On complete representations of algebras of logic

MOHAMED KHALED AND TAREK SAYED-AHMED, *Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.*
E-mail: rutahmed@gmail.com

Abstract

We show that there exists an atomic polyadic equality algebra of dimension n that is elementary equivalent to a completely representable algebra, but its diagonal free reduct (obtained by deleting diagonals and substitutions) is not completely representable.

Throughout this note, n is a finite ordinal >2. The classes Df_n, CA_n, PEA_n, of diagonal free cylindric algebras, cylindric algebras and polyadic equality algebras of dimension n are defined in [4]. In [5], definition 29, a set of coloured graphs $\mathfrak{G}$ is defined, which we now recall:

Definition 1. *A coloured graph is an undirected graph Γ such that every edge of Γ is coloured by a unique edge colour (below), and some $(n-1)$ - tuples have unique colours, too. The edge colours are :*
greens: $g_i(i=1,\ldots,n-2)$ and $g_0^i(i<\omega)$;
whites: $w_i(i=0,\ldots,n-2)$;
reds: $r_i^m(i<m<\omega.)$
The colours for $(n-1)$ - tuples are :
yellows: $y_S(S\subseteq\omega, S=\omega$ or S finite).

We will sometimes write $\Gamma(x,y)$ for the colour of an edge (x,y), and $\Gamma(a_1,\ldots,a_{n-1})$ for the colour of an $(n-1)$-tuple $a_1,\ldots,a_{n-1}$ in the coloured graph Γ.

Let $\Gamma,\triangle$ be coloured graphs, and $\psi:\Gamma\to\triangle$ be a map. ψ is said to be coloured graph embedding, or simply an embedding, if it is one to one and preserves all edges, and all colours, where defined, in both directions.

Let $i<\omega$ and let Γ be a coloured graph consisting of n nodes, $x_0,\ldots,x_{n-2},y$, such that (x_j,y) is an edge of Γ for each $j<n-1$. We call Γ an $i-$*cone* if for each $j<n-1$, the edge (x_j,y) is coloured g_j if $j>0$, and g_0^i if $j=0$, and no other edge of Γ (if any) are coloured green. The apex of the cone is y, its base $\{x_0,\ldots,x_{n-2}\}$. The tint of the cone is i. These are well-defined, as any Γ can be viewed as a cone in at most one way. Notice that a cone induces a linear ordering on its base, namely, $x_0,\ldots,x_{n-2}$.

Definition 2. *The class $\mathfrak{G}$ consists of all coloured graphs Γ with the following properties.*

(1) Γ *is a complete graph (all possible edges are present)*
(2) Γ *contains no triangles of the following types:*
- (g,g',g^*) *any green coloures* g,g',g^*
- (g_i,g_i,w_i) *any* $i=1,\ldots,n-2$
- (g_0^j,g_0^k,w_0) *any* $j,k<\omega$

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doi:10.1093/jigpal/jzp007

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- $(r_i^m, r_j^{m'}, r_k^{m*})$ *unless* $m=m'=m^*$ *and* $|\{i,j,k\}|=3$.

(3) If $a_1,\dots,a_{n-2}\in\Gamma$ *are distinct, and no edge* $(a_i,a_j)(i<j<n-1)$ *is coloured green, then the sequence* $(a_1,\dots,a_{n-2})$ *is coloured a unique shade of yellow. No other* $(n-1)$ *- tuples are coloured yellow.*

(4) If $D=\{d_0,\dots,d_{n-2},\delta\}\subseteq\Gamma$ *and* $\Gamma\restriction_D$ *(the coloured graph induced on D) is an i-cone with apex* δ*, inducing the ordering* $d_0,\dots,d_{n-2}$ *on its base, and the tuple* $(d_0,\dots,d_{n-2})$ *is coloured* y_S*, then* $i\in S$.

Clearly, $\mathfrak{G}$ is closed under isomorphism and under induced subgraphs. $\mathfrak{G}$ depends on n.

In the next definition we show how these graphs can be used to construct a polyadic atom structure.

