VARIOUS NOTIONS OF REPRESENTABILITY FOR CYLINDRIC AND POLYADIC ALGEBRAS

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Abstract

For $\beta$ an ordinal, let $\text{PEA}_\beta$ ($\text{SetPEA}_\beta$) denote the class of polyadic equality (set) algebras of dimension $\beta$. We show that for any infinite ordinal $\alpha$, if $A \in \text{PEA}_\alpha$ is atomic, then for any $n < \omega$, the $n$-neat reduct of $A$, in symbols $\mathcal{N}_n A$, is a completely representable $\text{PEA}_n$ (regardless of the representability of $A$). That is to say, for all non-zero $a \in \mathcal{N}_n A$, there is a $B_a \in \text{SetPEA}_n$ and a homomorphism $f_a: \mathcal{N}_n A \to B$ such that $f_a(a) \neq 0$ and $f_a(\sum X) = \bigcup_{x \in X} f_a(x)$ for any $X \subseteq A$ for which $\sum X$ exists. We give new proofs that various classes consisting solely of completely representable algebras of relations are not elementary; we further show that the class of completely representable relation algebras is not closed under $\equiv_\infty$, $\omega$. Various notions of representability (such as ‘satisfying the Lyndon conditions’, weak and strong) are lifted from the level of atom structures to that of atomic algebras and are further characterized via special neat embeddings. As a sample, we show that the class of atomic $\text{CA}_n$s satisfying the Lyndon conditions coincides with the class of atomic algebras in $\text{ElS}_n \mathcal{N}_\infty \text{CA}_\omega$, where $\text{El}$ denotes ‘elementary closure’ and $S_\omega$ is the operation of forming complete subalgebras.

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1. Introduction

It is well known that every Boolean algebra (satisfying a finite set of equations) is isomorphic to a field of sets, that is to say, every Boolean algebra is representable in some concrete sense, where the Boolean meets and joins are interpreted respectively as set theoretic intersections and unions. This result, better known in the literature as Stone’s Theorem, is equivalent (in ZFC) to the completeness of propositional logic. But in the case of cylindric and polyadic algebras of various dimensions the ‘representation problem’ is somewhat more involved. For example not every cylindric algebra of dimension > 1 is representable as a genuine field of sets with cylindrifications interpreted as projections, and in fact, the class of representable cylindric algebras of dimension > 2, though a variety, cannot be axiomatized by finite schema of equations. Nevertheless, Tarski proved that in certain significant cases, the finitely many cylindric algebra axioms enforce representability. In this connection, Tarski proved that every locally finite dimensional cylindric algebra of infinite dimension is representable, and this in turn is equivalent to the completeness theorem of first order logic proved earlier by Gödel. Here, ‘local finite dimensional’ is an algebraic condition reflecting the fact that formulas considered have finite length.

Independently, polyadic algebras were introduced by Halmos to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos’ representation theorem for locally finite dimensional polyadic algebras (with and without equality). The last Theorem is essentially equivalent to Tarski’s representation Theorem for locally finite cylindric algebras, a fortiori to Gödel’s completeness Theorem. Daigneault and Monk proved a tremendous extension of Halmos’ theorem. Removing the condition of local-finite dimensional, they show that, every polyadic algebra of infinite dimension without equality is representable [3], providing in this manner a purely algebraic proof of the completeness and compactness of Keisler’s logics where formulas of infinite length as well as infinite quantification are allowed. Complexity issues (such as undecidability of axiomatizations) were approached in [13]. The equational theory of $\text{PA}_\omega$ is not even recursively enumerable. Adding equality proves even more problematic [12]. The last two relatively recent results are due to Sági, and Németi and Sági, respectively.

