AER303 – Analysis of Aircraft Structures
Part 1 – Dynamics of Structures

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1

References

For part 1 – Structural Dynamics

- Rao, Mechanical Vibrations
- Douglas, Structural dynamics and vibration in practice.

2

Course Contents

We have mainly three topics:

Part 1 – Structural dynamics
Part 2 – Finite Element Method
Part 3 – Composite Structures

3

Course Hours and Grades

- Lectures : 3 hours
- Tutorials: 2 hours
- Total grades: 125
- Term work: 40
- Final exam: 85
- Final exam hours: 3 hours

4

Course Hours and Grades

Term activities:

- Assignments : 5 marks
- Attendance: 5 marks
- Midterm: 20 marks
- Final Project: 10 marks

5

Course Curriculum

For part 1 – Structural Dynamics

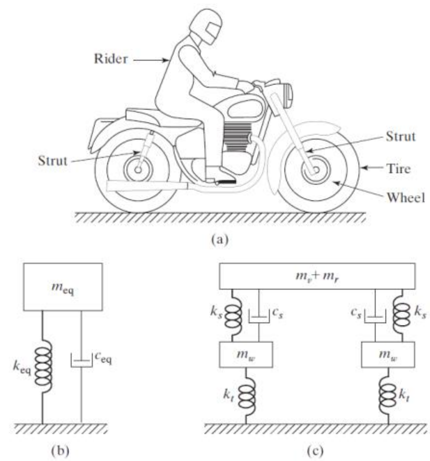
- Rao, Mechanical Vibrations
- Douglas, Structural dynamics and vibration in practice.

6

Structural dynamics (vibration)

A vibratory (dynamic) system has variables such as the excitations (inputs) and responses (outputs) are time dependent.

- It usually consists of masses, springs, and dampers.
- In some cases, the mass, spring and damper do not appear as separate components; they are inherent and integrated to the system.
- For example, in an airplane wing, the mass of the wing is distributed throughout the wing. Also,



7

History of vibration

- In *1802* the German scientist, ***Chladni*** (1756 -1824) developed the method of placing *sand* on a *vibrating plate* to find its mode shapes.
- ***Napoléon Bonaparte***, who attended the meeting, was very impressed and presented a sum of 3,000 francs, to be awarded to the first person to give a satisfactory *mathematical theory of the vibration of plates*.
- The competition executed 3 times till ***Sophie Germain***, (Lagrange was one of the judges) was finally awarded the prize in *1815*, although the boundary conditions were erroneous.
- The correct *boundary conditions* for the vibration of plates were given in *1850* by ***Kirchhoff*** (1824-1887).

8

History of vibration

- The *vibration of a rectangular flexible* membrane was solved for the first time by *Simeon Poisson (1781 1840).*
- The *vibration of a circular membrane* was studied by Clebsch (1833 1872) in 1862.
- Then, ***Rayleigh*** developed a method of finding the fundamental frequency of vibration of a conservative system using the principle of conservation of energy known as *Rayleigh's method*. An extension of the method, which can be used to find multiple natural frequencies, is known as the *Rayleigh-Ritz method*.
- In *1902* ***Frahm*** investigated the importance of *torsional vibration* study in the design of the *propeller shafts* of steamships. He also developed the *dynamic vibration absorber* in 1909.
- ***Aurel Stodola*** (1859-1943) developed a method for analyzing *vibrating beams* that is also applicable to *turbine blades*.
- ***Stephen Timoshenko*** (1878-1972) presented an improved theory of *vibration of beams*, which has become known as the *Timoshenko or thick beam theory*.
- A similar theory was presented by ***Mindlin*** for the *vibration analysis of thick plates* by including the effects of rotary inertia and shear deformation.

9

History of vibration

- Theory of *nonlinear vibrations* began to develop by ***Poincaré and Lyapunov*** at the end of the *nineteenth century*.
- The advent of high-speed digital computers in the *1950s* made it possible to treat moderately complex systems and to generate approximate solutions in semidefinite form, relying on classical solution methods but using numerical evaluation of certain terms that cannot be expressed in closed form.

10

Importance of the study of vibrations

- Most human activities involve vibration in one form or other.
- For example, we *hear* because *our eardrums vibrate* and *see* because *light waves* undergo vibration.
- *Breathing* is associated with the vibration of *lungs* and *walking* involves (periodic) oscillatory motion of legs and hands.
- *Human speech* requires the oscillatory motion of *tongues*.
- Whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation, there occurs a phenomenon known as *resonance*, which leads to excessive deflections and failure.

11

Importance of the study of vibrations What is Aeroelastic Flutter?

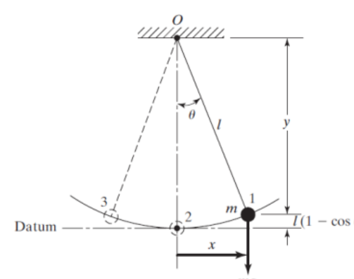
12

Aerospace applications in vibration

- Wing vibration
- Wind turbine blades vibration
- Compressor and turbines blades vibration
- Aircraft vibrations
- Spacecraft vibrations

13

Basic Concepts of Vibration

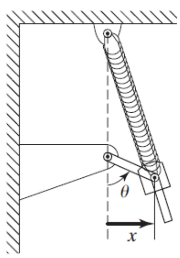


- *Vibration*: Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*.
- *Elementary Parts of Vibrating Systems*: includes a means for *storing potential energy* (*spring or elasticity*), a means for *storing kinetic energy* (*mass or inertia*), and a means by which *energy is gradually lost (damper).*
- The *vibration of a system* involves the transfer of its potential energy to kinetic energy and of kinetic energy to potential energy, alternately.
- If the system is *damped*, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

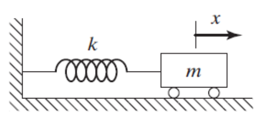
14

Basic Concepts of Vibration

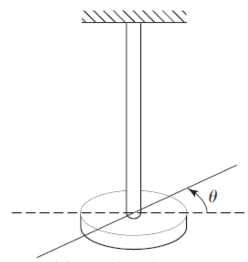
- ***Degrees of Freedom* :** The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any time.



