Limiting behavior of MHD flow over a porous rotating disk with Hall currents

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An electrically conducting fluid is driven by a rotating disk, in the presence of a magnetic field that is strong enough to produce significant Hall currents. The disk is porous, allowing mass transfer through suction or injection. The limiting behavior of the flow is studied, as the magnetic field strength grows indefinitely. The flow variables are properly scaled, and uniformly valid asymptotic expansions of the velocity components are obtained through parameter straining. The leading order approximations show sinusoidal behavior that is decaying exponentially, as we move away from the disk surface. The two-term expansions of the radial and azimuthal surface shear stress components, as well as the far field inflow speed, compare well with the corresponding finite difference solutions; even at moderate magnetic fields.

1 Introduction

The flow due to a rotating disk is one of the classical problems of fluid mechanics. It has obvious applications whenever a swirling flow is involved; for examples, in turbo-machinery, combustors, nuclear reactors, and computer storage devices. In some of these applications the presence of a strong magnetic field is inevitable.

The hydrodynamic problem was first introduced by von Kármán [1], who discovered its self-similar nature and introduced the appropriate similarity variables, which reduced the governing partial differential equations to ordinary ones. This simplified the problem considerably and invited many researchers to study different aspects of the flow, as indicated in the review article by Zandbergen and Dijkstra [2].

The magnetohydrodynamic (MHD) counterpart was formulated by Sparrow and Cess [3], and received renewed interest in the last decade [4–7].

In the present article, we study the MHD problem of the flow due to a rotating disk taking into consideration the Hall current and mass transfer. In particular, we investigate the limiting behavior of the flow as the magnetic field strength grows indefinitely. The straightforward perturbation analysis leads to secular behavior, which is removed by parameter straining [8]. Three-term uniformly valid asymptotic expansions for the velocity components are, thus, obtained. In contrast to the case when the two features of Hall current and mass transfer are absent [9], in which the velocity components decay exponentially in a monotonic fashion as we move away from the surface, the current problem exhibits exponential decay that is sinusoidal in nature. The flow involves alternating rounds of forward and backward velocity components.

Finite difference solutions are obtained and show qualitative adherence to the predicted limiting behavior even for moderate magnetic fields. Quantitatively, the two term expansions show excellent agreement with the numerical results.

2 Formulation of the problem

An otherwise quiescent, electrically conducting fluid is driven by an insulated porous disk of infinite extent, which is rotating in its plane with uniform angular speed ω about its axis of symmetry z. A strong uniform magnetic field is applied in the z-direction of unit vector k. The magnetic Reynolds number is small, so that the induced magnetic field can be neglected; and the applied field maintains its uniform magnetic flux density [10].

The fluid is incompressible of density ρ, viscosity μ, electrical conductivity σ, Hall coefficient h (= 1/en_e); where n_e is the number density for the electrons and −e is the electron charge. All these parameters are considered constant.

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The flow is governed by the steady continuity and Navier-Stokes equations [10]
\[
\nabla \cdot \mathbf{V} = 0,
\]
\[
\rho \mathbf{V} \cdot \nabla \mathbf{V} = \mu \nabla^2 \mathbf{V} - \nabla p + \mathbf{J} \times \mathbf{B},
\]
where \( p \) is the pressure, \( \mathbf{V} \) is the velocity vector, \( \mathbf{B} = B\mathbf{k} \) is the magnetic field, and \( \mathbf{J} \) is the electric current.

According to the generalized Ohm’s law \( \mathbf{J} \) is given by
\[
\mathbf{J} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B} - h\mathbf{J} \times \mathbf{B})
\]
which, in the absence of the electric field \( \mathbf{E} \), gives
\[
\mathbf{J} \cdot \mathbf{k} = 0.
\]

This is consistent with the disk being insulated.

