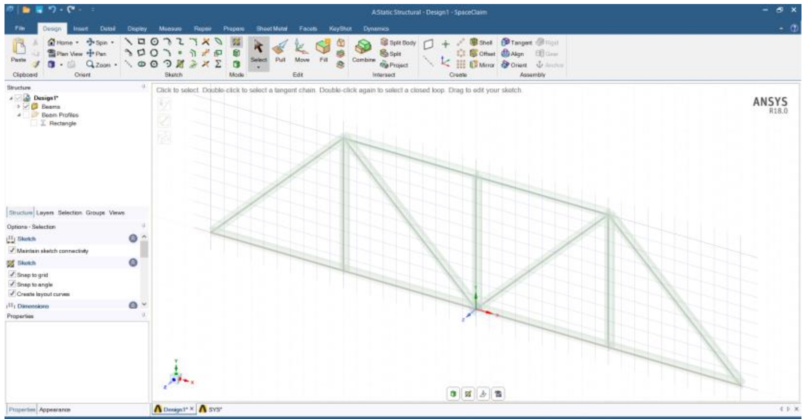


7. Right click on the profile name and click edit beam profile to change the profile dimensions



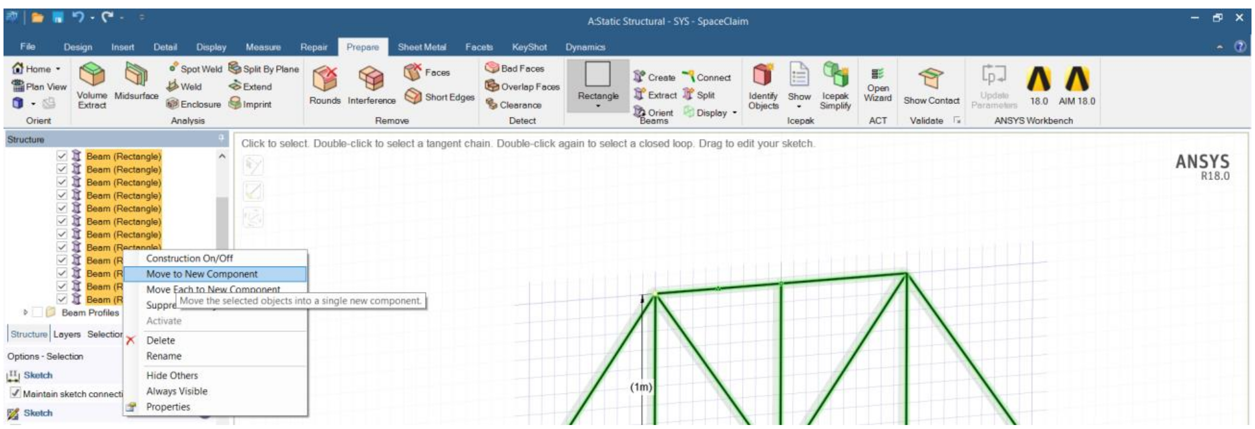
Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University

