TOLERANCE OPTIMIZATION: A DECOMPOSITION SCHEME, VARIANCE REDUCTION and FRACTIONAL APPROXIMATION

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ABSTRACT
Development of involved optimization algorithms is not an easy task for several reasons: First, every analyst is interested in a specific problem; Second, the capabilities of these methods may not be fully understood a priori; Third, coding of multi-purpose and more involved algorithms is not an easy job.

In this paper, the optimization problem employing the near to global optimum algorithm is studied (Gadallah, M.H., 2000). The focus is to exploit 2 ideas: First, the algorithm can be modified to act as a variance reduction technique; Second, the algorithm can be modified to tackle the problem of system decomposition. Both ideas are novel within the context of statistical design of experiments. The first, if fully proved experimentally could yield the simultaneous integration of nominal and variance optimization possible. The second, can be extended to deal with multi-dimensional highly constrained systems with ease. These two ideas are explained with the use of a simple example to illustrate the idea. An algorithm is developed that deal with the problem in several stages according to a predetermined decomposition scheme. The original objective and constraint functions are dealt with to suit each stage. Accordingly, all NP hard problems can ideally be transformed into NP complete ones with a consequence on the number of stages resulting from decomposition. Several decomposition scenarios are used and their results are compared numerically. Two orthogonal arrays and four composite arrays are used to plan experimentation; these are L27OA and L54OA and their sub-families. These arrays are compared with respect to their statistical measures.

The algorithm as such, is very promising optimization tool, especially for coupling system decomposition and variance reduction. Past work focused on either decomposition or statistical optimization. This work offers both capabilities. Several studies are reviewed and conclusions are drawn.

KEYWORDS: Decomposition Methods, Variance Reduction, Coded Designs, and Statistical DOE.

BACKGROUND
The tolerance consideration represents the transfer from nominal world to variation world. Although, specifically the tolerance on nominal dimensions represents an interest to manufacturing engineers to allocate the least cost tolerances on dimensions; the tolerance problem on variables is of prime interest to the whole engineering community. In the presence of variations, the nominal system setting is simply a matter of academic interest. Performance of realistic systems is different from the nominal systems. Since the ultimate objective is to optimize the system in the presence of variations, several attempts try to understand the problem. These include tolerance optimization, tolerance analysis, variation reduction, system decomposition, mathematical models and others. A full survey of these studies is certainly beyond the scope of this paper, although the diversity of the subjects certainly shows the importance of the matter.

Justin et al. (1992) described a suitable mathematical model and showed how it may be used to automate linear worst case tolerance analysis in assemblies.

Zhang and Wang (1993) presented an analytical model for tolerance allocation problem. The SA is used to solve the resulting nonlinear model. Johnson and Lee (1993) presented a truncated Monte Carlo simulation and GAs are used as analysis (i.e. multi-variate integration) and synthesis (i.e. optimization) tools respectively.
Kopardekan and Anand (1995) presented a neural based approach for the tolerance allocation problem considering machine capabilities and mean shifts. The network is trained using the back propagation learning method and used to predict the individual part tolerances.

Nigam and Turner (1995) reviewed various approaches to tolerance analysis. Issues such as non-ideal probability density functions of component tolerances and applications to solid modeling and geometric standards are discussed.

Webb and Parkinson (1995) presented a study on the properties and optimization of the linear variance function. The Linear variance function is a measure of how variations in variables and parameters are transmitted to design functions. Two things are of interest: a) design feasibility and b) sensitivity of the design to variations.

Altus et al. (1996) performed a study on the use of Genetic Algorithms for scheduling and decomposition of multi-disciplinary design problems. This study describes a method for structuring problem tasks with optimal ordering and decomposition into sub-problems. The discussion is restricted to the organization of computational subroutines of multi-disciplinary optimization studies.

Korngold and Gabriele (1997) developed a new algorithm to optimize multi-disciplinary coupled non-hierarchic systems with discrete variables. This algorithm decomposes into contributing disciplines and uses designed experiments within the disciplines to build local response surface approximations. The global design space is approximated by experimental data to obtain 1st and 2nd order sensitivity equations.