This is a profound achievement from the algebraic point of view for the class of locally finite dimensional algebras, the algebraic counterpart of first order logic, has some serious defects when treated as the sole subject of research in an autonomous algebraic theory. In universal algebra one prefers to deal with equationally defined classes, i.e. varieties. Classes of algebras which are not varieties are often introduced in discussions as specialized subclasses of varieties. One often treats fields as a special case of rings. This
is due to the tradition that in algebra, it is mainly the equational language and thus equational logic that is used. Finding an equational form for an algebraic entity is always a value on its own right. Another reason for this preference, is the fact that every variety is closed under the operations of forming subalgebras homomorphic images, and direct products. Local finite dimensional does not have the form of an equation nor can it be equivalently replaced by any family of equations, nor indeed any set of first order axioms.

However, this algebraic leap from the locally finite dimensional case to the general unrestricted case does not work - as far as preservation of representability is concerned- with infinite dimensional cylindric algebras nor with polyadic algebras with equality: For every infinite ordinal \( \alpha \), there are non-representable cylindric and polyadic equality algebras of dimension \( \alpha \). Furthermore, the class of representable polyadic equality algebras of infinite dimension is not first order axiomatizable (it is not closed under ultrapowers), least a variety [12]. This dichotomy between polyadic algebras and polyadic equality algebras in connection to deep representability results has been the ground for immense research illuminating the theory of both topics.

In the realm of representable algebras, there are several types of representations. Ordinary representations are just isomorphisms from Boolean algebras with operators to a more concrete structure (having the same signature) whose elements are sets endowed with set-theoretic operations like intersection and complementation [12, 17]. Complete representations, on the other hand, are representations that preserve arbitrary conjunctions whenever defined [8, 21]. More generally consider the following question: Given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? A complete representation would thus be one that preserves all existing meets (finite of course and infinite).

Here we are assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations. When the algebra in question is countable, and we have only countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set meets or non-principal types. When the algebra in question is atomic, then a representation omitting the non-principal type consisting of co-atoms, turns out to be a complete representation. This follows from the following result due to Hirsch and Hodkinson: A Boolean algebra \( \mathfrak{A} \) has a complete representation \( f: \mathfrak{A} \to \langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle \) (\( f \) is a 1-1 homomorphism and \( X \) a set) if and only if \( \mathfrak{A} \) is atomic and \( \bigcup_{x \in \text{At} \mathfrak{A}} f(x) = X \), where \( \text{At} \mathfrak{A} \) is the set of atoms of \( \mathfrak{A} \).

The notion of complete representations seems to be strikingly a second order one. This intuition is confirmed in [8] for cylindric algebras where it is proved that the classes of such completely representable cylindric algebras of dimension at least three and that of relation algebras are not elementary. The result is extended to other cylindric-like algebras in [21], like diagonal free cylindric algebras and polyadic algebras (with and without equality) of finite dimension \( > 2 \).
and relation algebras were proved by Hirsch and Hodkinson using so-called rainbow algebras \[8\] and analogous techniques are used in \[21\]; in this paper we present entirely different proofs for all such results and some more closely related ones using so called Monk-like algebras re proving the results in \[8, 21\]. Our proof depends essentially on some form of an infinite combinatorial version of Ramsey’s Theorem. But running to such conclusions – concerning (non-)first order definability – can be reckless and far too hasty; for in other non-trivial cases the notion of complete representations turns \textit{not to be} a genuinely second order one; it is definable in first order logic. The class of completely representable Boolean algebras is elementary; it simply coincides with the class of atomic, completely additive algebras. It is not hard to show that, like atomicity, complete additivity can indeed be defined in first order logic \[22\]. Nevertheless, again adding equality causes a noticeable discrepancy. Although complete additivity, which is a necessary condition for complete representability where sums are unions, is retrieved, there are polyadic equality algebras of infinite dimension that are not even representable, let alone completely representable \[12\].