Definition 3. *Consider the class* K_n *of surjective maps* $a: n(=\{0,\dots,n-1\})\to\Gamma_a$*, any* $\Gamma_a\in\mathfrak{G}$*. Many of these maps, though formally distinct, will differ only because the nodes in the image graphs will not be the same. So we define an equivalence relation* $\sim$ *on* K_n

$$a\sim b\Longleftrightarrow a(i)=a(j)\Leftrightarrow b(i)=b(j)$$

and

$$\Gamma_a(a(i),a(j))=\Gamma_a(b(i),b(j)),$$

if defined, and

$$\Gamma_a(a(k_0),\dots a(k_{n-2}))=\Gamma_b(b(k_0)\dots b(k_{n-2})),$$

if defined, for all $i,j\in n$ *and* $(n-1)$*-tuples* $\bar{k}$ *of elements of* n*. In other words,* a *and* b *define isomorphic coloured graphs. This is an equivalence relation on* K_n*. We define an atom structure* $\mathcal{C}_n'$ *with domain*

$$C_n'=\{[a]: a\in K_n\}.$$

For every $i,j\in n$ *and* $[a],[b]\in C_n'$ *we define* $E_{ij}\subseteq C_n'$*,* $T_i\subseteq^2 C_n'$ *and* $P_{ij}\subseteq^2 C_n'$ *as follows:*

$$[a]\in E_{ij}\Longleftrightarrow a(i)=a(j)$$

$$[a]T_i[b]\Longleftrightarrow a[n\backslash\{i\}]\sim b[n\backslash\{i\}].$$

That is $[a]T_i[b]$ *if and only if for some* $c\in[a]$*,* $b(j)=c(j)$ *for all* $j\neq i$*. Finally*

$$[a]P_{ij}[b]\Longleftrightarrow b\circ[i,j]\sim a$$

These definitions are sound (do not depend on the representatives). Now consider the atom structure

$$\mathcal{C}_n'=(C_n',T_i,P_{ij},E_{ij})_{i,j<n}.$$

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Let $\mathcal{D}_n$ be the complex algebra over $\mathcal{C}'_n$. That is $D_n = \wp(C'_n)$. The boolean operations are the usual set theoretic intersections and taking complements; the extra non boolean operations are defined for $X \in \wp(C'_n)$ as follows

$$\mathsf{c}_i X = \{[b] \in C'_n : \exists [a] \in X \ [a] T_i [b]\},$$

$$\mathsf{p}_{ij} X = \{[b] \in C'_n : \exists [a] \in X \ [a] P_{ij} [b]\},$$

$$\mathsf{d}_{ij} = E_{ij.}$$

Let $\mathcal{C}_n$ to be the subalgebra of $\mathcal{D}_n$ generated by the atoms, i.e. by the set $\{[a] : a \in K_n\}$.

Lemma 4. *The algebra $\mathcal{C}_n$ is generated by $n-1$ dimensional elements*

Proof. It suffices to show that $\{[a]\} = \prod\{\mathsf{c}_i\{[a]\} : i < n\}$ for any $a \in K_n$. Assume that $a : n \to \Gamma$ with $\Gamma \in \mathfrak{G}$. Clearly $\leq$ holds. For the other direction assume that $b : n \to \Delta$ and $[a] \neq [b]$. We show that b cannot be an element of the right hand side. Since a and b are not equivalent, we can assume that

1. $(\exists i, j < n)\Delta(b(i), b(j)) \neq \Gamma(a(i), a(j))$ or
2. $(\exists i_1, \dots i_{n-1} < n)\Delta(b(i_1), \dots b(i_{n-1})) \neq \Gamma(a(i_1), \dots a(i_{n-1}))$.

In the first case, let k be distinct from i and j. Then $[b] \notin \mathsf{c}_k\{[a]\}$. In the second case, choose $k \notin \{i_1, \dots i_{n-1}\}$ and proceed the same way. ∎

We shall need the notion of atomic networks which are basically finite approximations to complete representations.

Definition 5. *Let $\mathcal{D}$ be an atomic arbitrary n-dimensional polyadic-type algebra. Let $At\mathcal{D}$ denotes the set of its atoms. An atomic network N is a set of nodes $\triangle$ and a total function $N : {}^n\triangle \to At(\mathcal{D})$ such that*

- $N(\bar{\delta}) \leq \mathsf{d}_{ij}$ *iff* $\delta_i = \delta_j$ *(for any $\bar{\delta} \in {}^n\triangle$ and any $i, j < n$).*
- $N(\bar{\delta}^i_d) \leq \mathsf{c}_i N(\bar{\delta})$ *(for any $i < n, \bar{\delta} \in {}^n\triangle, d \in \triangle$).*
- $\mathsf{p}_{ij} N(\delta) = N(\delta \circ [i, j].)$