Skowronsksi (1998) extended the derivative estimation technique to tolerance analysis problems that use the acceptance fraction criterion. The accuracy of the technique is compared to the finite difference calculations. The paper concluded that the derivative estimation provides equivalent or better accuracy than finite difference.

Srikanth, K. et al. (1998) presented a fuzzy logic approach to model all uncertain parameters in the process for automatic optimal tolerance assignment. It also calculates the assembly rate. A comparison with the statistical method was discussed. A relatively smaller sample size is needed to estimate the assembly rate with reasonable accuracy.

Glancy and Chase (1999) presented a second order method for tolerance analysis that combines the advantages of the linearized method with the advantages of the Monte Carlo simulation. The SOTA method applies the system of moments to implicit variables. In an earlier paper (Gadallah, 2001), the author has presented an algorithm that can replace Monte Carlo expensive simulations. The author has proven that the algorithm can match the first 2 moments in very limited number of experiments.

Tappeta and Renaud (1997) provided a comprehensive overview and development of mathematical optimization strategies for multi-objective collaborative optimization (MOCO). The relative importance of each discipline design objectives is established a priori through weight assignments. This in effect transforms the multi-objective to a single objective function.

Decomposition synthesis in optimal design is the process of creating an optimal design model by selecting objectives and constraints so that it can be directly partitioned into an appropriate decomposed form. Most important, such synthesis results are not unique since there may be many partitions that satisfy the decomposition requirements (Krishnamachari and Papalambros, 1997). In another study, the authors stressed that casting a given problem into optimization model by selecting objectives and constraints is a subjective task.

Fujita et al. (2000) proposed a design optimization method for link mechanisms by combining non-hierarchic coupled system decomposition and mini-max relaxation. The essential concepts of the methods was detailed as:

1. Overall algorithm consists of iteration of partial optimization computations of decomposed sub problems and coordination of their results.
2. The successive quadratic programming method is used in respective phases and Hessian of the Lagrangian is inherited over such iteration to gradually increase approximation.
3. In partial optimization, the target sub-system is evaluated by strict analysis and the other sub-systems are approximated by 1st order sensitivity.

The partial results are coordinated by solving a coordination problem that is generated with the information on partial optimization results without executing any system analysis.

Neumann, A. (1994) reported on the new Y14.5 M standard on Dimensioning and Tolerancing. Gabriele, G. (1994) presented a set of proposed engineering design fundamentals in design education. Srinivasan and O’connor (1994) described three interpretations defined by the 1st 2 moments, the process capability indices or bounding cumulative distribution functions.

Wei and Lee (1995) discussed an approach that simplifies the traditional procedures of tolerance chart balancing. This takes into account the process capability. A mathematical model based on the modified rooted tree chart was developed.

Parkinson and Chase (2000) introduced the idea of adaptive robust design. Two notions, passive and active adaptive robust designs were defined. The variations that characterize the system change over time were also defined. Davidson et al. (2000) presented a mathematical model for tolerances of planar surfaces. T-map, tolerance map is a hypothetical volumes of points, which corresponds to all possible locations and variations of a segment of a plane which can arise from tolerances on size, form and orientation. This map is a convex set that can be resolved into component zones.
Venkataraman et al. (2000) developed a geometric design tool for surface micro-machined MEMS. This is analogous to generating a geometric model from the tool path. The model is queried to generate the process specific data. This is applied to 2-D cases to eliminate the complexity of the third dimension.

**PROBLEM STATEMENT**

The tolerance analysis and synthesis problem have been studied by Zhang and Wang, Johnson and Lee, Kopardekan and Anand, Skowronski, Nigam and Turner, Srikanth et al., Glancy and Chase. If the interest in this problem is very limited, others have studied the more general problem, which variation analysis, representation and optimization. Variation pattern, whether it follows a certain distribution, on variables, objective functions and constraints were also studied. Why this problem is of interest; mainly due to the need to incorporate the uncertainties of the real problem. Decomposition is some sort of approximations and algorithms either standalone or coupled with heuristics show the limitations of the existing method to deal with the dimensionality problem (Altus et al., Korngold and Gabriele, Tappeta and Renaud, Krishnamachari and Papalambros).

