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Analytical and numerical validation for solving the fractional Klein-Gordon equation using the fractional complex transform and variational iteration methods

DOI 10.1515/nleng-2016-0018
Received April 17, 2016; accepted May 21, 2016.

Abstract: In this paper, we implement the fractional complex transform method to convert the nonlinear fractional Klein-Gordon equation (FKGE) to an ordinary differential equation. We use the variational iteration method (VIM) to solve the resulting ODE. The fractional derivatives are presented in terms of the Caputo sense. Some numerical examples are presented to validate the proposed techniques. Finally, a comparison with the numerical solution using Runge-Kutta of order four is given.

Keywords: Nonlinear fractional Klein-Gordon equation; Caputo derivative; Fractional complex transform method; Variational iteration method.

1 Introduction

Fractional differential equations (FDEs) have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering and others [1]. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. This kind of equations is more complex than ordinary differential equations. In the field of numerical treatment of this kind, a great attention has been recently dedicated to the development efficient and accurate numerical methods [2]. Among of these methods are, variational iteration method [3], homotopy perturbation method [4], Adomian decomposition method [5], homotopy analysis method [6] and collocation methods [7].

The Klein-Gordon equation plays a significant role in mathematical physics and many scientific applications, such as solid-state physics, nonlinear optics, and quantum field theory. The equation has attracted much attention in studying solitons ([8], [9]) and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [10]. The study of numerical solutions of the Klein-Gordon equation has been investigated considerably in the last few years. For example, Wazwaz has obtained the various exact traveling wave solutions such as, compactions, solitons and periodic solutions by using the tanh method [11]. In the previous studies, the most papers have carried out different spatial discretization of the equation ([12], [13]). Where the numerical solution of this equation is given using, radial basis functions in [14], collocation and finite difference-collocation methods in [15], and the spline approach in [16]. Also, recently, an integral equation formalism in [17], the implicit RBF meshless approach [18] and the compact difference scheme [19] are given to solve numerically the FKGE. The main aim of this paper is devoted to implement the fractional complex transform method to convert the nonlinear FKGE to an ODE and then use the VIM to solve the resulting problem.

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory which will be used further in this work.

Definition 1.

The Caputo fractional derivative operator $D^\alpha$ of order $\alpha$ is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x f^{(m)}(t)(x-t)^{m-\alpha-1}dt, \quad \alpha > 0,$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$.

Similar to integer-order differentiation, Caputo fractional
derivative operator is a linear operation

\[ D^\alpha (\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x), \]

where \( \lambda \) and \( \mu \) are constants. For the Caputo’s derivative we have \( D^\alpha C = 0 \), if \( C \) is a constant [20] and

\[
D^\alpha x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < [\alpha]; \\
\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq [\alpha].
\end{cases}
\]

(1)

We use the ceiling function \([\alpha]\) to denote the smallest integer greater than or equal to \( \alpha \) and \( \mathbb{N}_0 = \{0, 1, \ldots\} \). Recall that for \( \alpha \in \mathbb{N} \), the Caputo differential operator coincides with the usual differential operator of integer order. For more details on fractional derivatives definitions and theirs properties see [21].

\section{Chain rule for fractional calculus}

In previous papers ([22]-[24]), the authors used the following chain rule

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha},
\]

to convert a fractional differential equation with Jumarie’s modification of Riemann-Liouville derivative into its classical differential partner. In [25], the authors showed that this chain rule is invalid by giving a counter example as follows: Assume, \( s = t^\alpha \), \( 0 < \alpha < 1 \) and \( u = s^m \), i.e., \( u = t^{ma} \), then

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = D^\alpha_t u = D^\alpha_t t^{ma} = \frac{\Gamma(1+ma)\Gamma(1-m\alpha-a)}{\Gamma(1+ma-a)}.
\]

Now we calculate \( \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha} \).

Since \( \frac{\partial u}{\partial s} = ms^{m-1} = m^{ma-a} \) and

\[
\frac{\partial^\alpha s}{\partial t^\alpha} = D^\alpha_t s = D^\alpha_t t^\alpha = \frac{\Gamma(1+a)\Gamma(1-a)}{\Gamma(1)} = \Gamma(1) = \Gamma(1+a).
\]

Then,

\[
\frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha} = m^{ma-a} \Gamma(1+a) = m^{ma-a} \Gamma(1+a).
\]

This shows that, \( \frac{\partial^\alpha u}{\partial t^\alpha} \neq \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha} \). In [25] the authors show that

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sigma_t \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha},
\]

(2)

where \( \sigma_t \) denotes the sigma index. This formula is called the modified chain rule. From the above example we can see that \( \sigma_t = \frac{\Gamma(1+ma)}{m^{ma-a} \Gamma(1)} \). For more details, see [26].