For some odd reason, historically the underlying intuition of the notion of complete representability progressed in a different direction. The correlation of (the first order property of) atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a while, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon’s error. But in retrospect, one can safely say by gathering and scrutinizing recent results that the first order definability of the the notion of complete representations heavily depends on the algebras required to be completely represented. In other words, the (possibly slippery) notion of ‘complete representations’ needs a context to be fixed one way or another, and it is surely unwise to declare a verdict without a careful and thorough investigation of the specific situation at hand. The elementary closure of the class of completely representable relation and cylindric algebras of dimension > 2 has been studied in some depth by Hirsch and Hodkinson. This class is characterized by the so-called Lyndon conditions. For each \(k\), there is a \(k\)th Lyndon condition, \(\rho_k\) which is a first order sentence coding a winning strategy in a deterministic \(k\)-rounded Ehrenfeucht–Fraïssé game between two players \(\exists\) and \(\forall\); the \(\rho_k\)s taken together axiomatize this class. All of the \(\rho_k\)s are needed for the axiomatization of this class, for it is not finitely axiomatizable. Fix finite \(n > 2\). Let \(\text{CRCA}_n\) denote the class of completely representable \(\text{CA}_n\)s and \(\text{LCA}_n = \text{ECRCA}_n\) be the class of algebras satisfying the Lyndon conditions. Lyndon’s error can be succinctly expressed using modern terminology as identifying the non-elementary class \(\text{CRCA}_2\) with its elementary closure, namely, \(\text{LCA}_n\). For a class \(K\) of Boolean algebras with operators, let \(K \cap \text{AT}\) denote the class of atomic algebras in \(K\).
By modifying the games coding the Lyndon conditions allowing $\forall$ to reuse the pebble pairs on the board, we will show (as indicated in the last part of the abstract) that $\text{LCA}_n = \text{ELCRCA}_n = \text{ElS}_n \text{Nr}_n \text{CA}_n \cap \text{At}$.

We will also show, as stated in the first part of the abstract, that for any atomic $\mathfrak{A} \in \text{PEA}_\alpha$ (even if non-representable), for any $n < \omega$, $\text{Nr}_n \mathfrak{A}$ is completely representable. We do not know whether an analogous situation holds for atomic $\text{CA}_\alpha$s, but for any infinite ordinal $\alpha$, and for any infinite cardinal $\kappa$, we will construct an atomless $\mathfrak{C} \in \text{CA}_\alpha$ such that for all $2 < n < \omega$, $\text{Nr}_n \mathfrak{C}$ is atomic having $2^n$ many atoms, but $\text{Nr}_n \mathfrak{C}$ is not completely representable. Using the constructed $\mathfrak{C}$, together with the proven fact that for any $n < \omega$ $\text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq \text{LCA}_n$ and the known one that $\text{LCA}_n = \text{ELCRCA}_n$ for all $n$, we immediately deduce that for $2 < n < \omega$, $\text{CRCA}_n$ is not elementary. Call an atomic algebra $\mathfrak{A} \in \text{CA}_n$ strongly representable $\iff$ if its Dedekind–MacNeille completion, namely, the complex algebra of its atom structure $(\text{CmAt}\mathfrak{A})$ is representable. If $\mathfrak{A}$ is strongly representable, then $\mathfrak{A}$ is representable, because $\mathfrak{A} \subseteq \text{CmAt}\mathfrak{A}$ and $\text{RCA}_n$ being a variety is closed under forming subalgebras. However, the converse is not true in general, due to the fact that $\text{RCA}_n$ is not atom-canonical: There is a countable atomic $\mathfrak{A} \in \text{RCA}_n$ such that $\text{CmAt}\mathfrak{A} \notin \text{RCA}_n$; furthermore this $\mathfrak{A}$ cannot possess a complete representation for a complete representation of $\mathfrak{A}$ induces a representation of $\text{CmAt}\mathfrak{A}$. Hirsch and Hodkinson prove that, like $\text{CRCA}_n$, the class of strongly representable $\text{CA}_n$s, call it $\text{SRCA}_n$ is not elementary either. If $\mathfrak{A} \in \text{LCA}_n$, then $\text{CmAt}\mathfrak{A} \in \text{LCA}_n$, since $\mathfrak{A}$ and $\text{CmAt}\mathfrak{A}$ share the same atom structure satisfying the Lyndon conditions relativized to atoms. Thus the following inclusions are valid: $\text{CRCA}_n \subseteq \text{LCA}_n \subseteq \text{SRCA}_n \subseteq \text{RCA}_n \cap \text{At}$, and it can be shown that the inclusions are strict. The classes of atom structures of these classes are studied by Hirsch and Hodkinson [10]. In analogy to the class of algebras satisfying the Lyndon conditions, such various notions of representability will be connected to neat embedding properties. Analogous results are obtained for relation algebras by replacing $\text{Nr}_n$ by the operation $\text{Ra}$ of forming relation algebra reducts of cylindric algebras.