Given a problem with a number of variables, a full solution would require a search space, if continuous in nature will require huge solution effort. Discretization of the solution space such that it contains the optimal solution is certainly an efficient idea. This is equivalent to modeling the continuous space into a discrete space employing orthogonal arrays. In case the full problem is modeled using a given array, imagine that we can reduce this array into a smaller size array; this smaller size array can be viewed as a decomposed set of the original space. Literature is very limited as related to the construction of various size arrays.

The various size arrays have different nature related to the convergence and variance computations. The tolerance domain and the cost-process domain are modeled using inner-out inner arrays. At each iteration, there are m x n solution points for the tolerance and process cost respectively, where n and m are size of inner and outer arrays receptively. For instance, when L27OA is used, there will be 27 solution points coupled with the fact that one solution point is of very little value, the procedure discussed here can result in what is called the expected value and variance of the process cost.

Answering the question which array to use for best variance value would need experimentation with several existing arrays. Another point to raise, a full factorial array for a larger number of variables. This raises another difficulty; these arrays can not be employed for any problem in a generic way. Even with the use of fractional arrays, the computational effort is still high. The problem studied has 5 variables (2 variables have 2 levels and 3 variables have 3 levels). Several solutions can be obtained:

**Solution 1:** Model of the problem by approximating 5 variables into 3 level variables. This requires 3^5=243 experiments and an L243OA is needed to model the tolerance domain.

**Solution 2:** Model of the problem by approximating the 5 variables into 3-3 level variables and 2-2 level variables. This requires 2^2 x 3^3 = 108 experiments and an L108OA is needed to model the tolerance domain. Accordingly, when the L243OA (which is the Full Factorial Array in this case) is used, several fractionation can be developed, these are 1/3 FFE and 1/9 FFE to result in L81OA and L27OA respectively. Similarly, when the L108OA (which is Full Factorial Array in this case) is used, several fractionations can be developed, these are 1/3 FFE and 1/9 FFE to result in L54OA and L27OA respectively. The L27OA can be either 1/9 FFE or 1/3 FFE and this proves our notion presented earlier.

**THE TOLERANCE MODEL**

Tolerance Model: The cost tolerance model used for optimization is

\[
C(T_{ij}) = a + b/T_{ij} \quad T_{lj}^1 \leq T_{ij} \leq T_{lj}^u
\]

(1)

Where: \( a, b \) (b > 0 are process constants, define the tolerance range for a given process);

The tolerance synthesis problem is a discrete problem because the domain is not continuous. The cost tolerance relations are not continuous and can split from one range to the other (corresponding to different processes). Besides, the tolerance design domain is approximated by a finite number of experiments (equal to the size of orthogonal array used to host the search).

The optimization problem can be given as:

\[
\min X_{ij}, T_{ij} \quad f(X_{ij}, T_{ij}) = \min X_{ij}, T_{ij} \sum_{i=1}^{n} \sum_{j=1}^{P_i} [X_{ij}, C(T_{ij})]
\]

(2)

Subject to:

\[
\sum_{i,j \in C_k} X_{ij} \cdot T_{ij} \leq T_K \quad \text{worst case analysis}
\]

\[
\sum_{i,j \in C_k} X_{ij} \cdot T_{ij}^2 \leq T_K^2 \quad \text{Statistical case analysis}
\]

(3)

Where:

\( f = \) Total machining cost of design dimensions in assembly;

\( T_K = \) Tolerance of resultant design element k;

\( C_k = \) Dimension chain for resultant design element k;

\( T_{ij} = \) Manufacturing tolerance on the i-component using process j;

\( X_{ij} = 1 \) if process j is chosen to produce component dimension i, 0 otherwise;

\( C(T_{ij}) = \) Cost of producing tolerance on dimension i by process j;