(a) Slider-crank-
spring mechanism



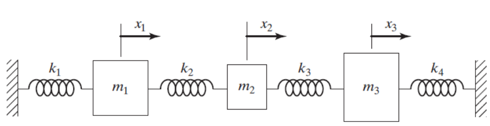
(b) Spring-mass system



(c) Torsional system

Basic Concepts of Vibration

Degrees of Freedom



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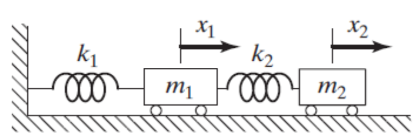

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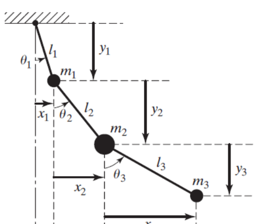

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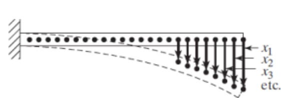
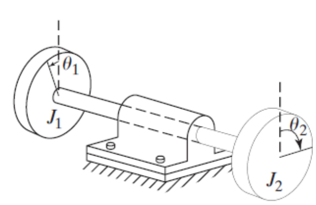


A cantilever beam
(an infinite-number-of-degrees-of-freedom system).



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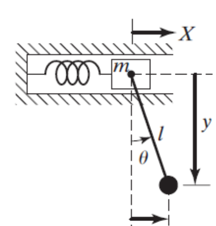
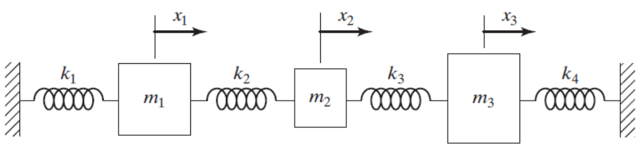
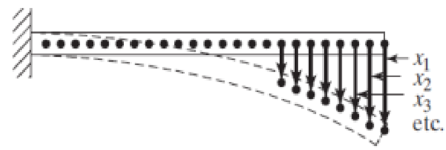


(c)

Classification of Vibration
1. Discrete and Continuous Systems

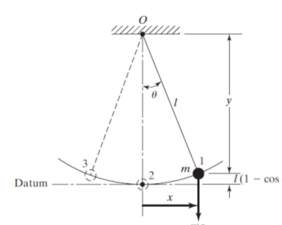
Systems with a *finite number of degrees of freedom* are called discrete or lumped systems, and those with an *infinite number of degrees of freedom* are called *continuous or distributed* systems.





A cantilever beam (an infinite-number-of-degrees-of-freedom system).

17



Classification of Vibration
2. Free and Forced Vibration

- *Free Vibration:* results when a system is left to vibrate on its own after an initial disturbance. No external force acts on the system. Example: the oscillation of a simple pendulum.

- *Forced Vibration:* results when a system is subjected to an *external force* (often, a repeating type of force). Example: the oscillation that arises in machines such as diesel engines.

18

Classification of Vibration
3. Undamped and Damped Vibration

- *Undamped vibration:* results if no energy is lost or dissipated in friction or other resistance during oscillation.
- *Damped vibration:* if any energy is lost in this way,

- The amount of damping is so small that it can be disregarded for most engineering purposes.
- However, consideration of damping becomes extremely important in analyzing vibratory systems near *resonance*.

19

Classification of Vibration
4. Linear and Nonlinear Vibration

- *linear vibration* if all the basic components of a vibratory system the spring, the mass, and the damper behave linearly.

- Otherwise, the vibration is called *nonlinear vibration*.

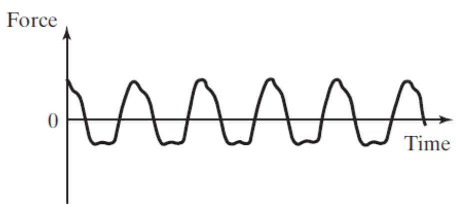
- Most vibratory systems tend to behave *nonlinearly* with increasing amplitude of oscillation, a knowledge of nonlinear vibration is desirable in dealing with practical vibratory systems.

20

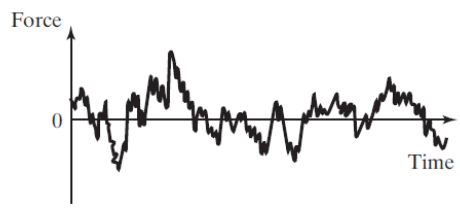
Classification of Vibration
4. Deterministic and Random Vibration

- *<u>Deterministic vibration:</u>* if the value or *<u>magnitude</u>* of the *<u>excitation</u>* (force or motion) is known at any given time.

- In some cases, the excitation is *<u>nondeterministic or random</u>;* the value of the excitation at a given time cannot be predicted.