Making use of the rotational symmetry, we formulate the problem for a fixed meridional plane, and introduce the \( r \)-axis in the radial direction. The velocity components: \( u \) in the \( r \)-direction, \( v \) in the azimuthal direction, and \( w \) in the \( z \)-direction, as well as the pressure \( p \) are dependent on \( r \) and \( z \) only. The governing equations become [6]
\[
\begin{align*}
    u_r + \frac{u}{r} + w_z &= 0, \quad (1a) \\
    \rho \left( u w_{rr} - \frac{w^2}{r} + w u_z \right) &= \mu \left[ u_{rr} + \left( \frac{u}{r} \right)_r + u_{zz} \right] - p_r - \frac{\sigma B^2}{1 + m^2} (u - mv), \quad (1b) \\
    \rho \left( u w_{rr} + \frac{w}{r} + w u_z \right) &= \mu \left[ v_{rr} + \left( \frac{v}{r} \right)_r + v_{zz} \right] - \frac{\sigma B^2}{1 + m^2} (v + mu), \quad (1c) \\
    \rho (u w_{rr} + w w_z) &= \mu \left[ w_{rr} + \frac{1}{r} w_r + w_{zz} \right] - p_z, \quad (1d)
\end{align*}
\]
where subscripts following a comma denote differentiation, \( m = \sigma h B \) is the Hall parameter. Note that \( m \) may be positive or negative in accordance with the sign of \( B \); i.e. depending on whether the magnetic field is directed away from or toward the disk.

At the surface, \( z = 0 \), the adherence conditions \( u = 0 \) and \( v = \omega r \) apply; together with the injection condition \( w = w_s \). Far from the disk, as \( z \sim \infty \), the fluid has pressure \( p_\infty \) and velocity components \( u \sim 0 \) and \( v \sim 0 \).

The problem admits von Kármán’s similarity transformations \( z = (\mu/\rho \omega)^{1/2} \zeta \), \( u = \omega r F(\zeta) \), \( v = \omega r G(\zeta) \), \( w = (\mu/\rho) \zeta^{1/2} H(\zeta) \), and \( p = p_\infty + \omega^2 Q(\zeta) \); leading to the following problem
\[
\begin{align*}
    H_{,\zeta} + 2 F &= 0, \quad (2a) \\
    F_{,\zeta \zeta} - H F_{,\zeta} - F^2 + G^2 - \beta (\dot{\eta} F - \dot{m} G) &= 0, \quad (2b) \\
    G_{,\zeta \zeta} - H G_{,\zeta} - 2 F G - \beta (\dot{\eta} G + \dot{m} F) &= 0, \quad (2c) \\
    Q_{,\zeta} &= H_{,\zeta \zeta} - H H_{,\zeta}, \quad (2d) \\
    \zeta = 0 : \quad F = 0, \quad G = 1, \quad H = H_s, \quad (2e-g) \\
    \zeta \sim \infty : \quad F \sim 0, \quad G \sim 0, \quad Q \sim 0, \quad (2h-j)
\end{align*}
\]
where \( (\dot{\beta}, \dot{m}, \dot{\eta}) = (\beta, m, 1)/(1 + m^2)^{1/2} \), with \( \beta = \sigma B^2/\rho \omega \) being the magnetic interaction number.

### 3 Asymptotic analysis

We are interested in the limiting behavior of the velocity components as \( \beta \sim \infty \) with fixed \( m \); i.e. as \( \beta \sim \infty \) with fixed \( \dot{\eta} \) and \( \dot{m} \). That \( G(0) = 1 \) irrespective of the value of \( \dot{\beta} \) means that \( G = O(\dot{\beta}^0) \). The driving term in Eq. (2c) is the azimuthal component of the Lorentz force \( \dot{\beta} \dot{m} G \). It can be partially balanced by the corresponding diffusion term \( G_{,\zeta \zeta} \) in a contracting region in which \( \zeta = O(\dot{\beta}^{-1/2}) \). In Eq. (2b) we then have \( \dot{\beta} \dot{m} F \sim F_{,\zeta \zeta} \); expressing a partial balance between the radial component of the Lorentz force and the corresponding diffusion term. Moreover, since \( \dot{\beta} \dot{m} G \gg G^2 \), the driving force
for the radial flow is not the centrifugal force expressed by $G^2$, but, rather, the Hall effect expressed by $\dot{\beta} \dot{n} G$. Requiring $F_\zeta \sim \dot{\beta} \dot{n} G$, for further balance of radial force components, leads to $F = O(\dot{\beta})$. Then, Eq. (2a) gives $H = O(\dot{\beta}^{-1/2})$, while Eq. (2d) gives $Q = O(\dot{\beta}^0)$.

