ELC 2050: Fields & Wave Propagation

Department of Electronics and Electrical Communications Engineering

Introduced By:

Eng. Mohamed Ossama Ashour

E-mail: vert4231@gmail.com

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Agenda

- Introduction to Magnetostatics
- Current & Current Density Vector
- Conduction Current Density Vector
- Boundary conditions for current density
- Resistance of a Conductor

Introduction

- Previously, we saw problems with charge at rest. Now, we will consider charges in **steady motion**, which is known as a steady electric current.
- Steady motion means that charges move with a constant velocity (motion doesn't vary with time).
- Steady electric current is the source of the static magnetic field.
- Conducting materials will also be studied, in which charge carriers are free to move under electric field action. This movement leads to two types of currents: **conduction currents** and **convection currents**.

Current & Current Density Vector

Current

- Current $'I'$ is the charge that goes through surface S per time unit.

$$I \equiv \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} \equiv \frac{dq}{dt}$$

Current Density Vector

- The total current $'I'$ flowing through an arbitrary surface S is the flux of the $'J'$ vector through S.

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Conduction Current Density Vector

$$\mathbf{J} = \sigma \mathbf{E}$$

- This relation is known as Ohm’s law, and conductors that follow this law are called ohmic conductors or isotropic linear conductors
- The constant of proportionality $\sigma > 0$ that appears in Ohm’s law is called the conductivity. conductivity is measured in $\Omega^{-1}m^{-1}$. Ω^{-1} is known as siemens (S).

Boundary conditions for current density

- When current obliquely crosses an interface between two media with **different conductivities**, the current density vector changes both in direction and in magnitude.

- The normal component of $\underline{J}$ is continuous across an interface.

$$J_{2N} = J_{1N}$$

- the tangential component of $\underline{J}$ is continuous across an interface.

$$E_{1T} = E_{2T} \longrightarrow \frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Boundary conditions for current density

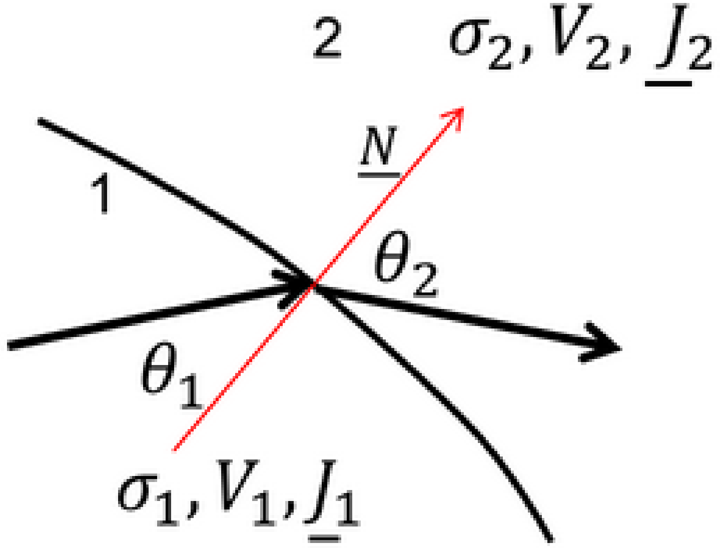
Special case (The refraction of J at a lossy dielectric or conductor interface)

- For Normal components:

$$J_{N1} = J_1 \cos\theta_1 = J_2 \cos\theta_2 = J_{N2}$$

- Also, The ratio of the tangential components can be expressed as

$$\frac{J_{\tan 1}}{J_{\tan 2}} = \frac{J_1 \sin\theta_1}{J_2 \sin\theta_2} = \frac{\sigma_1}{\sigma_2}$$



$$\frac{\sigma_1}{\tan\theta_1} = \frac{\sigma_2}{\tan\theta_2}$$

Resistance of a Conductor

Steps of solution

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a potential difference V_0 between conductor terminals.
3. Find electric field intensity E within the conductor. (If the material is homogeneous, having a constant conductivity, the general method is to solve Laplace's equation for V in the chosen coordinate system, and then obtain $E = -\nabla V$)
4. Find the total current

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma \mathbf{E} \cdot d\mathbf{s}$$

, where S is the cross-sectional area over which J flows.