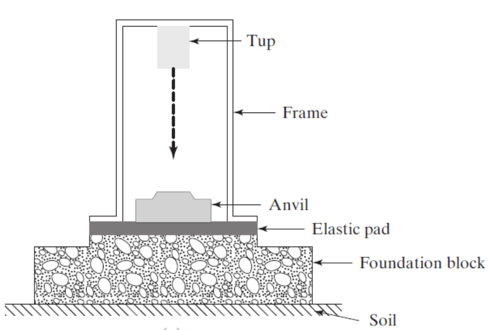
(a) A deterministic (periodic) excitation



(b) A random excitation

Vibration Analysis Procedure

1. Determination of the *<u>mathematical model</u>* represents the important features of the system to derive its governing equation.



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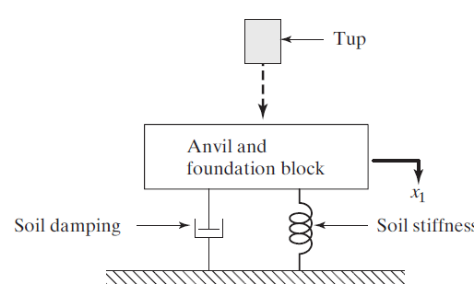

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Vibration Analysis Procedure

2. *Derivation of Governing Equations*: once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The *equations of motion* can be derived conveniently by drawing the *free-body diagrams* of all the masses involved.

3. *Solution of the Governing Equations:* The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transform methods, matrix methods, and numerical methods.

23

Vibration Analysis Procedure

4. *Interpretation of the Results:* the solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system.

These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

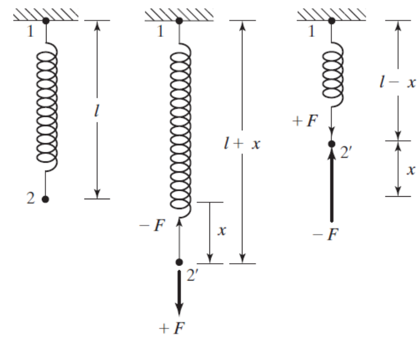
24

Elements of vibration body - Spring

A *spring* is a type of mechanical link, which in most applications is assumed to have *negligible mass and damping*.

- In fact, any elastic or deformable body or member, such as a cable, *bar, beam, shaft or plate*, can be considered as a *spring*.
- The *work done* (U) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

$$U = \frac{1}{2}kx^2$$



$$F = kx$$

Equivalent spring stiffnesses – Rod

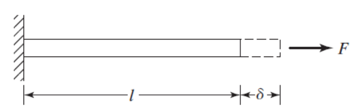
Assume we have a rod of length l and subjected to a force F.

The elongation (or shortening) δ of the rod under the axial tensile (or compressive) force F can be expressed as

$$\delta = \frac{\delta}{l} l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{AE}$$

where $\varepsilon = \frac{change\ in\ length}{original\ length} = \frac{\delta}{l}$ is the strain and $\sigma = \frac{force}{area} = \frac{F}{A}$ is the stress induced in the rod.
Using the definition of the spring constant k, we obtain from Eq. (E.1):

$$k = \frac{force\ applied}{resulting\ deflection} = \frac{F}{\delta} = \frac{AE}{l}$$



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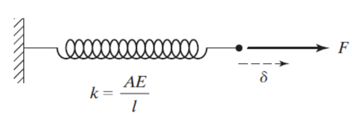
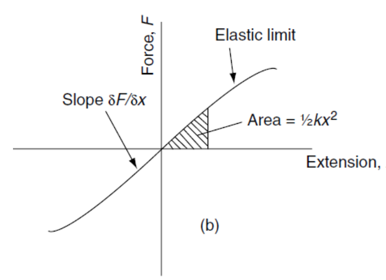
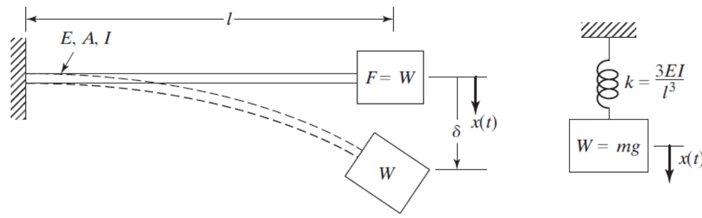




(b)

Equivalent spring stiffnesses – Beam

Assume we have a beam with length l, and subjected to a tip weight W.



(a) Cantilever with end force (b) Equivalent spring

We assume, for simplicity, that the self weight (or mass) of the beam is negligible and the concentrated load F is due to the weight of a point mass ($W = mg$). From strength of materials [1.26], we know that the end deflection of the beam due to a concentrated load $F = W$ is given by

$$\delta = \frac{Wl^3}{3EI}$$

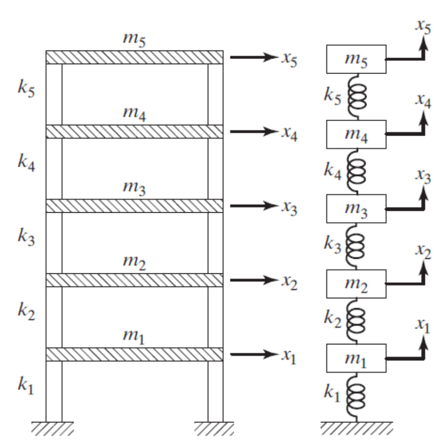
where E is the Young's modulus and I is the moment of inertia of the cross section of the beam about the bending or z-axis (i.e., axis perpendicular to the page). Hence the spring constant of the beam is (Fig. 1.25(b)):

$$k = \frac{W}{\delta} = \frac{3EI}{l^3}$$

27

Elements of vibration body – Mass or Inertia

- The *mass or inertia* element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes.
- From *Newton's second law* of motion, the product of the mass and its acceleration is equal to the force applied to the mass.
- *Work* is equal to the force multiplied by the displacement in the direction of the force, and the work done on a mass is stored in a form of kinetic energy.



