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MHD flow due to a linearly stretching sheet with induced magnetic field

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Abstract The full MHD equations, governing the flow due to a linearly stretching sheet in the presence of a transverse magnetic field, can be cast in a self similar form involving two parameters—the magnetic Prandtl number P_{m} and the magnetic interaction number β. The leading-order problem, as $P_{\mathrm{m}} \sim 0$, is of hierarchical type allowing the solution first for the velocity field and then for the induced magnetic field. Solutions of the full problem tend readily to the hierarchical solutions as P_{m} gets smaller, thus justifying the use of the hierarchical approach even at not so small P_{m}.

1 Introduction

Magnetohydrodynamics is concerned with the interaction between flows of electrically charged fluids and electromagnetic fields. Charged fluid particles moving in an electromagnetic field are acted upon by a body force—Lorentz force, which affects their motion. The motion of the particles, on its turn, induces changes to the electromagnetic fields. This interactive process is governed by the fluid flow continuity and Navier–Stokes equations, coupled with the electromagnetic Maxwell's equations and Ohm's law [1].

In the absence of an externally applied electric field, and under the assumption of no surplus charge, the induced electric field is obviated. The electromagnetic equations reduce to Gauss' law for magnetism, Ampere's law, and Ohm's law.

The further assumption of diminishing magnetic Reynolds number R_{m} implies a negligible induced magnetic field. The applied magnetic field is, thus, unaltered. This assumption is feasible in several applications and has been adopted in many publications since the 1950s [2,3]. It simplifies the governing equations, as it uncouples the velocity field equations from the electromagnetic field equations. One can first solve the flow equations for the velocity and then solve the electromagnetic equations for the induced magnetic field, in a hierarchical approach. However, fluid mechanicists are usually concerned with the first step only.

A question may arise as how reliable this hierarchical approach is, and how accurate its results are. Should not one solve the coupled equations simultaneously, thus adhering to the magnetohydrodynamic MHD interactive doctrine?

This article is meant to respond to this question. The simple example of the MHD flow due to the linear stretching of a non-conducting sheet, in the presence of a transverse magnetic field, is considered. Under the assumption of small $R_{\mathrm{m}} \sim 0$, Pavlov [4] obtained a closed form solution for the velocity field, within the boundary layer approximation, i.e., for growing Reynolds number $R_e \sim \infty$. The same solution was shown by Andersson [5] to be an exact solution of the Navier-Stokes equations, i.e., for $R_e < \infty$. Herein, the MHD problem is shown to depend on the product $R_e^{-1} R_{\mathrm{m}} = P_{\mathrm{m}}$, the magnetic Prandtl number, which is much smaller than unity [6]. The hierarchical approach is derived as corresponding to the leading term in the limit

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as $P_{\mathrm{m}} \sim 0$. Solutions of the hierarchical approach for both the velocity and magnetic fields are compared to those of the full MHD problem (coupling approach).

2 Mathematical model

An electrically conducting incompressible Newtonian fluid is driven by the stretching of a non-conducting non-porous sheet occupying the xz-plane. The stretching speed along the x-direction is ωx, where the stretching rate ω is constant. In the farfield, the fluid is essentially quiescent under pressure p_∞ and is permeated by a stationary magnetic field of uniform strength B in the transverse y-direction. Constants are the fluid density ρ, kinematic viscosity ν, permeability μ, and conductivity σ.

The non-dimensional equations governing this steady MHD flow are

$$\nabla \cdot \mathrm{V} = 0, \tag{1}$$
$$\mathrm{V} \cdot \nabla \mathrm{V} + \nabla p = R_e^{-1} \nabla^2 \mathrm{V} + \beta(\mathrm{V} \times \mathrm{B}) \times \mathrm{B}, \tag{2}$$
$$\nabla \times \mathrm{B} = R_{\mathrm{m}} \mathrm{V} \times \mathrm{B}, \tag{3}$$
$$\nabla \cdot \mathrm{B} = 0 \tag{4}$$

where $R_e = \omega L^2/\nu$, $R_{\mathrm{m}} = \sigma\mu\omega L^2$, and $\beta = \sigma B^2/\rho\omega$ are the Reynolds number, magnetic Reynolds number, and magnetic interaction number, respectively. Other symbols are non-dimensionalized, the gradient operator ∇ by a suitably chosen length L, the velocity V by ωL, the pressure p by $\rho\omega^2 L^2$, and the magnetic field B by B.