\section{Reducing the nonlinear FKGE to ordinary differential equation}

In this section, to demonstrate the effectiveness of our approach, we will apply the complex transformation of Li and He to construct an approximate solution for the nonlinear fractional Klein-Gordon equation. Consider the following general form of FKGE

\[
D^\alpha_t u(x, t) + a D^\beta_x u(x, t) + bu(x, t) + cu''(x, t) = 0, \\
x \in (0, 1), \quad t > 0,
\]

(3)

where \( D^\alpha_t \) denotes the fractional derivative of order \( \alpha \) with respect to \( t \), \( D^\beta_x \) denotes the fractional derivative of order \( \beta \) with respect to \( x \), \( u(x, t) \) is unknown function, and \( a, b, c \) and \( \gamma \) are known constants with \( \gamma \in \mathbb{R}, \gamma \neq \pm 1 \). We also assume the following initial conditions

\[
u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad x \in (0, 1), \quad (4)
\]

and the following boundary conditions \( u(0, t) = u(1, t) = 0 \).

Li and He proposed a fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus ([23], [24]).

Now, we will reduce the given nonlinear fractional Klein-Gordon equation (3) to an ordinary differential equation in the following steps.

1. Take the following fractional complex transform:

\[
u(x, t) = U(\xi), \quad \xi = \frac{K t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{L x^{2\beta}}{\Gamma(1 + 2\beta)},
\]

(5)

where \( K \) and \( L \) are constants.

2. By using the fractional modified chain rule (2), we can obtain the following forms:

\[
D^\alpha_t u = \sigma_t \frac{d^2 u}{d\xi^2} \frac{D^\alpha_t \xi}{\xi}, \quad D^\beta_x u = \sigma_\xi \frac{d^2 u}{d\xi^2} \frac{D^\beta_x \xi}{\xi}.
\]

(6)

3. With the help of formula (1), we find that:

\[
D^\alpha_t \xi^{2\alpha} = \Gamma(1 + 2\alpha), \quad D^\beta_x \xi^{2\beta} = \Gamma(1 + 2\beta).
\]

(7)

4. Use the equations (6) and (7) in Eq.(5) we can obtain:

\[
D^\alpha_t u = \sigma_t K U'', \quad D^\beta_x u = \sigma_\xi L U'',
\]

(8)

where the prime denotes the derivative with respect to \( \xi \).
5. Without loss of generality we can take \( \sigma_I = \sigma_x = \ell \) where \( \ell \) is a constant. So, Eq.(3) will convert to the following ordinary differential equation:

\[
(K\ell + aL\ell)U'' + bU + cU'' = 0.
\]

6. Rewrite the above ODE in the form:

\[
U''(\xi) + \chi_1 U'(\xi) + \chi_2 U'(\xi) = 0,
\]

where \( \chi_1 = \frac{b}{(K\ell + aL\ell)} \) and \( \chi_2 = \frac{c}{(K\ell + aL\ell)} \).

4 Procedure of solution with VIM

In this section, we implement VIM for solving nonlinear ODE (9) with suitable boundary conditions. According to VIM, we construct the following recurrence formula

\[
U_{n+1}(\xi) = U_n(\xi) + \int_0^\xi \lambda(\tau) [U''_n(\tau) + \chi_1 U_n(\tau) + \chi_2 U'_n(\tau)] d\tau,
\]

where \( \lambda \) is a general Lagrange multiplier. Making the above correction functional stationary

\[
\delta U_{n+1}(\xi) = \delta U_n(\xi) + \int_0^\xi \lambda(\tau) \delta U''_n(\tau) d\tau + \chi_2 U'_n(\tau) d\tau,
\]

\[
= \delta U_n(\xi) + \int_0^\xi [\lambda(\tau) \delta U''_n(\tau)] d\tau + \chi_2 \int_0^\xi \delta U'_n(\tau) d\tau,
\]

\[
= \delta U_n(\xi) + \int_0^\xi [\lambda(\tau) \delta U''_n(\tau)] d\tau + \chi_2 \int_0^\xi \delta U'_n(\tau) d\tau + \chi_2 \delta U_n' \bigg|_{\tau = \xi} - \chi_2 \delta U_n \bigg|_{\tau = 0} = 0,
\]

where \( \delta U_n \) is considered as restricted variation, i.e., \( \delta U_n = 0 \), yields the following stationary conditions (by comparison the two sides in the above equation)

\[
\lambda''(\tau) = 0, \quad 1 - \lambda'(\tau) \bigg|_{\tau = 0} = 0, \quad \lambda(\tau) \bigg|_{\tau = \xi} = 0. \tag{11}
\]