28

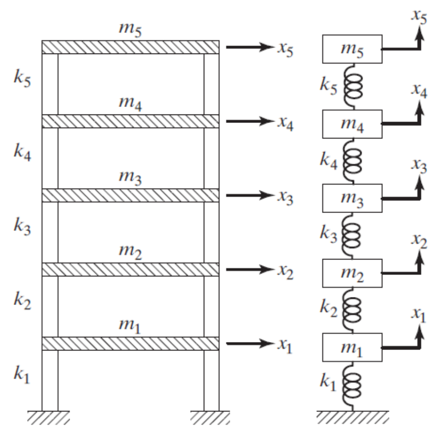
Elements of vibration body – Mass or Inertia

- The relationship between mass, m, and acceleration, $\ddot{x}$, is given by Newton's second law.
- This states that when a force acts on a mass, the rate of change of momentum (the product of mass and velocity) is equal to the applied force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}x}{\mathrm{d}t}\right) = F$$

For constant mass, this is usually expressed in the more familiar form:

$$F = m\ddot{x}$$



29

Elements of vibration body – Damping

- *Mass and stiffness* are *conservative* in that they can only store, or conserve, energy.
- In many practical systems, the vibrational energy is gradually converted to heat or sound.
- Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases.

- The mechanism by which the vibrational energy is gradually converted into heat or sound is known as *damping*.

- A *damper* is assumed to have *neither mass nor elasticity*, and damping force exists only if there is relative velocity between the two ends of the damper.
- *Damping dissipates energy*, which is lost from the system.
- It is difficult to determine the causes of damping in practical systems.

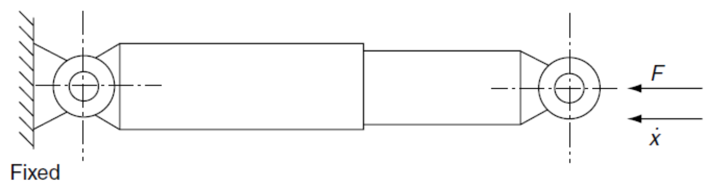
30

Elements of vibration body – Damping

- The figure below shows a *discrete damper* of the type often fitted to vehicle suspensions.
- Such a device typically produces a damping force, *F*, in response to closure velocity, $\dot{x}$, by forcing fluid through an orifice.
- This is inherently a square-law rather than a linear effect but can be made approximately linear by a *special valve*, which opens progressively with increasing flow.
- The damper is then known as an *automotive damper*,

- The force and velocity are related by:

$$F = c\dot{x}$$



31

Elements of vibration body – Damping types

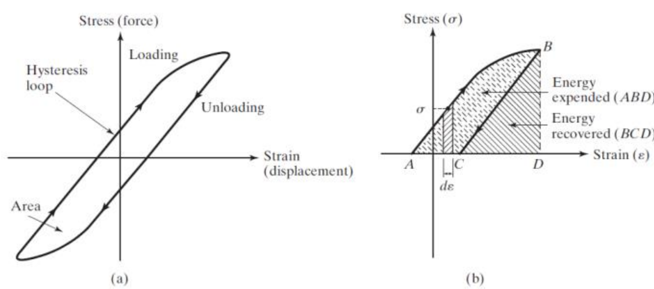
- **Viscous Damping:** Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated.
  film in a bearing.
  Typical examples of viscous damping include
  (1) fluid film between sliding surfaces,
  (2) fluid flow around a piston in a cylinder,
  (3) fluid

- **Coulomb or Dry-Friction Damping:** the damping force is constant.
  It is caused by *friction* between rubbing surfaces that either are dry or have insufficient lubrication.

32

Elements of vibration body – Damping types

Material or Solid or Hysteretic Damping: When a material is deformed, energy is absorbed and dissipated by the material.
The effect is due to friction between the internal planes, which slip or slide as the deformations take place.
When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated in the figure.
The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping



33

Elements of vibration body – Dampers

- The damping may be deliberately added to a mechanism or structure to suppress unwanted oscillations.
- A viscous damper can be obtained, when a plate moves relative to another parallel plate with a viscous fluid in between the plates.
- If there is no relative flow between the structure and the fluid, only radiation damping is possible, and the energy loss is due to the generation of sound.
- Damping can be generated by magnetic fields. The damping effect of a conductor moving in a magnetic field is often used in measuring instruments

34

Elements of vibration body – Construction of Viscous Dampers – Example

- Assume two plates with a fluid in between as shown in figure.
- Let one plate be fixed and let the other plate be moved with a velocity *v* in its own plane.
- The fluid layers in contact with the moving plate move with a velocity *v*, while those in contact with the fixed plate do not move.
- The velocities of intermediate fluid layers are assumed to vary linearly between 0 and *v*, as shown in Fig. According to Newton's law of viscous flow, the shear stress developed in the fluid layer at a distance *y* from the fixed plate is given by

$$\tau = \mu \frac{du}{dy} \tag{E.1}$$

where $du/dy = v/h$ is the velocity gradient. The shear or resisting force (F) developed at the bottom surface of the moving plate is

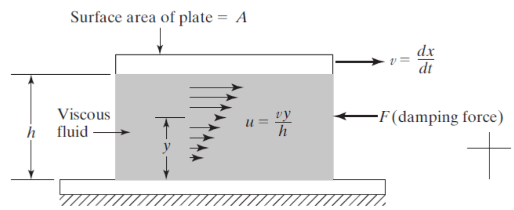
$$F = \tau A = \frac{\mu A v}{h} \tag{E.2}$$

where A is the surface area of the moving plate. By expressing F as

$$F = cv \tag{E.3}$$

the damping constant c can be found as

$$c = \frac{\mu A}{h} \tag{E.4}$$



35

Vibration representation in time domain

Let the motion of a given point be described by the equation: $x = X \sin \omega t$

where x is the displacement from the equilibrium position, X the displacement magnitude of the oscillation, ω the frequency in rad/s and t the time. The quantity X is the *single-peak* amplitude, and x travels between the limits $\pm X$, so the *peak-to-peak* amplitude (also known as *double* amplitude) is $2X$.