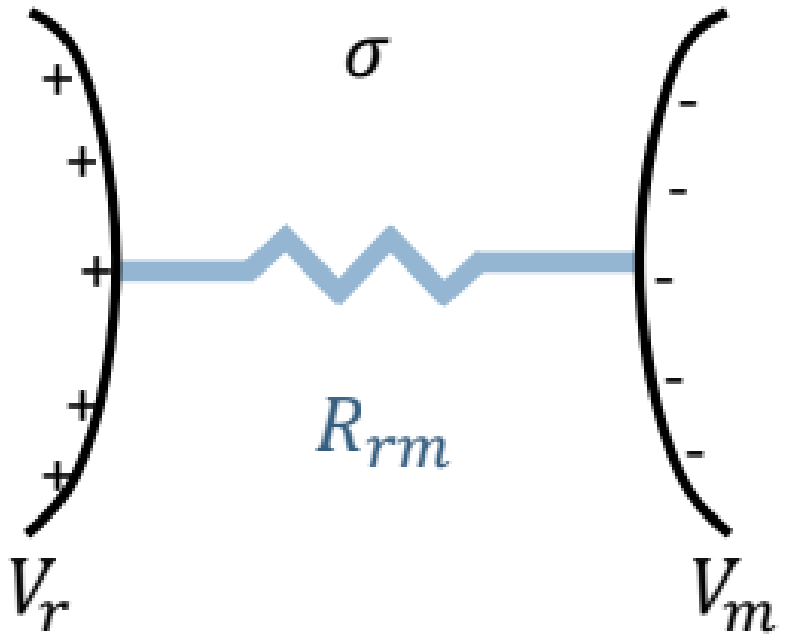
5. Find resistance R by taking the ratio $\frac{V_0}{I}$.

Resistance of a Conductor

Steps of solution

Or by using the following relation:

$$R_{rm} = 1 \Big/ \int_S \frac{1}{\int_r^m \frac{d\ell_{\parallel}}{\sigma \, dS_{\perp}}} \quad \Omega$$



Or by getting capacitance first then using the following relation:

$$RC = \frac{\epsilon}{\sigma}$$

Sheet 3

1. At steady flow the current density vector at the boundary surface of two conducting media makes an angle of 45° with respect to the surface in medium (1) and is of 10 Amp/m2. Calculate the magnitude and direction of the current density vector at the boundary in medium (2). Calculate also the surface charge densities (free and polarization) at the boundary. The constants of the media are:
   (a) Medium (1): $\sigma_1 = 0.01$ mho/m, $\varepsilon_{r1} = 1$
   (b) Medium (2): $\sigma_2 = 1$ mho/m, $\varepsilon_{r2} = 2$

2. A conducting 45° sector of thickness d has inner and outer radii r_1 and r_2. If the conductivity is σ mho/m, obtain expressions for the resistance R between the equipotential surfaces: (i) $r = r_1$ & $r = r_2$, (ii) $z = 0$ & $z = d$ and (iii) $\varphi = 0$ & $\varphi = \pi/4$.

Sheet 3 Answers

1)

(i) B.C : $J_{2N} = J_{1N} \to \frac{J_1}{\sqrt{2}} = J_2 * \cos\alpha$

$\frac{\sigma_1}{\tan\theta_1} = \frac{\sigma_2}{\tan\theta_2} \Rightarrow \tan\theta_2 \approx 100 \to \theta_2 \approx 89.42^\circ \to J_2 = 707.14$ Amp/m²

$D = \epsilon E$; $\epsilon_0 \epsilon_r$

(ii) $J_{N1} = 10/\sqrt{2} \to E_{N1} = J_N/\sigma_1 = 707.106 \to D_{N1} = \epsilon E_{N1} = 62.5 \times 10^{-\,}$

$J_{N2} = 10/\sqrt{2} \to E_{N2} = J_N/\sigma_2 = 5\sqrt{2} = 7.07 \to D_{N2} = \epsilon E_{N2} = 1.25 \times 10^{-10}$

$\epsilon_0 \epsilon_r$

$\rho_{SF} = D_{2N} - D_{1N} = 61.349 * 10^{-10}\, C$

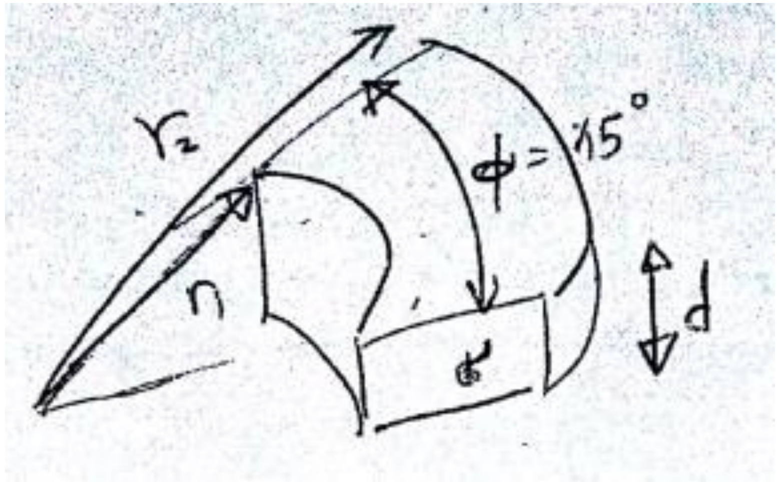
(iii) $\frac{\rho_s^{TA}}{\epsilon_0} = E_{2N} - E_{1N} \to \rho_s^{T} = \epsilon_0 (E_{2N} - E_{1N}) = 61.98 \times 10^{-10}\, C$