A complete representation is a representation that preserves infinitary meets and joins whenever defined. In [5] it is proved that an algebra has a complete representation if and only if it has an atomic one. An atomic representation of $\mathcal{A}$ is a representation, i.e a map $f : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is a set algebra with unit ${}^nU$ such that ${}^nU = \bigcup\{f(x) : x \text{ is an atom of } A\}$. We now prove the main result of this paper:

Theorem 6. *$\mathcal{C}_n$ is elementary equivalent to a completely representable PEA_n, hence is representable, but its Df reduct has no complete representation.*

Proof. A "graph game" is defined between two players $\exists$ (female) and $\forall$ (male) in [5], cf. lemma 30. It is shown that for all $k < \omega$, $\exists$ has a winning strategy in the graph game of length k [5] proposition 33 while $\forall$ has a winning strategy for the graph game of length ω [5] proposition 32. Another game on networks $G_k(\mathfrak{D})$, $\mathfrak{D}$ is an atomic polyadic algebra, has k rounds, $k \leq \omega$, and is defined as follows. In the zero'th round, $\forall$ picks any atom a of $\mathfrak{D}$. $\exists$ has

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to respond with a finite atomic network N_0 such that $N_0(\bar{d})=a$ for some n-tuple of nodes $\bar{d}\in{}^{n_0}N$. Without loss, $|N_0|\leq n$. In any further round, let the last network played be N. ∀ picks an index $i<n$, a "face" $F=(f_0,f_1,\ldots,f_{n-2})\in N^{n-1}$, and an atom $b\leq \mathsf{c}_iN(f_0,\ldots f_{i-1},x,f_i\ldots,f_{n-2})$ (the choice of $x\in N$ is arbitrary, as the second part of the definition of an atomic network together with the fact that $\mathsf{c}_i(\mathsf{c}_ix)=\mathsf{c}_ix$ for all $x\in\mathcal{D}$ ensures that the right-hand side does not depend on it). ∃ must respond, if possible, with a network $N\subseteq N^+$ with at most one more node, such that there is a node $d\in N^+$ with $N^+(f_0,\ldots,f_{i-1},d,f_i\ldots,f_{n-2})=b$. If she can do this in every round, she has won the play. It is proved in [5], lemma 31, that ∃ has a winning strategy in the graph games of all finite lengths if and only if she has a winning strategy in the games $G_k(\mathcal{C}_n)$ and that ∃ has a winning strategy in the graph game of length ω if and only if she has a winning strategy in the game $G_\omega(\mathcal{C}_n)$. $\mathcal{C}_n$ is clearly countable. Now for all $k<\omega$, ∃ has a winning strategy σ_k in $G_k(\mathcal{C}_n)$. Let $\mathcal{B}$ be a non-principal ultrapower of $\mathcal{C}_n$. Then ∃ has a winning strategy σ in $G_\omega(\mathcal{B})$, essentially she uses σ_n in the nth component of the ultraproduct so that at each round of $G_\omega(\mathcal{B})$ ∃ is still winning in co-finitely many components, this suffices to show that she has still not lost. Now we can use an elementary chain argument to construct countable elementary subalgebras of $\mathcal{B}$ containing $\mathcal{C}_n$. $\mathcal{C}_n=\mathcal{A}_0\leq\mathcal{A}_1\ldots\leq\mathcal{B}$. For this let $\mathcal{A}_{i+1}$ be a countable elementary subalgebra of $\mathcal{B}$ containing $\mathcal{A}_i$ and all elements of $\mathcal{B}$ that σ selects in play of $G_\omega(\mathcal{B})$ in which ∀ only chooses elements from $\mathcal{A}_i$. Let $\mathcal{A}'=\bigcup_{i\in\omega}\mathcal{A}_i$. This is a countable elementary subalgebra of $\mathcal{B}$ and ∃ has a winning strategy in $G_\omega(\mathcal{A}')$. We prove that $\mathcal{A}'$ has a complete representation, this will prove that $\mathcal{C}_n$ is representable (since the class of representable algebras is a variety and $\mathcal{C}_n\equiv\mathcal{A}'$.) Consider a play $N_0\subseteq N_1\subseteq\ldots$ of $G_\omega(\mathcal{A}')$ in which ∃ uses her strategy, so that all the N_t are atomic networks, and ∀ eventually picks up every face $(f_0,\ldots f_{n-2})$ every $i<n$ and every atom b. That is ∀ plays every possible legal move in some stage of the play. He can do this because there are countably many nodes that appear in the play and countably many atoms in $\mathcal{A}'$. If ∃ uses her winning strategy to this particular game, then the limit network $M=\bigcup_{t<\omega}(N_t)$ will satisfy the following condition (*)