For a two-dimensional flow, $\mathrm{V} = [u(x, y), v(x, y), 0]$ and $\mathrm{B} = [r(x, y), 1 + s(x, y), 0]$, where (u, v) are components of the velocity and (r, s) are components of the induced magnetic field in the (x, y) directions, respectively. The governing equations become

$$u_x + v_y = 0, \tag{5}$$
$$u u_x + v u_y + p_x = R_e^{-1}(u_{xx} + u_{yy}) + \beta[(1 + s) r v - (1 + s)^2 u], \tag{6}$$
$$u v_x + v v_y + p_y = R_e^{-1}(v_{xx} + v_{yy}) + \beta[(1 + s) r u - r^2 v], \tag{7}$$
$$s_x - r_y = R_{\mathrm{m}}[(1 + s) u - r v], \tag{8}$$
$$r_x + s_y = 0 \tag{9}$$

with the adherence conditions at the sheet

$$y = 0: \quad u = x, \quad v = 0 \tag{10}$$

and the farfield conditions

$$y \sim \infty: \quad u \sim 0, \quad p \sim p_\infty. \tag{11}$$

The conditions on r and s are addressed below.

The problem admits the similarity transformations :

$y = R_e^{-1/2}\eta$, $v = -R_e^{-1/2} f(\eta)$, $u = x f'$, $s = R_e^{-1} R_{\mathrm{m}} g(\eta)$, $r = -R_e^{-1/2} R_{\mathrm{m}} x g'$, and $p = p_\infty - \frac{1}{2} R_e^{-1} R_{\mathrm{m}} \beta x^2 g'^2 - R_e^{-1}[f' + \frac{1}{2} f^2 - \frac{1}{2} f^2(\infty)]$, where primes denote differentiation with respect to η. The problem becomes

$$f''' + f f'' - f'^2 - \beta f' = -P_m \beta[g'^2 + (1 + P_m g) f g' - (2 + P_m g) f' g], \tag{12}$$
$$g'' = f' + P_{\mathrm{m}}(g f' - f g'), \tag{13}$$
$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{14}$$

where $P_{\mathrm{m}} = R_e^{-1} R_{\mathrm{m}}$ is the magnetic Prandtl number. For practical applications, P_{m} is much smaller than unity [6]. Considering the limiting behavior as $P_{\mathrm{m}} \sim 0$ is, therefore, justified. Note that $P_{\mathrm{m}} \sim 0$ may be attributed to $R_{\mathrm{m}} \sim 0$ or $R_e \sim \infty$, independently.

3 The limiting behavior for diminishing $P_m \sim 0$

The flow variables expand as follows:

$$f \sim f_0 + P_m f_1 + \cdots , \tag{15}$$
$$g \sim g_0 + P_m g_1 + \cdots . \tag{16}$$

We introduce these expansions into Eqs. (12)–(14) and equate like powers of P_m. The problem for f_0 is

$$f_0''' + f_0 f_0'' - f_0'^2 - \beta f_0' = 0, \tag{17}$$
$$f_0(0) = 0, \quad f_0'(0) = 1, \quad f_0'(\infty) = 0. \tag{18}$$

This is the problem formulated by Andersson [5] who also gave its solution

$$f_0 = (1 - e^{-a\eta})/a, \quad a = \sqrt{1+\beta}. \tag{19}$$

Then, g_0 satisfies

$$g_0'' = f_0' = e^{-a\eta}. \tag{20}$$

Problems governing higher-order terms can be formulated. However, our interest is restricted to the leading-order terms only, insofar as responding to the question posed in the Introduction is concerned. The approach is hierarchical. We determine f_0 and then use it to determine g_0.