8. Return to the design tab, select all the truss members and click create to define the elements cross-section



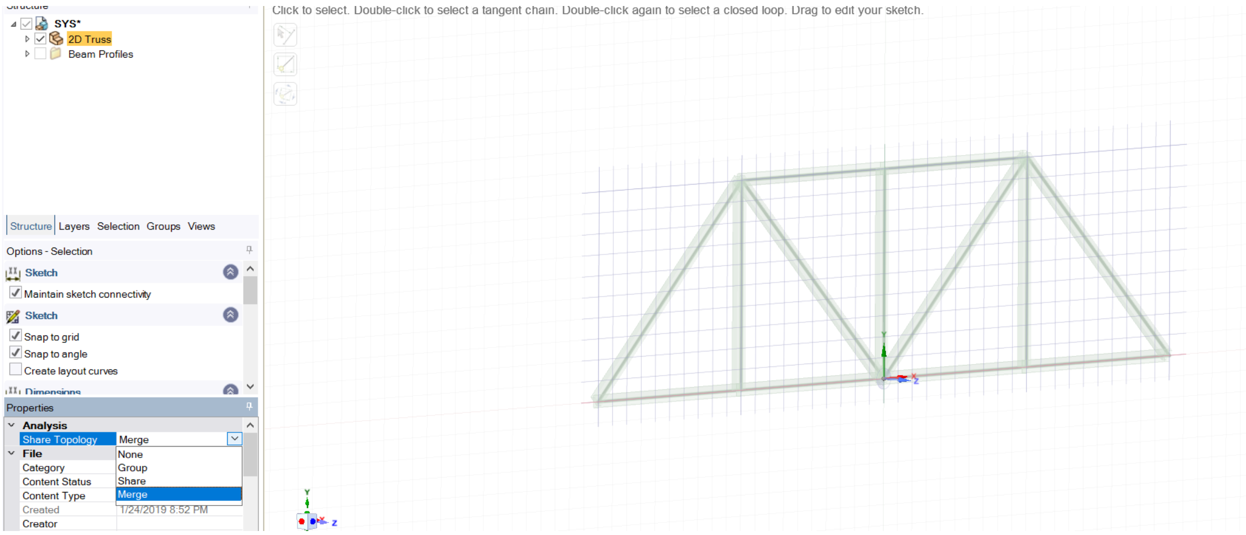
9. Merge the truss nodes all together

Select all member – right click – select create new component



10. Name the new component 2D truss – in the component options under share topology – select Merge

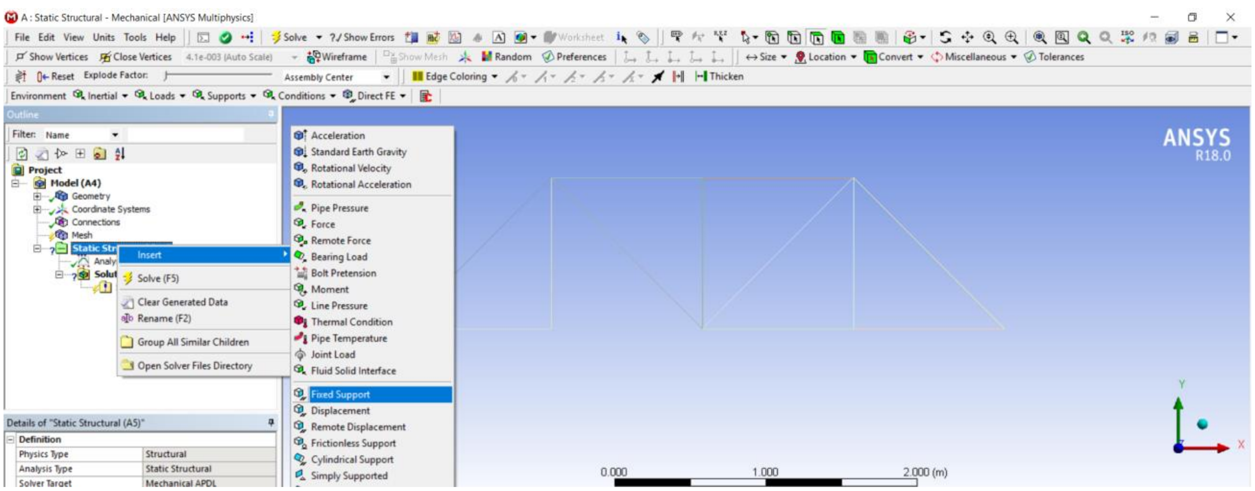
Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University



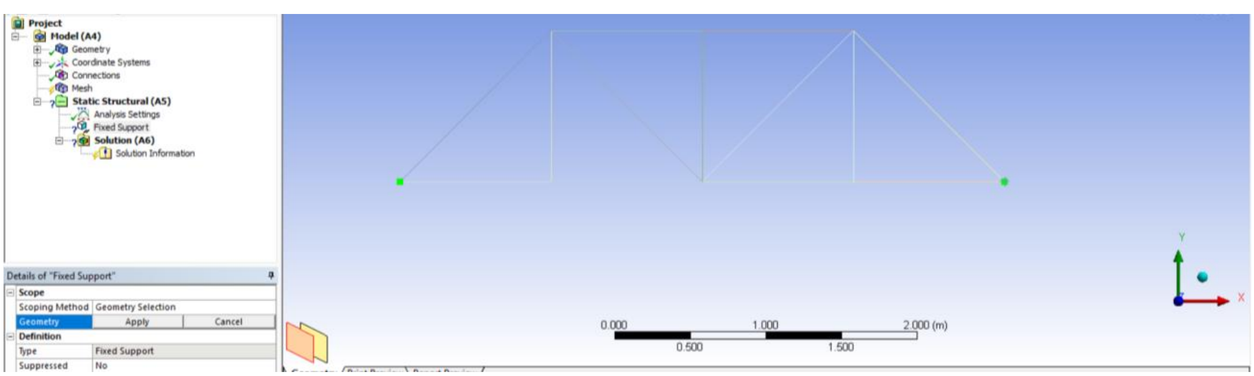
This concludes the geometry, then we start the FEA