$\rho_s^P = \rho_s^T - \rho_s^F = 0.633\, C$

Sheet 3 Answers

2)

$$R^{-1} = \int_S \frac{1}{\int_{1}^{2} \frac{dl_{\parallel}}{\sigma\, dS_{\perp}}}$$



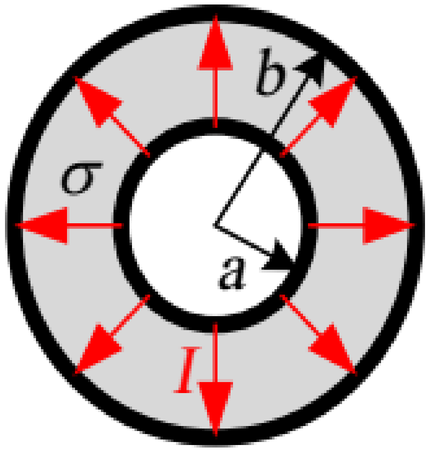
(i) $$G = \int_{\phi=0}^{\frac{\pi}{4}} \int_{z=0}^{d} \frac{1}{\int_{r=r_1}^{r=r_2} \frac{d\rho}{\sigma \rho\, d\phi\, dz}} = d * \frac{\pi}{4} * \sigma / \ln\left(\frac{r_2}{r_1}\right) \rightarrow R = \frac{4\ln\left(\frac{r_2}{r_1}\right)}{\sigma \pi d}$$

(ii) $$G = \int_{\phi=0}^{\frac{\pi}{4}} \int_{r=r_1}^{r=r_2} \frac{1}{\int_{z=0}^{z=d} \frac{dz}{\sigma \rho\, d\rho\, d\phi}} = \frac{\sigma * \frac{\pi}{4}}{d} \frac{(r_2^2 - r_1^2)}{2} \rightarrow R = \frac{8d}{\sigma \pi (r_2^2 - r_1^2)}$$

$$= \frac{\ell}{\sigma A} = \frac{d}{\sigma * \frac{\pi}{8}(r_2^2 - r_1^2)}$$

(iii) $$G = \int_{r=r_1}^{r_2} \int_{z=0}^{d} \frac{1}{\int_{\phi=0}^{\phi=\frac{\pi}{4}} \frac{\rho\, d\phi}{\sigma\, d\rho\, dz}} = \frac{\sigma \ln\left(\frac{r_2}{r_1}\right) d}{\pi/4} \rightarrow R = \frac{4\sigma \ln\left(\frac{r_2}{r_1}\right) d}{\pi}$$

3. A resistor is made of two coaxial cylindrical electrodes of radii a = 0.5 mm and b = 1 mm. Its cross-section is shown in the figure. The resistor has length l = 2 mm. The medium between the electrodes has specific conductivity σ = 50 S/m. The total current flowing through the resistor is I = 2 A.
   - (a) Find the current density **J** as a function of the distance ρ to the axis of the inner cylinder.
   - (b) Find the electric field vector **E** as a function of the distance ρ to the axis of the inner cylinder.
   - (c) Find the voltage between the two electrodes.
   - (d) Find the resistance of this cylindrical resistor.
   - (e) Find the power dissipated in the resistor.



Sheet 3 Answers

3)

(a)

$$I = \iint_S \vec{J} \cdot d\vec{s} = J_\rho \times 2\pi\rho \times \ell$$

$$\Rightarrow J_\rho(\rho) = \frac{I}{2\pi\ell} \cdot \frac{1}{\rho} = \frac{2}{2\pi \times 2 \times 10^{-3}} \times \frac{1}{\rho}$$

$$\Rightarrow J_\rho(\rho) = \frac{1000}{2\pi} \frac{1}{\rho} = \frac{500}{\pi} \frac{1}{\rho} \approx \frac{159.155}{\rho}, \quad A/m^2$$

$$\Rightarrow \vec{J}(\rho) = \hat{a}_\rho \frac{159.155}{\rho}, \quad A/m^2$$

(b)

$$\vec{J} = \sigma \vec{E}$$

$$\Rightarrow \vec{E}_{(\rho)} = \frac{\vec{J}}{\sigma} = \hat{a}_\rho \frac{500}{\pi \sigma} \cdot \frac{1}{\rho} = \hat{a}_\rho \frac{10}{\pi} \times \frac{1}{\rho}, \quad \sigma = 50$$

$$\Rightarrow \vec{E}(\rho) \approx \frac{3.183}{\rho} \hat{a}_\rho, \quad V/m$$

(c)

$$V_{ab} = \int_a^b E_\rho(\rho)\, d\rho = \int_{5\times10^{-4}}^{10^{-3}} \frac{3.183}{\rho}\, d\rho$$

$$V_{ab} \approx 3.183 \ln\left(\frac{10^{-3}}{5\times10^{-4}}\right) \approx 2.21\,V$$

Sheet 3 Answers

(d)

$$R = \frac{V}{I} \approx \frac{2.21}{2} \approx 1.1\ \Omega$$

(e)

$$P = RI^2 \approx 1.1 \times 4 = 4.4\ W$$