New stretched variables

$$\eta = \dot{\beta}^{1/2} \zeta, \ V = \dot{\beta}^{1/2} H$$

are introduced; transforming problem (2) to the form

$$2F + V' = 0, \quad (4a)$$

$$F'' - \dot{n} F + \dot{n} G = \dot{\beta}^{-1} (VF' + F^2 - G^2), \quad (4b)$$

$$G'' - \dot{n} G - \dot{n} F = \dot{\beta}^{-1} (VG' + 2FG), \quad (4c)$$

$$Q' = V'' - \dot{\beta}^{-1} VV', \quad (4d)$$

$$\eta = 0: \ F = 0, \ G = 1, \ V = V_s, \quad (4e-g)$$

$$\eta \sim \infty: \ F \sim 0, \ G \sim 0, \ Q \sim 0, \quad (4h-j)$$

where the primes denote differentiation with respect to $\eta$.

We expand the flow variables in powers of $\dot{\beta}^{-1}$ in the form

$$Z \sim Z_0 + \dot{\beta}^{-1} Z_1 + \dot{\beta}^{-2} Z_2 + \cdots, \quad (5)$$

where $Z$ stands for $F$, $G$, $V$, and $Q$. The problems for $Z_n$, $n = 0, 1, 2, 3, \ldots$ are linear. For $Z_0$, we get the solutions $F_0 = ES, G_0 = EC, V_0 = V_s = 2(\gamma - E(\gamma C + \alpha S))/((\alpha^2 + \gamma^2))$, and $Q_0 = -2ES$, where $\alpha$ and $\gamma$ satisfy $\alpha^2 - \gamma^2 = \dot{n}$ and $2\alpha \gamma = \dot{m}$; with $E = e^{-\alpha \eta}, S = \sin \gamma \eta,$ and $C = \cos \gamma \eta$, for short. For $Z_1$, the solutions involve secular terms of the form $\eta ES$ and $\eta EC$, the removal of which is affected by straining the parameters $\dot{m}$ and $\dot{n}$ in the form (5); and the procedure can be continued to higher orders.

The following expansions up to $O(\dot{\beta}^{-2})$ are obtained

$$\dot{n} \sim (\alpha^2 - \gamma^2) + \dot{\beta}^{-1} \alpha v_0 + \dot{\beta}^{-2} \alpha v_1 + \cdots, \quad (6a)$$

$$\dot{m} \sim 2\alpha \gamma + \dot{\beta}^{-1} \gamma v_0 + \dot{\beta}^{-2} \gamma v_1 + \cdots, \quad (6b)$$

$$F \sim ES + \dot{\beta}^{-1} [E(Cf_1 + Sg_1) - E^2 f_1] + \dot{\beta}^{-2} [E(Cf_2 + Sg_2) + E^2 \bar{\epsilon} + E^3(C\bar{\alpha} + S\bar{b})] + \cdots, \quad (6c)$$

$$G \sim EC + \dot{\beta}^{-1} [E(Cg_1 - Sf_1) - E^2 g_1] + \dot{\beta}^{-2} [E(Cg_2 - Sf_2) + E^2 \bar{\epsilon} + E^3(C\bar{\alpha} + S\bar{b})] + \cdots, \quad (6d)$$

$$V \sim [v_0 + 2\lambda E(\gamma C + \alpha S)] + \dot{\beta}^{-1} [v_1 + 2\lambda E(\gamma g_1 + \alpha f_2)] + \gamma f_1 / \alpha + \dot{\beta}^{-2} [v_2 + 2\lambda E(\gamma g_2 + \alpha f_2) + \gamma f_1 / \alpha + \dot{\beta}^{-2} v_2 + \cdots] \quad (6e)$$

where symbols appearing on the right-hand sides are given in Appendix A.