ANSYS FEA

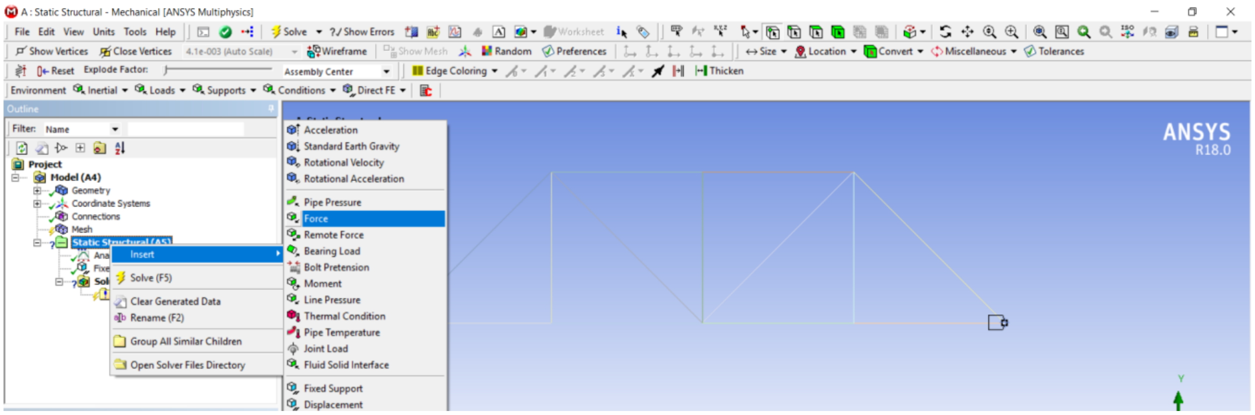
1. Right click on static structure – insert – fixed support



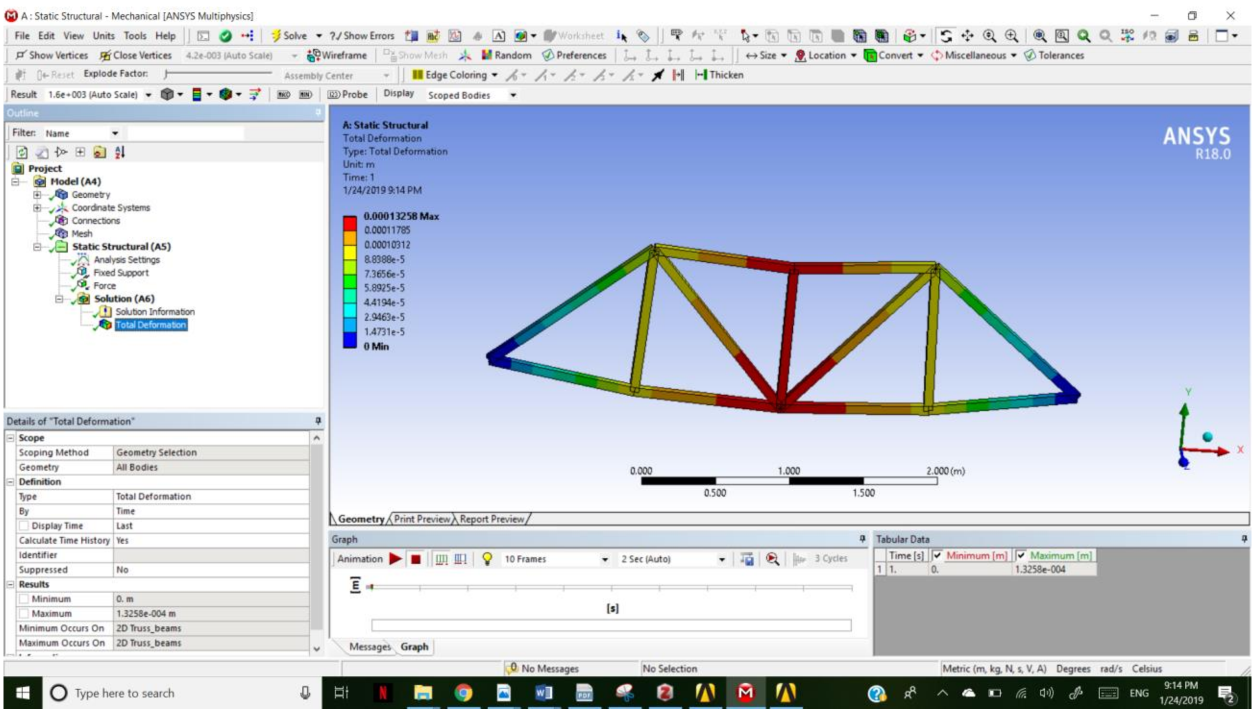
2. Select the two fixed points and click apply



3. Right click on static structure – insert – force



4. Select the loaded points and define the load value
5. Right click on solution – select the solutions you want to display
6. Click solve to analyze your model



Mohamed Abdou Mahran Kasem, PhD
Aerospace Engineering Department, Cairo University