4 Boundary conditions on the induced magnetic field components *r* and *s*

In the farfield as $\eta \sim \infty$, the condition on f' indicates that f approaches a constant value $f(\infty)$. There, the magnetic field B must be parallel to the velocity V $\propto$ $[0, f(\infty), 0]$, so that the current density J $\propto$ V $\times$ B vanishes. This requires $r \sim 0$ as $y \sim \infty$, i.e., $g'(\infty) = 0$. Then, g may approach a constant value $g(\infty)$. On the other hand, Eq. (20) integrates to $g_0' = -e^{-a\eta}/a$, leading to $g_0'(0) = -1/a$. So we cannot specify $g'(0)$, g_0' being the leading-order term of g'. We may specify either $g(0)$ or $g(\infty)$. To specify $g(0)$ is rather clueless. We choose to specify $g(\infty)$. In particular, we set $g(\infty) = 0$, i.e., $s \sim 0$ as $y \sim \infty$, which may be interpreted as follows. We consider B to be the farfield total magnetic field, imposed and induced, and $B(r, s, 0)$ the deviation thereof. The magnetic field at the surface $B[r(x, 0), 1 + s(x, 0), 0]$ is transmitted to the sheet, where the induced part of B is ascertained.

5 Numerical method

To test the accuracy of the leading-order terms f_0 given above and $g_0 = e^{-a\eta}/a^2$, we compare them with those obtained by the coupling approach, wherein we solve Eqs. (12) and (13) simultaneously, together with the specified boundary conditions.

Since a closed form solution is not possible, we seek an iterative numerical solution, making use of the hierarchical approach. In the n^{th} iteration, we solve, for ${}_n f(\eta)$, Eq. (12) with its right-hand side evaluated using the previous iteration solutions ${}_{n-1} f(\eta)$ and ${}_{n-1} g(\eta)$, together with conditions (14). Then we solve, for ${}_n g(\eta)$, Eq. (13) with the known ${}_n f(\eta)$, together with the conditions $g(\infty) = 0$ and $g'(\infty) = 0$. The iterations continue until the maximum error in $f(\eta_\infty)$, $f''(0)$, $g(0)$ and $g'(0)$ becomes less than 10^{-10}. For the first iteration, we zero the right-hand side of Eq. (12) which corresponds to ${}_0 g(\eta) = 0$. Then ${}_1 f(\eta)$ is given by $f_0 = (1 - e^{-a\eta})/a$, and ${}_1 g(\eta)$ is given by $g_0 = e^{-a\eta}/a^2$. These results are used to test the correctness of the numerical procedure.

The numerical solution of the problems for ${}_n f(\eta)$ and ${}_n g(\eta)$ utilizes Keller's two-point, second-order accurate, finite-difference scheme [7]. A uniform step size $\Delta\eta$=0.01 is used on a finite domain $0 \le \eta \le \eta_\infty$. The value of $\eta_\infty = 30$ is chosen sufficiently large in order to insure the asymptotic satisfaction of the farfield conditions. (For large $\beta > 1$, we use $\Delta\eta = 0.01\beta^{-1/2}$ and $\eta_\infty = 30\beta^{-1/2}$.) The nonlinear terms in the problem for ${}_n f(\eta)$ are quasi-linearized, and an iterative procedure is implemented, terminating when the maximum error in ${}_n f(\eta_\infty)$ and ${}_n f''(0)$ becomes less than 10^{-10}.

6 Results and discussion

The results presented below are intended to explore how close the hierarchical approach is to the coupling approach. The subjects of comparison are the surface shear, the entrainment rate, and the x and y components of the induced magnetic field at the surface, which are represented, respectively, by $f''(0)$, $f(\eta_\infty)$, $g(0)$ and $g'(0)$.

The problem involves two parameters, the magnetic Prandtl number P_m and the magnetic interaction number β.

Table 1 presents the coupling results for successively decreasing values of P_m, with $\beta = 1$. The last line corresponding to $\beta = 0$ presents the hierarchical results. The tendency of the coupling results to approach the hierarchical results as $P_m \sim 0$ is obvious, justifying the use of the small R_m (small P_m) hierarchical approach, not only for describing the velocity field, but also for describing the magnetic field.