Of interest are the radial and azimuthal components of the shear stress at the surface, as well as the farfield speed. They are represented, respectively, by

$$F'(0) \sim \gamma + \dot{\beta}^{-1}(\gamma g_1 + \alpha f_1) + \dot{\beta}^{-2}(-\alpha f_2 + \gamma g_2 - 2\alpha \bar{\epsilon} - 3\alpha \bar{a} + \gamma \bar{b}) + \cdots, \quad (7a)$$

$$G'(0) \sim -\alpha + \dot{\beta}^{-1}(\alpha g_1 - \gamma f_1) + \dot{\beta}^{-2}(-\alpha g_2 - \gamma f_2 - 2\alpha \bar{\epsilon} - 3\alpha \bar{a} + \gamma \bar{b}) + \cdots, \quad (7b)$$

$$V(\infty) \sim v_0 + \dot{\beta}^{-1} v_1 + \dot{\beta}^{-2} v_2 + \cdots = V_\infty. \quad (7c)$$

The expansion for the pressure can be obtained from

$$Q = -2F + \frac{1}{2} \dot{\beta}^{-1}(V_\infty^2 - V^2) \quad (8)$$

which is the result of integrating Eq. (4d) and the use of Eq. (4a) and Condition (4j). This expansion is not given here, for brevity.
Expansions (6) describe how the flow behaves as $\beta \sim \infty$ with fixed $m$. They reveal a sinusoidal behavior that decays exponentially, as we move away from the surface. In contrast, when the MHD effect of Hall currents is neglected, the exponential decay is monotonic. This is detailed in Appendix B.

Expansions (6) describe, as well, the limiting case when $\beta \sim \infty$ with $m = O(\beta^{1/2})$, in which $\hat{n} = O(\beta^{-1})$. All we need is to set $\alpha = |\gamma|$, as inferred from Expansion (6a). It should, however, be noted that, now, as $\hat{\beta} = O(\beta^{1/2})$, the stretching (3) is weaker; being $O(\beta^{1/4})$. Moreover, the perturbations to the basic flow proceed in powers of $\beta^{-1/2}$.

4 Results and discussion

The asymptotic expansions obtained above are tested against corresponding numerical results. The problem described by Eqs. (4) is solved numerically in double precision, using Keller’s two-point, second-order-accurate, finite-difference scheme [11]. A uniform step size $\Delta \eta = 0.01$ is used on a finite domain $0 \leq \eta \leq \eta_{\infty}$. The value of $\eta_{\infty}$ is chosen sufficiently large in order to insure the asymptotic satisfaction of the farfield conditions (3hj). (As pointed out by Pantokratoras [12], a small value of $\eta_{\infty}$ can lead to erroneous results.) The non-linear terms are quasi-linearized, and an iterative procedure is implemented; terminating when the maximum errors in $F''(0), G'(0)$, and $V(\infty)$ become less than $10^{-10}$.

The numerical results exhibit the attenuating sinusoidal behavior predicted by the asymptotic analysis. This is clearly illustrated in Fig. 1 showing the radial and azimuthal velocity profiles $F(\eta)$ and $G(\eta)$, when $\beta = 20$, $m = -10$, and $V_s = 1.4$. However, at such low values of $\beta$, a fixed period of oscillation is not sustained. When $\beta$ is increased to 100 the $F$ and $G$ profiles, respectively, cross the zero line first at $\eta \approx 4.69$ and 2.34 then at $\eta \approx 9.43$ and 7.06; thus sustaining the same period $\tau \approx 9.43$. The two profiles cross the zero line several times later, but with much smaller magnitudes; maintaining the same period, though. Moreover, cases of negative and positive $m$ with the same $|m|$ and $\beta$ are found to almost have the same period; as predicted by the asymptotic analysis; cf. Eq. (10) below. For example, when $\beta = 100$ and $V_s = 1.0$, the cases of $m = \pm 2.5, \pm 5$, and $\pm 10$ have periods $\tau \approx 11.2, 9.9, \text{and} 9.4$, respectively.