1.1. Layout

In Section 1, we recall the basic notions and concepts to be used in the sequel and we fix the notation. In Section 2, we prove the first result in the abstract, and we prove analogous results for various reducts of $\text{PEA}_\alpha$s. In Section 3, we show that the class of completely representable cylindric algebras and relation algebras are not elementary; for the latter class we go further by showing that it is not even closed under $\equiv_{\infty,\omega}$. In the last section we relate various notions of representability to neat embeddings proving the last result in the abstract and (much) more.
2. Preliminaries

We use throughout the paper fairly standard or/and self-explanatory notation following mainly the notation of [1,6] for cylindric algebras (and its relatives) and the textbook [9] for relation algebras. We shall have the occasion to deal with (in addition to CA\(\alpha\)s), the following cylindric-like algebras [1]: Sc short for Pinter’s substitution algebras, QA(QEA) short for quasi-polyadic (equality) algebras and PA(PEA) short for polyadic (equality) algebras. For \(K\) any of these classes and \(\alpha\) any ordinal, we write \(K_\alpha\) for the variety of \(\alpha\) dimensional \(K\) algebras, and \(RK_\alpha\) for the class of representable \(K_\alpha\)s, which happens to be a variety too (that is not finitely axiomatizable for \(\alpha > 2\)). The standard reference for all the classes of algebras mentioned previously is [6]. For a given ordinal \(\alpha\), each class in \(\{Sc_\alpha, QA_\alpha, QEA_\alpha, PA_\alpha, PEA_\alpha\}\) consists of BAOs with extra operators, as shown in figure 1, where \(d_{ij}\) is a nullary operator (constant), \(c_i, s_i^j, s_{[i,j]}\) are unary operators, for \(i, j < \alpha\) and \(\tau: \alpha \to \alpha\). Each class is defined by a finite schema of equations schema; in particular, each such class is a variety. Existing in a somewhat scattered form in the literature, equations defining \(Sc_\alpha\), \(QA_\alpha\), and \(QEA_\alpha\) are given explicitly in the appendix of [11]. For \(PA_\alpha\) and \(PEA_\alpha\), \(\alpha\) an infinite ordinal, we follow the equational axiomatization adopted in [6]. QA\(_n\) (QEA\(_n\)) are
class | extra operators
--- | ---
Sc\(_\alpha\) | \(c_i, s_i^j: i, j < \alpha\)
CA\(_\alpha\) | \(c_i, d_{ij}: i, j < \alpha\)
PA\(_\alpha\) | \(c_i, s_\tau: i < \alpha, \tau \in ^\alpha\alpha\)
PEA\(_\alpha\) | \(c_i, d_{ij}, s_\tau: i, j < \alpha, \tau \in ^\alpha\alpha\)
QA\(_\alpha\) | \(c_i, s_i^j, s_{[i,j]}: i, j < \alpha\)
QEA\(_\alpha\) | \(c_i, d_{ij}, s_i^j, s_{[i,j]}: i, j < \alpha\)