Table 2 demonstrates the effect of β. The results of the two approaches are presented over a wide range of β, with $P_m = 0.1$. Corresponding to each entry of β, the first line gives the coupling results, while the second

Table 1 Tendency of coupling results toward hierarchical results as $P_m \sim 0$; $\beta = 1$

P_m	$f''(0)$	$f(\eta_\infty)$	$g(0)$	$g'(0)$
10^{-1}	−1.4322	0.69822	0.51249	−0.73400
10^{-2}	−1.4160	0.70622	0.50125	−0.70976
10^{-3}	−1.4144	0.70702	0.50013	−0.70737
10^{-4}	−1.4142	0.70710	0.50001	−0.70713
10^{-5}	−1.4142	0.70711	0.50000	−0.70711
0	−1.4142	0.70711	0.50000	−0.70711

Table 2 Hierarchical results versus coupling results, at different values of β; $P_m = 0.1$

β	$f''(0)$	$f(\eta_\infty)$	$g(0)$	$g'(0)$
10^{-3}	−1.0006	0.99944	1.1097	−1.1104
	−1.0005	0.99950	0.99900	−0.99950
10^{-2}	−1.0055	0.99450	1.0976	−1.1037
	−1.0050	0.99504	0.99010	−0.99504
10^{-1}	−1.0535	0.94920	0.99020	−1.0432
	−1.0488	0.95346	0.90909	−0.95346
1	−1.4322	0.69822	0.51249	−0.73400
	−1.4142	0.70711	0.50000	−0.70711
10	−3.3303	0.30027	0.090984	−0.30300
	−3.3166	0.30151	0.090909	−0.30151
10^2	−10.055	0.099455	0.0099011	−0.099553
	−10.050	0.099504	0.0099010	−0.099504
10^3	−31.640	0.031605	0.00099900	−0.031609
	−31.639	0.031607	0.00099900	−0.031607

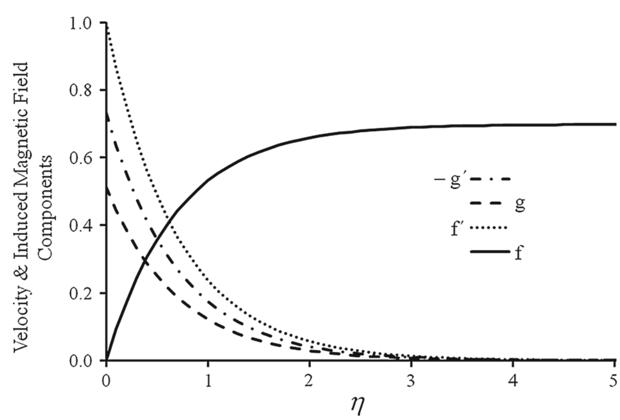

Fig. 1 Velocity (f', f) and induced magnetic field (g', g) components, $\beta = 1.0$, $P_m = 0.1$

line gives the hierarchical results. For $f''(0)$ and $f(\eta_\infty)$, the results get closer as β deviates from unity. The difference, at $\beta = 1$, is less than 1.3 %. For $g(0)$ and $g'(0)$, the results get closer as β increases. The maximum difference, at the smaller β, is less than 10 %.

Plots of the velocity components (f', f) and the induced magnetic field components (g', g), when $\beta = 1.0$ and $P_m = 0.1$, are shown in Fig. 1. The leading-order counterparts cannot be distinguished from these plots.

7 Conclusions

The problem of the flow due to a linearly stretching sheet in the presence of a transverse magnetic field has been shown to admit self similarity of the full MHD governing fluid flow and electromagnetic equations. The problem has been shown to involve two parameters, the magnetic Prandtl number P_m and the magnetic interaction number β. Leading-order problems for the velocity field and the magnetic field, as $P_m \sim 0$, have been formulated and solved. Numerical solutions of the full MHD problem have shown a tendency toward the leading-order solution as $P_m \sim 0$.

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