$$\begin{align*}
F'(0) &= \gamma_0 + \hat{\beta}^{-1}(\gamma_1 + \alpha_0 f_{10} + \gamma_0 g_{10}) + O(\hat{\beta}^{-2}), \\
G'(0) &= -\alpha_0 + \hat{\beta}^{-1}(-\alpha_1 + \alpha_0 g_{10} - \gamma_0 f_{10}) + O(\hat{\beta}^{-2}), \\
V(\infty) &= v_{00} + \hat{\beta}^{-1}(v_{10} + v_{10}) + O(\hat{\beta}^{-2}).
\end{align*}$$

Results of the two-term asymptotic expansions (i.e., up to $O(\hat{\beta}^{-1})$) proved to be in close agreement with the numerical results, even at values of $\beta$ as low as 20. This is illustrated in Figs. 2a,b for $F_\gamma(0) = \hat{\beta}^{1/2}F'(0)$ and in Figs. 3a,b for $G_\gamma(0) = \hat{\beta}^{1/2}G'(0)$ with different values of positive and negative $m$, and in Fig. 4 for $H(\infty) = \hat{\beta}^{-1/2}V(\infty)$ with different values of $V_s$. 

![Fig. 1 Velocity profiles $F(\eta)$ and $G(\eta)$; $\beta = 20$, $m = -10$, $V_s = 1.4$.](www.zamm-journal.org)
The period of the sinusoidal oscillations $\tau = \frac{2\pi}{|\gamma|}$ takes the form

$$\tau = 4\pi \left[ \frac{1}{2} \left\{ \hat{n} + (\hat{m}^2 + \hat{n}^2)^{1/2} \right\} \right]^{1/2} / |\hat{n}| + O(\hat{\beta}^{-2}).$$  

(10)

For $|m| = 2.5, 5,$ and 10 Eq. (10) gives $\tau = 11.21, 9.91,$ and $9.36,$ respectively, which are close to the numerically predicted values quoted above, for the same $|m|$ with $\beta = 100$ and $V_s = 1.0; \text{ noting, in passing, that, to } O(\hat{\beta}^{-2}), \tau \text{ is independent of } V_s.$
5 Conclusion

The limiting behavior of the MHD flow due to a porous rotating disk has been studied, as the magnetic field grows indefinitely, taking into consideration the Hall currents. Three-term uniformly-valid asymptotic expansions have been derived using parameter straining. The velocity components show sinusoidal behavior that attenuates exponentially, as we move away from the disk. (When the Hall current is neglected, the exponential decay becomes monotonic.) Finite difference solutions have also been calculated. The two-term asymptotic and the numerical solutions have shown excellent agreement both qualitatively and quantitatively.

Appendix A

Given below are the expressions and relations that define the different symbols appearing in Expansions (6). For conciseness, let

\[ E = e^{-\alpha \eta}, \quad C = \cos \gamma \eta, \quad S = \sin \gamma \eta, \quad \lambda = (\alpha^2 + \gamma^2)^{-1} \]  

(A1–4)

then

\[ v_0 = V_s - 2\lambda \gamma. \]  

(A.5)

\( f_1, g_1, \) and \( v_1 \) are obtained from

\( (3\alpha^2 + \gamma^2)f_1 + 2\alpha \gamma g_1 = \lambda(\alpha^2 - \gamma^2), \)  

(A.6)

\(- 2\alpha \gamma f_1 + (3\alpha^2 + \gamma^2)g_1 = 2\lambda \alpha \gamma, \)  

(A.7)

\[ v_1 = -2\lambda(\gamma g_1 + \alpha f_1) + f_1/\alpha. \]  

(A.8)

\( \bar{c} \) and \( \tilde{c} \) satisfy

\[ (3\alpha^2 + \gamma^2)\bar{c} + 2\alpha \gamma \tilde{c} = v_0(\gamma g_1 + \alpha f_1) - 2\lambda(\alpha^2 - \gamma^2)g_1, \]  

(A.9)

\(- 2\alpha \gamma \bar{c} + (3\alpha^2 + \gamma^2)\tilde{c} = v_0(\alpha g_1 - \gamma f_1) - 4\lambda \alpha \gamma g_1. \]  