Fig. 1. Non-Boolean operators for the classes
term definitionally equivalent to \(PA_n(PEA_n)\) for finite \(n\), but for \(n \geq \omega\), they are not quite like \(PA_n(PEA_n)\), for their signature contains only substitutions indexed by replacements and transpositions. Let \(\alpha < \beta\) be ordinals. If \(\mathcal{B} \in CA_\beta\), then the \(\alpha\)-neat reduct of \(\mathcal{B}\), is denoted by \(Nr_\alpha \mathcal{B}\) [6, Definition 2.6.28]. If \(\mathfrak{A} \subseteq Nr_\beta \mathcal{B}\), we say that \(\mathfrak{A}\) neatly embeds in \(\mathcal{B}\), and that \(\mathcal{B}\) is a \(\beta\)-dilation of \(\mathfrak{A}\), or simply a dilation of \(\mathfrak{A}\) if \(\beta\) is clear from context. For \(K \subseteq CA_\beta\), \(Nr_\alpha K = \{Nr_\alpha \mathcal{B}: \mathcal{B} \in K\}(\subseteq CA_\alpha\).

For operators on classes of algebras, \(S, P, H\) denote the operations of forming subalgebras, products and homomorphic images, respectively. Following [6, Definition 3.1.2], we let \(Cs_\alpha\) denote the class of set algebras of dimension \(\alpha\), and \(Gs_\alpha\) denote the class of generalized set algebras of dimension \(\alpha\), so that \(IGs_\alpha = RCA_\alpha\) and \(SPCs_\alpha = RCA_\alpha\); here \(I\) denotes the operation
of closing under isomorphic copies. An algebra $\mathfrak{A} \in \mathcal{CA}_\alpha$ is \textit{completely representable} $\iff$ there exists an isomorphism $f: \mathfrak{A} \to \mathfrak{B}$, with $\mathfrak{B} \in \mathcal{G}_\alpha$ such that $f(\bigwedge X) = \bigcap_{x \in X} f(x)$, whenever $\bigwedge X$ exists in $\mathfrak{A}$. This implies that $\mathfrak{A}$ is atomic and that $f$ is atomic in the sense that $\bigwedge x \in \text{At}_A f(x) = 1_B$, where (recall that) $\text{At}_A$ denotes the set of atoms of $\mathfrak{A}$ [8].

The definition of neat reducts for $\mathcal{Sc}$, $\mathcal{QA}_s$, and $\mathcal{QEA}_s$ is entirely analogous to $\mathcal{CA}_s$. For $\beta$ an infinite ordinal and $\alpha < \beta$, the $\alpha$-neat reduct of $\mathfrak{A} \in \mathcal{PA}_\beta$ or $\mathfrak{A} \in \mathcal{PEA}_\beta$, denoted also by $\text{Nr}_\alpha \mathfrak{A}$, is defined to have domain $\text{Nr}_\alpha \mathfrak{A} = \{a \in \mathfrak{A} : c(\beta \sim \alpha) a = a\}$, cf. [6, Definition 5.4.16]. Observe that this domain is different from $\mathfrak{B} = \{a \in \mathfrak{A} : c_i a = a, \forall i /\in \alpha\}$. It is the case that $\text{Nr}_\alpha \mathfrak{A} \subseteq \mathfrak{B}$ always, but the converse inclusion may not be true if $\beta \sim \alpha$ is infinite. The notion of complete representations for all other cylindric-like algebras dealt with in this paper is defined similarly to the $\mathcal{CA}$ case.