\( T_{lj}^1, T_{lj}^u \) = Lower and upper tolerance limits;

APPLICATION OF TOLERANCE MODEL to PROBLEM

**P1:**

\[
\min X_{ij}, T_{ij} \quad f(X_{ij}, T_{ij}) = \min X_{ij}, T_{ij} \sum_{i=1}^{n} \sum_{j=1}^{P_i} [X_{ij}, C(T_{ij})]
\]

(4)
There are 5 variables and 13 process combinations. The first scenario is to take variable 1, 3 and 5 in the first decomposition stage. Variables 2 and 4 can be taken in the second stage. The second scenario is to take variable 1, 2 and 4 in the first stage and variables 3 and 5 in the second stage respectively. The algorithm is implemented and experimentation allowed testing the difference between various Decomposition Scenarios. The cost-tolerance data for this problem is given in Gadallah, M.H., 2000. This equation can be written further as a first stage:

Minimize

$\{ (X_{ij}, T_{ij}) = (1.30 + 0.012 / T_{11}, 2.0 + 0.016 / T_{12}, 1.12 + 0.029 / T_{13})$

$\{ 3.0 + 0.003 / T_{31}, 2.0 + 0.008 / T_{32})$

$\{ 6.0 + 0.004 / T_{51} / 1.50 + 0.010 / T_{52} - 4.70 + 0.047 / T_{53} \} \}$

Subject to:

$T_1 + T_3 + T_5 \leq 0.015(6)$

And a second stage:

$\{ (X_{ij}, T_{ij}) = \min \{ \{ -0.33 + 0.00992 / T_{21}, -8.0 + 0.042 / T_{21}, 21 \} \}

\{ -4.0 + 0.008 / T_{41}, 3.0 + 0.012 / T_{42} \}

\{ 2, 4 \}

\{ 22, -2.0 + 0.012 / T_{23} \}$

Subject to original constraint:

$T_1 + T_2 + T_3 + T_4 + T_5 \leq 0.255$

and $T_1 + T_3 + T_5 \leq 0.015(8)$

In stage 1, the optimum design variables are used with the optimum process combinations to stage 2. Instead of dealing with $3^5 \times (3^3 \times 2^2)$, the problem is split into 2 subproblems of sizes $(3^3 \times 3^3 \times 2^1)$ and $(3^2 \times 3^2)$ simultaneously.

RULES for DECOMPOSITION

Generally, there are rules on breaking the original problem into a set of smaller problems. In this paper, since we deal with a heuristic algorithm, we don’t use a specific breaking rule. We experiment with few decomposition scenarios; the least cost scenario is taken as the optimum one. The author is motivated with the fact that the algorithm is numerically stable and experimentation is not expensive, although the number of function evaluations and iterations to optimum are generally high.

Another problem with 10 variables and 28 process combinations is too big to solve as one concrete problem. This needs further explanation. We have $3^{10} = 531,441$ combinations for the tolerance domain and $3^8 \times 2^2 = 26,244$ combinations for the process domain. This is clearly an expensive computational approach to solve using either computational local or global optimization techniques.

IDEA-1 The first idea is to decompose in a way with variables having the same number of process functions. For instance, if $X_1$ can be produced using 2 processes, then the process domain can be modeled using 2 levels. Similarly, if $X_1$ can be produced using 3 processes, then the process domain can be modeled using 3 levels. In this sense, more processes correspond to more levels for the variables.

IDEA-2 The second idea is to decompose in a way that allows the elimination of complete constraint functions. For instance, if a problem requires 2 stages; the first stage requires at least one binding constraint. We call the variables of this stage the entering variables. Once the optimum values of entering variables are reached, the second stage starts. This second stage will be subject to the original set of constraints less the constraints of stage 1.

IDEA-3 The third idea is to decompose in a way that allows mathematical manipulation of constraints. In this case, each stage will have new generated constraints. In this study, experimentation with various ideas allows more insight to the decomposition problem using our suggested approach.