Since $\sin \omega t$ repeats every 2π radians, the period of the oscillation, T, say, is $2\pi/\omega$ seconds, and the frequency in hertz (Hz) is $1/T = \omega/2\pi$. The velocity, dx/dt, or $\dot{x}$, of the point concerned,

$$\dot{x} = \omega X \cos \omega t$$

The corresponding acceleration, d^2x/dt^2, or $\ddot{x}$,

$$\ddot{x} = -\omega^2 X \sin \omega t$$

36

Vibration representation in time domain

$$\dot{x} = \omega X \cos \omega t$$

can be written as

$$\dot{x} = \omega X \sin\left(\omega t + \frac{\pi}{2}\right)$$

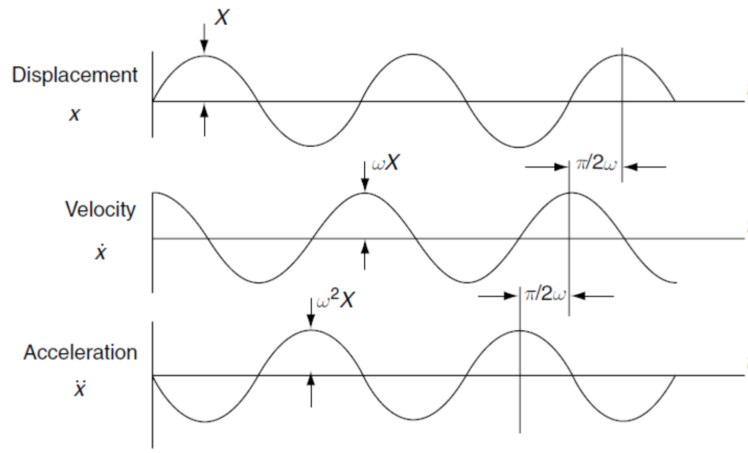
or

$$\dot{x} = \omega X \sin\left[\omega\left(t + \frac{\pi}{2\omega}\right)\right]$$

$$\ddot{x} = -\omega^2 X \sin \omega t$$

can be written as

$$\ddot{x} = -\omega^2 X \sin(\omega t + \pi)$$



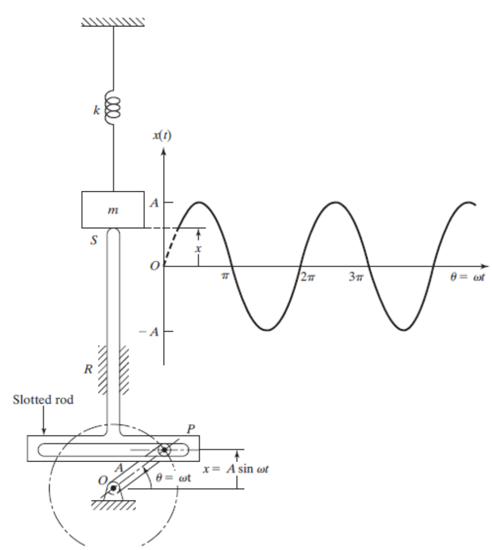
37

Vibration representation in time domain

The velocity is said to 'lead' the displacement by $t = \frac{\pi}{2\omega} = \frac{T}{4}$.
Or a phase angle of $\frac{\pi}{2}$ radians, or 90.

the acceleration 'leads' the displacement by a time $t = \frac{\pi}{\omega} = \frac{T}{2}$, or a phase angle of π radians or 180.

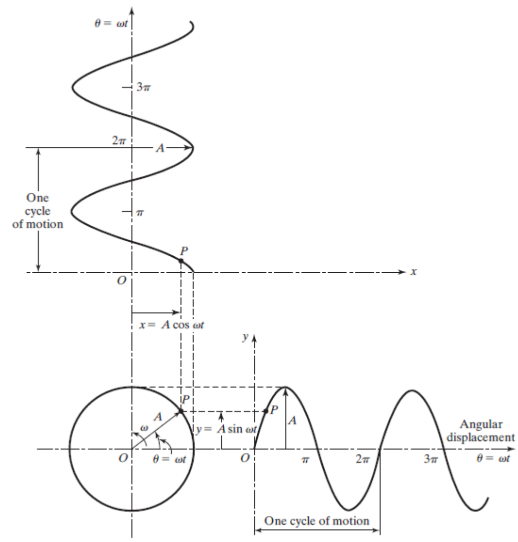
This shifts the velocity and acceleration plots to the left by these amounts relative to the displacement: the lead being in time, not distance along the time axis.



38

Vibration representation in time domain

$y = A \sin \omega t$ $x = A \cos \omega t$



- In general the motion shown in figure is called a harmonic motion.
- The harmonic function has the form $a_n \cos(n\omega t)\, or b_n \sin(n\omega t)$ which we called harmonics of order n.
- It has a period $\frac{T}{n}$.