Eqs.(11) are called Lagrange-Euler equation and its natural boundary conditions, the Lagrange multiplier, therefore

\[
\lambda(\tau) = \tau - \xi. \tag{12}
\]

Now, by substituting from (12) in (10), the following variational iteration formula can be obtained

\[
U_{n+1}(\xi) = U_n(\xi) + \int_0^\xi (\tau - \xi) [U''_n(\tau) + \chi_1 U_n(\tau) + \chi_2 U'_n(\tau)] d\tau. \tag{13}
\]

Now, we start with initial approximation

\[
U_0(\xi) = U(0) + \frac{U'(0)}{1!} \xi + A + B\xi,
\]

for some constants \( A = U(0) \) and \( B = U'(0) \). By using the above iteration formula (13), we can directly obtain the first components of the solution of (9) as follows

\[
U_0(\xi) = A + B\xi,
\]

\[
U_1(\xi) = A + Bx + 2.1761A x^2 + 2.1761A^2 x^3 + 0.7254Bx^4 + 2.1761A^2 Bx^3 + 1.0881AB^2 x^5 + 0.21763B^2 x^5, \ldots,
\]

and so on. The unknown variables \( A \) and \( B \) are computed if we satisfy the boundary conditions.

5 Numerical simulation

In this section, we solve numerically the nonlinear fractional Klein-Gordon equation where we use the complex transformation method to reduce it as ODE, then we solve the resulting ODE using VIM. Some numerical examples are presented to validate the solution scheme [27]. To achieve this propose, we consider the following three cases.

Case study 1:

In this case, we take the values of the constants as follows

\[
a = -1, \quad b = 1, \quad c = 1, \quad y = 3, \quad K = 0.25, \quad L = 0.5, \quad \ell = 1,
\]

with different values of \( a \) and \( B \) (\( a = 0.5, 0.7, 1.0 \) and \( B = 0.5, 0.7, 1.0 \)). In this case, the values of \( A \) and \( B \) are \( A = 0.0, \quad B = 0.05 \).

The obtained numerical results by means of the proposed methods are shown Figure 1. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate solution using the proposed method, VIM with \( n = 5 \).

Case study 2:

In this case, we take the values of the constants as follows

\[
a = 0.9, \quad B = 0.9, \quad y = 3, \quad K = 0.25, \quad L = 0.5, \quad \ell = 1,
\]

with different values of \( a \), \( b \) and \( c \) (\( a = -1, 1, -1, \quad b = 1, -1, 1 \) and \( c = 1, -1, -1 \)). In this case, the values of \( A \) and \( B \) are \( A = 0.0, \quad B = 0.05 \).

The obtained numerical results by means of the proposed methods are shown Figure 2. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate
solution using the proposed method, VIM with \( n = 5 \).

**Case study 3:**

In this case, we take the values of the constants as follows

\[ a = -1, \ b = 1, \ c = 1, \ \alpha = 0.75, \ \beta = 0.75, \ y = 3, \ \ell = 1, \]

with different values of \( K \) and \( L \) (\( K = 0.25, 0.50, 0.25 \), and \( L = 0.50, 0.25, 0.75 \)). In this case, the values of \( A \) and \( B \) are \( A = 0.0, \ B = 0.05 \).

The obtained numerical results by means of the proposed methods are shown Figure 3. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate solution using the proposed method, VIM with \( n = 5 \).

6 Conclusion and remarks

In this article, the properties of the fractional complex transform method are used to reduce the nonlinear fractional Klein-Gordon equation to ordinary differential equation. The resulting ODE is solved by using variational iteration method. The obtained approximate solution using the suggested methods is in excellent agreement with the numerical solution using the forth order Runge-Kutta method and show that these approaches can be solved the problem effectively and illustrates the validity and the great potential of the proposed technique. All computation in this paper are done using Matlab 8.0.

References

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