For every face $(f_0,\ldots f_{n-2})\in{}^{n-1}M$ for all $i<n$ for every atom b of A if $b\leq\mathsf{c}_iM(f_0,\ldots,x,f_{n-2})$, then there exists a node l such that $b=M(f_0,\ldots l,\ldots f_{n-2})$. For $r\in\mathcal{A}'$, let

$$h(r)=\{\bar{d}\in{}^nM:\exists t<\omega(\bar{d}\in{}^nN_t \text{ and } N_t(\bar{d})\leq r)\}.$$

For every $\bar{d}\in{}^nM$ there is a $t<\omega$ and an atom $a=N_t(\bar{d})$ such that $\bar{d}\in h(N_t(\bar{d}))$, so that h is an atomic representation. We check the boolean operations. We have $\bar{d}\in h(r+s)$ iff $\exists t<\omega$, such that $\bar{d}\in{}^nN_t$ and $N_t(\bar{d})\leq r+s$. Because $N_t(\bar{d})$ is an atom this is equivalent to $\exists t<\omega$, $\bar{d}\in{}^nN_t$ and $N_t(\bar{d})\leq r$ or $N_t(\bar{d})\leq s$. Equivalently $\bar{s}\in h(r)\cup h(s)$. Complement is just as easy: $\bar{d}\in h(-r)$ iff $d\notin h(r)$. Indeed assume that $\bar{d}\in h(-r)$. Then $\exists t_1<\omega$, $N_{t_1}(\bar{d})\leq -r$. $\bar{d}$ cannot be in $h(r)$ for else we get $t_2<\omega$ such that $N_{t_2}(\bar{d})\leq r$. By taking $t=max\{t_1,t_2\}$ it follows that $N_t(\bar{d})=0$ which is impossible because $N_t(\bar{d})$ is an atom. The reverse inclusion is the same. Now h preserves cylindrifications by (*). Indeed, the following follows from (*): there exists $t_1<\omega$ such that $N_{t_1}(\bar{d})\leq\mathsf{c}_ir$ iff there exists $t_2\in\omega$ such that $N_{t_2}(\bar{d}^i_u)\leq r$ for some u. Now we check substitutions. Let $\bar{d}\in h(\mathsf{p}_{ij}r)$. Then there exists $t<\omega$ such that $N_t(\bar{d})\leq\mathsf{p}_{ij}r$. Hence $\mathsf{p}_{ij}N_t(\bar{d})=N_t[\bar{d}\circ[i,j])\leq r$. The other inclusion is similar. Preserving diagonals follows from the definition of atomic networks. Now we show that the diagonal free reduct of $\mathcal{C}_n$ has no complete representation. Assume, seeking a contradiction, that it has a complete representation. Then since $\mathcal{C}_n$ is generated by $n-1$ dimensional elements, we have by theorems 5.1.51 and 5.4.26 in [4] that $\mathcal{C}_n$ as a polyadic equality algebra has a complete representation h. Then

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$\exists$ can use h as a guide and win $G_\omega(\mathcal{C}_n)$, which contradicts the above. Let the base of h be U. Any finite subset $N \subseteq U$ defines an atomic network by defining $N(\bar{d})$ to be the unique atom $a \in \mathcal{C}_n$ such that $\bar{d} \in h(a)$. Such an atom exists since the representation is atomic. It is easy to see that, so defined, N is a network. $\exists$ ensures that each network is played this way. For $t=0$ let $\forall$ choose an atom a. $\exists$ chooses $\bar{d} \in {}^nU$ with $\bar{d} \in h(a)$. She defines the atomic network by stating that its notes are $d_0, d_1 \ldots d_n$. Given the inductive hypothesis $N_t \subseteq U$ at round t, $\forall$ chooses $(f_0, \ldots f_{n-2}) \in {}^{n-1}N_t$, $i < n$ and atom $b \leq \mathsf{c}_i N_t(f_0, x, \ldots f_{n-2})$. Then $(f_0, x, \ldots f_{n-2}) \in h(\mathsf{c}_i b)$. Hence there exists $z \in U$ such that $(f_0, z, \ldots f_{n-2}) \in h(b)$. $\exists$ selects such a z and forms N_{t+1} by stating that its nodes are those of N_t together with z. This completes the proof. ∎

It follows that not only the class of completely representable cylindric algebra is not elementary as proved in [5], but for any class with signature between Df_n and PEA_n the class of completely representable algebras is not elementary.