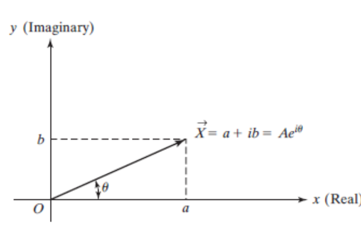
39

Representation in Complex Domain

Expressing simple harmonic motion in complex exponential form considerably simplifies many operations, particularly the solution of differential equations. It is based on Euler's equation, which is usually written as:

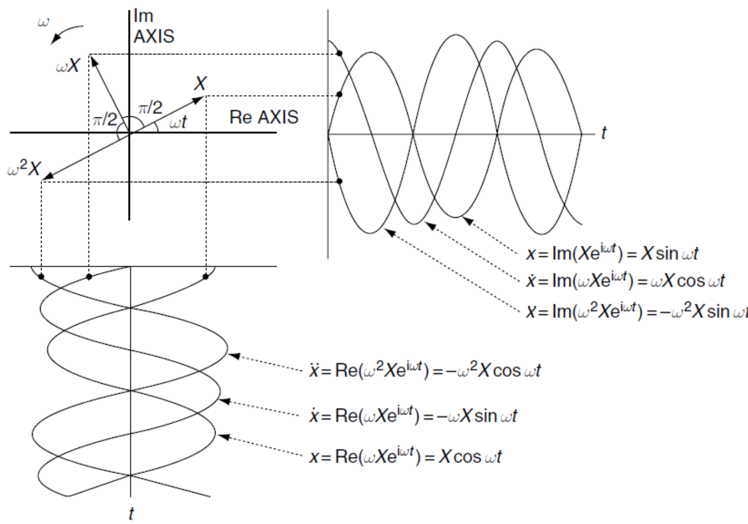
$$e^{i\theta} = \cos\theta + i\sin\theta$$

where e is the well-known constant, θ an angle in radians and i is $\sqrt{-1}$.



40

Representation in Complex Domain



41

Representation in Complex Domain

Noting that $i^2 = -1, i^3 = -i, i^4 = 1, \ldots, \cos\theta$ and $i \sin\theta$ can be expanded in a series as

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots \qquad (1.39)$$

$$i \sin\theta = i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots$$

Equations (1.39) and (1.40) yield

$$(\cos\theta + i\sin\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta}$$

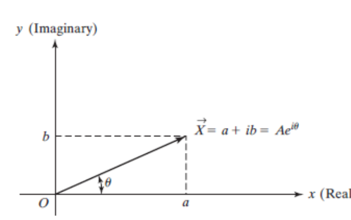
and

$$(\cos\theta - i\sin\theta) = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \cdots = e^{-i\theta}$$

$$\vec{X} = A\cos\theta + iA\sin\theta$$

$$A = (a^2 + b^2)^{1/2}$$

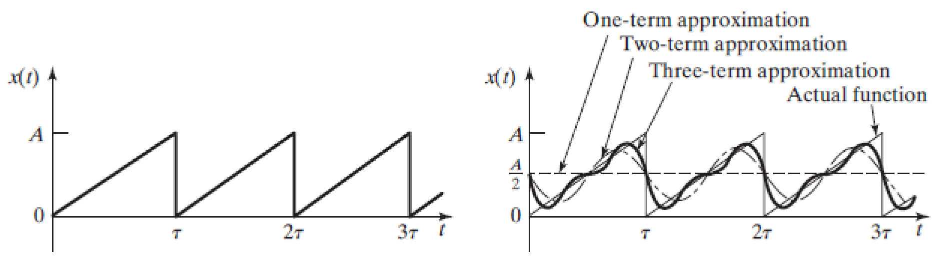
$$\theta = \tan^{-1}\frac{b}{a}$$



42

Fourier Series representation

- Although harmonic motion is simplest to handle, the motion of many vibratory systems is not harmonic.
- However, in many cases the vibrations are periodic for example, the type shown in Fig.
- Fortunately, any periodic function of time can be represented by Fourier series as an infinite sum of sine and cosine terms



43

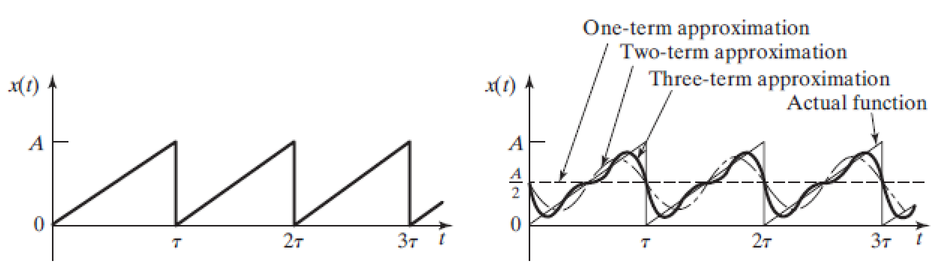
Fourier Series representation

If $x(t)$ is a periodic function with period τ, its Fourier series representation is given by

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\,\omega t + \cdots$$

$$+ b_1 \sin \omega t + b_2 \sin 2\,\omega t + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$



44

Fourier Series representation

where $\omega = 2\pi/\tau$ is the fundamental frequency and $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constant coefficients. To determine the coefficients a_n and b_n, we multiply Eq. (1.70) by $\cos n\omega t$ and $\sin n\omega t$, respectively, and integrate over one period $\tau = 2\pi/\omega$—for example, from 0 to $2\pi/\omega$. Then we notice that all terms except one on the right-hand side of the equation will be zero, and we obtain