(A.10)

\( \bar{a}, \tilde{a}, \bar{b}, \) and \( \tilde{b} \) satisfy

\[ 8\alpha^2 \bar{a} - 6\alpha \gamma \tilde{b} + 2\alpha \gamma \bar{a} = (4\lambda \alpha^2 - 1)\gamma f_1/\alpha + 2g_1, \]  

(A.11)

\[ 6\alpha \gamma \bar{a} + 8\alpha^2 \tilde{b} + 2\alpha \gamma \bar{b} = (4\lambda \alpha^2 - 1)f_1, \]  

(A.12)

\(- 2\alpha \gamma \bar{a} + 8\alpha^2 \tilde{a} - 6\alpha \gamma \bar{b} = 4\lambda \alpha \gamma g_1 - f_1, \)  

(A.13)

\(- 2\alpha \gamma \tilde{b} + 6\alpha \gamma \bar{a} + 8\alpha^2 \tilde{b} = (4\lambda \alpha^2 - 2)g_1 + \gamma f_1/\alpha. \)  

(A.14)

\( \hat{a} \) and \( \hat{b} \) satisfy

\[ 3\alpha \hat{a} - \gamma \hat{b} = 2\hat{a}, \]  

(A.15)

\[ \gamma \hat{a} + 3\alpha \hat{b} = 2\hat{b} \]  

(A.16)

and finally \( f_2, g_2, \) and \( v_2 \) are obtained from

\[ f_2 = -\tilde{c} - \bar{a}, \]  

(A.17)

\[ g_2 = -\tilde{c} - \bar{a}, \]  

(A.18)

\[ v_2 = -2\lambda(\gamma g_2 + \alpha f_2) - \tilde{c}/\alpha - \bar{a}. \]  

(A.19)
Appendix B

When the MHD effect of Hall current is neglected, by setting $m = 0$ so that $(\hat{m}, \hat{n}) = (0, 1)$, Expansions (6c)–(6e) reduce to the following expansions

\[ F \sim \hat{\beta}^{-1}(E - E^2)/3 + \hat{\beta}^{-2}(E - E^2)(2V_s/9) + \cdots, \quad (B.1) \]

\[ G \sim E + \hat{\beta}^{-2}(E - E^3)/24 + \cdots, \quad (B.2) \]

\[ V \sim V_s - \hat{\beta}^{-1}(1 - 2E + E^2)/3 - \hat{\beta}^{-2}(1 - 2E + E^2)(7V_s/18) + \cdots, \quad (B.3) \]

where $E = e^{-\alpha \eta}$ and $\alpha$ is given by

\[ \alpha \sim 1 - \hat{\beta}^{-1}V_s/2 + \hat{\beta}^{-2}(V_s^2/8 + 1/6) + \cdots. \quad (B.4) \]

The corresponding expansions for the shear stresses at the disk and the inflow speed are

\[ F'(0) \sim \hat{\beta}^{-1}1/3 + \hat{\beta}^{-2}V_s/18 + \cdots, \quad (B.5) \]

\[ G'(0) \sim -1 + \hat{\beta}^{-1}V_s/2 - \hat{\beta}^{-2}(V_s^2/8 + 1/12) + \cdots, \quad (B.6) \]

\[ V(\infty) \sim V_s - \hat{\beta}^{-1}1/3 - \hat{\beta}^{-2}7V_s/18 + \cdots. \quad (B.7) \]

It is noted that the sinusoidal behavior disappears and the velocity components decay exponentially in a monotonic manner. Moreover, the radial velocity component $F$ turns out to be $O(\hat{\beta}^{-1})$.

Further, when the mass transfer is omitted by setting $V_s = 0$, the expansions proceed in powers of $\hat{\beta}^{-2}$, and $V$ turns out to be $O(\hat{\beta}^{-1})$ – i.e., $H = O(\hat{\beta}^{-3/2})$ – in total agreement with the expansions of El-Mistikawy et al. [9], who studied the limiting behavior of this reduced problem through coordinate straining.

References