Remark We note that the Df part of our theorem follows directly from the result in [5], together with lemma 4, and theorem 5.1.51 in [4]. Indeed in [5] $\mathcal{C}_n$ is proved to be elementary equivalent to $\mathcal{A}$ such that $\mathcal{A}$ has a complete representation but $\mathcal{C}_n$ does not. Let $\mathfrak{Rd}_{df}$ stand for the diagonal free reduct. So if one takes the diagonal free reducts, then $\mathfrak{Rd}_{df}\mathcal{C}_n$ is still elementary equivalent to $\mathfrak{Rd}_{df}\mathcal{A}$, the latter has a complete representation but the former does not by theorem 5.1.51. One thus might be tempted to think that the PEA part of our theorem follows from [5] together with lemma 4 and theorem 5.4.26 in [4]. But this is *not* true. Let us see why. In this case we need to *expand* the cylindric algebras of [5] to be polyadic algebras. But this done in the definition of $\mathcal{C}_n$ above. However we have not proved that $\mathcal{C}_n$ is actually a PEA_n, all we know (before Theorem 6) is that it has the similarity type of PEA_n, the fact that it is a PEA_n and indeed a representable one follows from the proof of Theorem 6. Even if we prove that $\mathcal{C}_n$ is a PEA_n before Theorem 6, which is not a trivial matter (but can be done), all we know is that it does not have a complete representation as a PEA_n. But then how shall we find an $\mathcal{A} \in PEA_n$ such that $\mathcal{A} \equiv \mathcal{C}_n$ and $\mathcal{A}$ has a complete representation. [5] provides us only with a CA_n with the required properties, and we do not know anything about the generators of this CA_n, nor does the proof therein shows that it is a reduct of a PEA_n. In other words, we cannot directly use Theorem 5.4.26. So it seems to us that the longer proof of Theorem 6 is required to prove our result. Besides Theorem 6 shows that *any* class of algebras with signature between Df_n and PEA_n is not elementary. Other algebras of logic to which our result applies are Pinter's substitution algebras (SC), and Halmos polyadic algebras (PA).

Using the above one can easily show that there is a *simple* countable representable atomic PEA_n whose Df reduct is not completely representable. Such a result can be used to prove that both the omitting types theorem and Vaught's theorem on the existence of atomic models [2] fail for finite variable fragments of first order logic as long as the number of variables >2 [1]. In contrast we have every atomic representable $\mathcal{A} \in PEA_2$ is completely representable. We sketch a proof. By theorem 3.2.65 in [4] $\mathcal{A}$ has no defective atoms, since it is representable. Define Dat , small, big, A_{ab} as in lemma 3.2.59. For $a \in Dat$, let $X_a = \{(a, i) : i < \mu\}$ where $\mu = |At\mathcal{A}| + \omega$. Let $U = \bigcup_{a \in Dat} X_a$. A mapping ϕ from $At\mathcal{A}$ to $\wp({}^2U)$ is defined by defining $\phi \restriction A_{ab}$ in [4] lemma 5.4.3. Then for any $x \in A$, let

$$f(x) = \bigcup \{\phi(a) : a \leq x, a \in Dat\}.$$

Then clearly f is an atomic representation. The preservation of all operations other than the diagonal elements can be done as in theorem 5.4.32. The preservation of diagonal elements

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can be done by adapting the corresponding part in lemma 3.2.59. On the other hand, using the techniques in lemma 3.2.59, theorem 3.2.65, lemma 5.1.46, theorem 5.4.33 in [4], it can be proved without much difficulty that for $n \leq 2$ and $K \in \{Df, CA, PA, PEA, SC\}$ the class of completely representable algebras in K_n is elementary. In fact this class coincides with the class of atomic representable algebras. In the cases of Df, PA, SC the completely representable algebras are, like boolean algebras, exactly the atomic algebras. For $K \in \{CA, PEA\}$ atomic algebras can contain *defective* atoms under the diagonal element, cf. [4] lemma 3.2.59, that prevents building an ordinary representation in which the unit is a disjoint union of cartesian squares. Representable algebras, however, do not contain such defective atoms.

To summarize we have:

Theorem 7. *Let $n \in \omega$. Let K be a signature between Df_n and PEA_n. Then the class of completely representable K algebras is elementary if and only if $n \leq 2$, in which case this class coincides with the (elementary) class of atomic representable algebras.*

We believe that Theorem 7 is a significant addition to the results in [5]. The if part is new even for cylindric algebras.

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Received 15 October 2008

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