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt \tag{1.71}$$

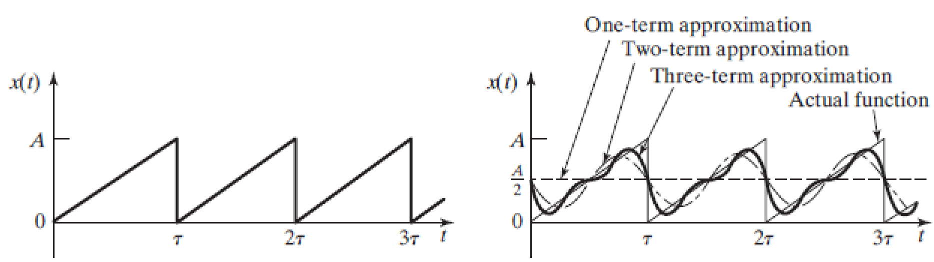
$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt \tag{1.72}$$

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt \tag{1.73}$$

45

Fourier Series representation

- The physical interpretation of the previous equations is that any periodic function can be represented as a sum of harmonic functions.
- Although the series are an infinite sum, we can approximate most periodic functions with the help of only a few harmonic functions.



46

Fourier Series representation – Example

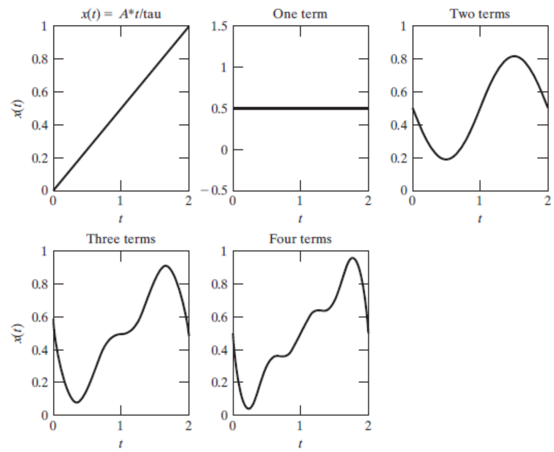
Plot the periodic function

$$x(t) = A\frac{t}{\tau}, \qquad 0 \le t \le \tau$$

and its Fourier series representation with four terms

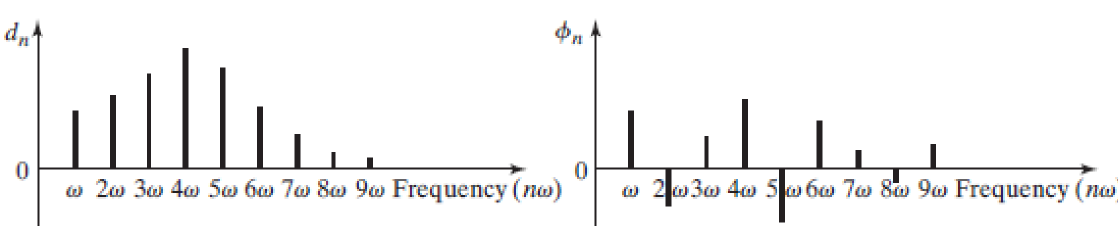
$$\bar{x}(t) = \frac{A}{\pi}\left\{\frac{\pi}{2} - \left(\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t\right)\right\}$$

for $0 \le t \le \tau$ with $A = 1$, $\omega = \pi$, and $\tau = \frac{2\pi}{\omega} = 2$.



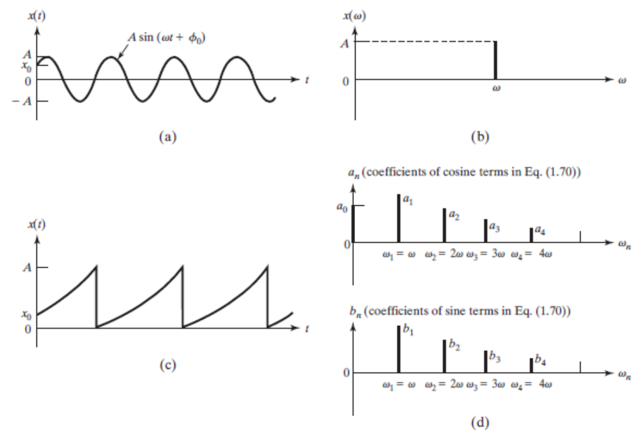
Frequency Spectrum – frequency domain

When plotting the harmonic functions in forms of amplitude in terms of the frequency, the resulting curve is called frequency spectrum or special diagram, as shown in figure.



Time & Frequency-Domain Representations

- The Fourier series expansion permits the description of any periodic function using either
- A time-domain or a frequency-domain representation.
- For example, a harmonic function given in time domain can be represented by the amplitude and frequency in the frequency domain.
- Similarly, a periodic function, such as a triangular wave, can be represented in time or frequency domains, as shown.



49

Methods of Solution

50

Methods of Solution

1. Newton's second law.
2. Energy methods

   - Rayleigh's method
   - The principle of virtual work (or virtual displacements)
   - Lagrange's equations

3. Numerical methods - FEM

51

Rayleigh's energy method

- It is applicable only to single-DOF systems if the kinetic and potential energies in the system can be calculated.
- The motion at every point in the system (i.e. the mode shape in the case of continuous systems) must be known or assumed.
- Since, in vibrating systems, the maximum kinetic energy in the mass elements is transferred into the same amount of potential energy in the spring elements, these can be equated, giving the natural frequency.
- It should be noted that the maximum kinetic energy does not occur at the same time as the maximum potential energy.

$$T_{\max} = \int_0^L \tfrac{1}{2} m \dot{y}^2$$

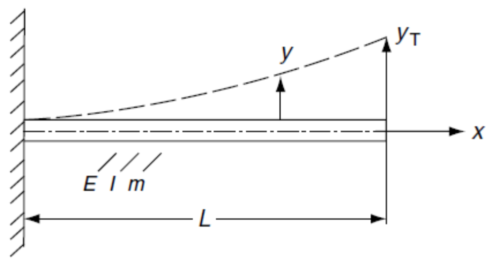
the basis of the Rayleigh method is that $T_{\max} = U_{\max}$.

$$U = \tfrac{1}{2} \int_0^L EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right)^2 \cdot \mathrm{d}x$$

52

Rayleigh's energy method – Example

Use Rayleigh's energy method to find the natural frequency of the fundamental bending mode of the uniform cantilever beam shown in Figure.