DISCUSSION

Table 1 gives the 2-stage decomposition for the problem discussed. Stage 1 refers to the $1^{st}$ stage of decomposition process. Since only one constraint function is binding the optimization model, the value of constraint is given under the heading Constraint Values. The number of iteration per stage is also calculated. Several optimal process combinations result in the same objective value. For instance, $\{2, -, 1, -, 3\}$ means the variables 1, 3, 5 should be manufactured using process 2, 1 and 3 respectively. The optimum returned is $12.732$ in 110 iterations. Stage 2 deals with 2 variables and one constraint only. Variables 2 and 4 should be manufactured using process 3 and 2 respectively. The optimum returned is $18.543$ in 84 iterations. The “-” means that the corresponding variable is left until stage 2.

Delta1 and Z are two parameters discussed in another paper by Gadallah, 2000 and 2001. The 2-stage decomposition for different values of delta1 and Z (delta1 = 0.0008, 0.0007 and 0.000065, Z = 0.075, 0.060 and 0.05) are shown. The optimum objective functions and the optimum process combinations are numerically stable. The number of iterations generally changes as the value of delta1 and Z change. Accordingly, several experimentation of the stage decomposition is needed and the optima obtained are compared; the best of which is chosen.

The solution also includes the 2-stage decomposition for the same problem, with the exception that variables 1, 2 and 4 are the entering variables. The optimum returned in the $1^{st}$ stage is almost the same as earlier trials; the optimum returned in the $2^{nd}$ stage is higher than earlier trials. The number of trials to reach the optimum is comparable. The optimum process combinations are the same. This shows and confirms that different decompositions don’t yield unique solutions. This is in line with earlier results of Krishnamachari and Papalambros, 1997.
Questions can be asked: 
- How far can the modeler go with the fractionation without changing the nature of the problem? 
- What is the relationship between a nonlinear function and 2 & 3 level array? 
- What is the difference between 2-level and 3-level arrays and their effect on variance reduction? 
- What is the relationship between a nonlinear function and 2 & 3 level array?

It is obvious that the minimum number of experiments will be preferred; however, their information content varies. Since we deal with 2 search domains, Full and Fractionation of arrays have an effect on the optimum. Combination 1, 2, 3, 4, 5 and 6 represent different combinations of arrays. For instance, combination 1 has a full inner array ($= 3^5 = 243$ experiments) and maximum fractional outer array ($= ¼ x 108 = 27$). Combination 2 has $⅓$ FFE for the inner array and $⅕$ FFE for the outer array. Combination 3 has $⅕$ FFE for the inner array and $⅕$ FFE for the outer array. Combination 4 has $⅕$ FFE for the inner array and $½$ FFE for the outer array. Combination 5 has $⅕$ FFE for the inner array and FFE for the outer array. Combination 6 has FFE for the inner array and FFE for the outer array. Results for some of these experiments (Gadallah, 2001) indicated that the number of experiments and cost of solution increase with the increase in number of experiments.

Figure 3 shows the process cost versus combination number using standard L27OA. At each iteration, there are 27x27 experiments calculated. For the initial run, the variance of process cost is $\$ 40 - \$ 120. As the solution proceeds, the variance reduces to be in the range $\$ 20 - \$ 40 in iteration 5.

Figure 4 shows the number of cells versus dimensional tolerances per iteration. The purpose was to explore more meaningful results. As an observation, in iterations 1 and 2, each dimensional tolerance form a positive shifted normal distribution. In iteration 3, a uniform distribution is formed. In iterations 4 and 5, dimensional tolerances form a negative normal distribution.

Figure 5 shows the process cost versus combination number using L54OA. Of prime difference, the number of experiments is 27x54, which is double the number of experiments performed using L27OA.

Figure 6 shows the process cost versus combination number using the 5-point composite L27OA. This composite array serves to reduce the variance of larger size arrays without performing huge number of experiments. Figures 7 and 8 show similar responses using the 11-point and 9-point composite L27 OA respectively.