Rayleigh's energy method – Example

Assuming that the vibration mode shape is given by:

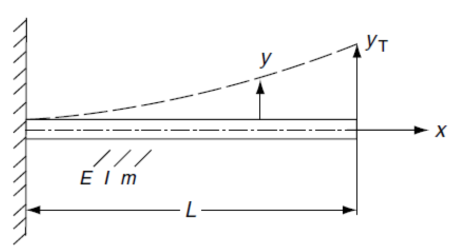
$$\frac{y}{y_T} = \left(\frac{x}{L}\right)^2$$

where y_T is the single-peak amplitude at the tip; y the vertical displacement of beam at distance x from the root; L the length of beam; m the mass per unit length; E the Young's modulus and I the second moment of area of beam cross-section.

$$y = y_T \frac{x^2}{L^2} \sin \omega t$$

Then,

$$\dot{y} = \omega_1 y_T \frac{x^2}{L^2} \cos \omega t$$



Rayleigh's energy method – Example

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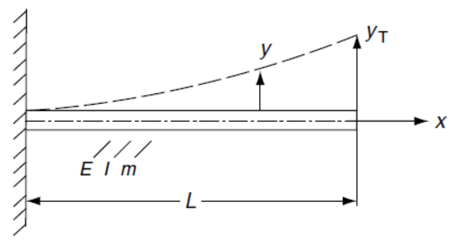


55

Rayleigh's energy method – Example

The maximum kinetic energy, T_{max}, occurs when $\cos \omega t = 1$, i.e. when $\dot{y} = \omega_1 y_T x^2 / L^2$, and is given by:

$$T_{max} = \int_0^L \tfrac{1}{2} m \dot{y}^2 = \tfrac{1}{2} \frac{m}{L^4} \omega_1^2 y_T^2 \int_0^L x^4 \cdot dx = \tfrac{1}{10}\, m\, L\, \omega_1^2 y_T^2$$

To find the maximum potential energy, we need an expression for the maximum curvature as a function of x.

From Eq. (A),

$$y = y_T \frac{x^2}{L^2} \sin \omega t$$

so

$$\frac{dy}{dx} = y_T \frac{2x}{L^2} \sin \omega t$$

and

$$\frac{d^2 y}{dx^2} = \frac{2 y_T}{L^2} \sin \omega t$$

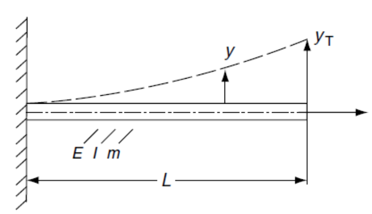
therefore

$$\left(\frac{d^2 y}{dx^2}\right)_{max} = \frac{2 y_T}{L^2}$$

$$U = \tfrac{1}{2} \int_0^L EI \left(\frac{d^2 y}{dx^2}\right)^2 \cdot dx$$

The maximum potential energy U_{max} is

$$U_{max} = \frac{2EI \cdot y_T^2}{L^3}$$



56

Rayleigh's energy method – Example

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2 y}{dx^2}\right)^2 \cdot dx$$

The exact solution is $\frac{3.52}{L^2}\sqrt{\frac{EI}{m}}$,

The maximum potential energy $U_{\max}$ is

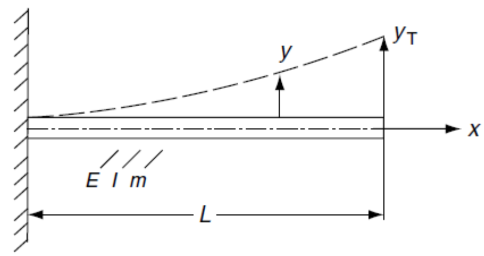
$$U_{\max} = \frac{2EI \cdot y_T^2}{L^3}$$

$$T_{\max} = U_{\max}.$$

$$\frac{1}{10} m L \omega_1^2 y_T^2 = \frac{2EI \cdot y_T^2}{L^3},$$

which simplifies to

$$\omega_1 = \frac{\sqrt{20}}{L^2}\sqrt{\frac{EI}{m}} = \frac{4.47}{L^2}\sqrt{\frac{EI}{m}}$$



57

Lagrange's Equations

- Lagrange's equations were published in 1788 and remain the most useful and widely used energy-based method to this day, especially when expressed in matrix form.
- Its basic form, for a system without damping, is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$i = 1, 2, 3, \ldots, n.$$

where

T is the total kinetic energy in the system;

U is the total potential energy in the system;

q_i are generalized displacements, as discussed in Section 1.2. These must meet certain requirements, essentially that their number must be equal to the number of degrees of freedom in the system; that they must be capable of describing the possible motion of the system; and they must not be linearly dependent.

Q_i are generalized external forces, corresponding to the generalized displacements q_i. They can be defined as those forces which, when multiplied by the generalized displacements, correctly represent the work done by the actual external forces on the actual displacements.

58