**CONCLUSION**

In this paper, 3 novel ideas are introduced and preliminary results are given, these are:
1. The decomposition scheme, as solution outlet for large-scale problems;
2. The variance reduction, resulting from the use of standard and non-standard orthogonal arrays;
3. The coupling effect of fractionation on the resulting optimum. The algorithm developed in (Gadallah, 2000 and

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**THE USE OF ORTHOGONAL ARRAYS in VARIANCE REDUCTION**

The near to global optimum algorithm (Gadallah, 2000) is modified to act as a variance reduction technique and used to simulate the effect of different arrays. The same example problem P1 is used for illustration. The process cost domains used several arrays; these are L27OA and L54OA. The L27OA is equivalent to $¼$ Full factorial Experiments. Similarly, the L54OA is equivalent to $½$ Full Factorial Experiments. An example problem with 5 variables was used to illustrate the procedure. Orthogonal arrays are arrays designed to host the variation of variables (and or their tolerances simultaneously equal number of times). When this requirement is met, the array is termed balanced array. Another property, orthogonality means the independence of variables during search. The use of orthogonal arrays in the engineering community is very limited especially in the area of quality by design. The author has experienced the potential of these arrays to minimize the variance within the norm of the optimization. In this paper, we are going to use orthogonal arrays and their derivatives (Full Factorial Experiments and Fractional Factorial Experiments) to minimize variance and bring the capability of dimensions to target values. These issues are of prime importance and are missing concerns in most of existing optimization tools.

Figure 2 shows an illustrative concept of fractional inner and fractional outer array and their effect on optimum. Several questions can be asked:

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**Figure 1: Objective Function versus System Level Iterations for Different Parameters and Entering Variables.**

Figure 1 shows the system objective function versus system level iteration for problem P1. The first curve details the optimization of stage 1 and second curve details the optimization of stage 2 respectively for different delta1 and Z respectively. The optima obtained is very close to those of the near to global optimum (Gadallah, 2000) and those obtained by Simulated Annealing (Zhang and Wang, 1993).

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**Figure 2: Illustrative Concept of Fractional Inner and Outer Array and Their Effect on Optimum.**

- Several questions can be asked:
  - How far can the modeler go with the fractionation without changing the nature of the problem?
  - What is the relationship between a nonlinear function and 2 & 3 level array?
  - What is the difference between 2-level and 3-level arrays and their effect on variance reduction?
  - How far can the modeler go with the fractionation without changing the nature of the problem?
show the potential of the method in several areas. These are:
- Optimization in 1-Dimension, 2-Dimension and more
- Precision, cost and quality of solution are linked together in this algorithm. Precision is essential in any optimization method. Full Factorial Experiment is the most expensive but accurate solution method. We should mention that the size of the Full Factorial Experiments is dependent on the number of variables per problem. The more the fractionation, the less the cost of experiments and the quality of solution. The author believes that this is the most tangible difference from any other algorithm.
- Variance reduction of the cost of processes L27OA and L54OA are used in this paper. The approach is very data intensive from the limited experimentation.
- The decomposition procedure discussed in this paper is illustrated for a linear set of constraints. Future work can tackle nonlinear constraints.
- We admit that the choice of L27OA and its multiples was merely ad hoc. We meant to reduce the experimentation effort, especially for an understood problem.
- We will continue to explore further the capability of this algorithm. So far, the algorithm, its modifications and results are novel. The engineering optimization community should welcome such algorithm.
- Comparison of results between L27OA and L54OA may not give the full picture without the use of larger arrays such as L81OA and L108OA. This is a task for future research. Further, we believe that the variance may not be enough to compare various arrays. The \(3^rd\) and \(4^th\) moments can be calculated further.

REFERENCES

<table>
<thead>
<tr>
<th>Stage</th>
<th>O-T</th>
<th>O-O-F</th>
<th>C-V</th>
<th>NoI</th>
<th>O-P-C</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 1</td>
<td>0.0040, 0.0049, 0.0059</td>
<td>$12.732</td>
<td>0.0149</td>
<td>110</td>
<td>([2, - , 1, - , 3], [2, - , 2, - , 1])</td>
<td>Delta1 = 0.0009, Z = 0.050, Tolerance on entering variables 1, 3 and 5 in 1st stage.</td>
</tr>
<tr>
<td>Stage 2</td>
<td>0.0040, 0.0045, 0.0049, 0.0055, 0.0059</td>
<td>$18.543</td>
<td>0.0249, 0.0149</td>
<td>84</td>
<td>([2, 3, 2, 1, 3], [2, 3, 2, 2, 1])</td>
<td>Tolerance on variables 2 and 4 in 2nd stage.</td>
</tr>
</tbody>
</table>

| Stage 1 | 0.0040, 0.0049, 0.0060 | $12.738 | 0.0149 | 69 | \([2, - , 2, - , 1]\) | Delta1 = 0.0008, Z = 0.075 |
| Stage 2 | 0.0040, 0.0045, 0.0049, 0.0055, 0.0060 | $18.547 | 0.0249, 0.0149 | 56 | \([2, 3, 2, 1]\) |

| Stage 1 | 0.0039, 0.0049, 0.0059 | $12.780 | 0.0149 | 125 | \([2, - , 2, - , 1]\) | Delta1 = 0.0007, Z = 0.060 |
| Stage 2 | 0.0039, 0.0045, 0.0049, 0.0055, 0.0059 | $18.541 | 0.0249, 0.0149 | 93 | \([2, 3, 2, 1]\) |

| Stage 1 | 0.0040, 0.0049, 0.0059 | $12.732 | 0.0149 | 256 | \([2, - , 2, - , 1]\) | Delta1 = 0.00065, Z = 0.050 |
| Stage 2 | 0.0040, 0.0045, 0.0049, 0.0055, 0.0059 | $18.544 | 0.0249, 0.0149 | 153 | \([2, 3, 2, 1]\) |

| Stage 1 | 0.0034, 0.0044, 0.0054 | $12.787 | 0.0149 | 33 | \([1, 2, - , 1, -]\) | Z=0.10 Delta1= 0.00090 |
| Stage 2 | 0.0034, 0.0044, 0.0057, 0.0054, 0.0058 | $19.545 | 0.0248, 0.0149 | 54 | \([1, 2, 2, 1, 3]\) |
| Stage 1 | 0.0034, 0.0044, 0.0054 | $12.695 | 0.0149 | 47 | \([1, 2, - , 1, -]\) | Delta1= 0.0008 Z= 0.075, Tolerance on entering variables 1, 2 and 4 in 1st stage. Tolerance on variables 3 and 5 in 2nd stage. |
| Stage 2 | 0.0034, 0.0044, 0.0057, 0.0054, 0.0058 | $19.462 | 0.0247, 0.0149 | 63 | \([1, 2, 2, 1, 3]\) |

N-o-I = Number of Iterations, C-V = Constraint Value, O-O-F = Optimum Objective Function, O-T = Optimum Tolerance, O-P-C = Optimum Process Combination

Table 1: Two-Stage Decomposition Results for Example Problem

| 1. L243/L27 OA (1/1 x 1/4) | 2. L81/L27 OA (1/3 x 1/4) | 3. L27/L27 OA (1/9 x 1/4) | 4. L27/L54 OA (1/9 x 1/2) | 5. L27/L108 OA (1/9 x 1) | 6. L243/L108 OA (1 x 1) |

# of experiments, Cost of Solution (Direction of Increase)

Figure 2: Illustrative Concept of Fractional Inner and Fractional Outer Array & Effect on Optimum
Figure 3: Process Cost Versus Combination Number Using L27 OA (27 x 27 points in each series)

Figure 4: Number of Cells Versus Dimensional Tolerances per Iteration
Figure 5: Process Cost Versus Combination Number using L54OA (54 x 27 points in each series)

Figure 6: Process Cost Versus Combination Number (27 x 27 5-point Approximation)
Figure 7: Process Cost Versus Combination Number (27 x 27 11-point Approximation)

Figure 8: Process Cost Versus Combination Number (27 x 27 - 9-point Approximation)