Let $K$ be any class between $\mathcal{Sc}$ and $\mathcal{PEA}$. Then $\text{CRK}_\alpha$ denotes the class of completely representable $K_\alpha$s, and $\text{CRRA}$ denotes the class of completely representable RAs. It is known that for finite $\alpha > 2$, $\text{CRK}_\alpha$ and $\text{CRRA}$ are pseudo-elementary but not elementary; new proofs of the last fact will be given below. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\text{BAO}_s$ having the same signature. We write $\mathfrak{A} \subseteq_d \mathfrak{B}$ if $\mathfrak{A}$ is a dense subalgebra of $\mathfrak{B}$, that is to say, for all non-zero $b \in \mathfrak{B}$, there is a non-zero $a \in \mathfrak{A}$, such that $a \leq b$. We write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. By the latter, we understand that if $X \subseteq \mathfrak{A}$, then $\bigwedge^\mathfrak{A} X = 1 \implies \bigwedge^\mathfrak{B} X = 1$; in particular, it can (and will) be proved that $\mathfrak{A} \subseteq_d \mathfrak{B} \implies \mathfrak{A} \subseteq_c \mathfrak{B}$. Throughout the paper $S_d(S_c)$ denotes the operation applied to classes of $\text{BAO}_s$ of forming dense (complete) subalgebras, so that for any such $K$, the following inclusions hold: $K \subseteq S_d K \subseteq S_c K$. For a $\text{BAO}$ $\mathfrak{A}$, we denote its canonical extension by $\mathfrak{A}^+$; if $\mathfrak{A}$ is atomic, we denote its atom structure by $\text{At}_\mathfrak{A}$. We write $\mathfrak{Cm}$ for the operation of forming complex algebras.

If $K$ is a class of $\text{BAO}_s$, we denote by $K \cap \text{At}$ the class of atomic algebras in $K$; for example $\text{CRCA}_\alpha \cap \text{At} = \text{CRCA}_\alpha$. Observe that for any elementary class $K$, $K \cap \text{At}$ is elementary, as atomicity can be expressed by a first order sentence. For a class $K$ of first order structures, $\text{El}_K$ denotes the elementary closure of $K$. Suprema is denoted throughout the paper by $\bigvee$. By the Keisler Shelah ultrapower Theorem $\text{El}_K = \bigcup \text{Up}_K$, where $\bigcup$ is the operation of forming ultraproducts while $\bigcup$ is the operation of forming ultraroots. In particular, the class $K$ (in a first order signature) is elementary $\iff K = \text{El}_K \iff K$ is closed under $\text{Up}$ and $\text{Ur} \iff K$ is closed under $\text{Up}$ and elementary equivalence.
3. Completely representable algebras of every finite dimension from a single atomic infinite dimensional PEA

Although for any infinite ordinal \( \alpha \), there are atomic PEA\(_\alpha\)s that are not representable, let alone completely representable [3, 6], if we take finite dimensional neat reducts of any atomic PEA\(_\alpha\) the situation changes drastically. From one such (atomic) PEA\(_\alpha\) \((\alpha \geq \omega)\), we obtain a plethora of completely representable algebras; for each \( n < \omega \) a completely representable QEA\(_n\) together with its complete subalgebras.

**Theorem 3.1.** Let \( \alpha \) be an infinite ordinal and \( n < \omega \). If \( \mathcal{D} \in \text{PEA}_\alpha \) is atomic, then any complete subalgebra of \( \mathcal{N}_r \mathcal{D} \) is completely representable. Expressed more succinctly, \( S_n \mathcal{N}_r (\text{PEA}_\alpha \cap \text{At}) \subseteq \text{CRQEA}_n \).

**Proof.** We identify set algebras with their domain. Assume that \( \mathfrak{A} \subseteq \mathcal{N}_r \mathcal{D} \), where \( \mathcal{D} \in \text{PEA}_\alpha \) is atomic. Let \( c \in \mathfrak{A} \) be non-zero. We will find a homomorphism \( f: \mathfrak{A} \to \varphi(\mathcal{N}U) \) such that \( f(c) \neq 0 \), and \( \bigcup_{y \in Y} f(y) = \varphi(\mathcal{N}U) \), whenever \( Y \subseteq \mathfrak{A} \) satisfies \( \sum_{i=1}^{\mathfrak{A}} Y = 1 \). Assume for the moment (to be proved in a while) that \( \mathfrak{A} \subseteq \mathcal{D} \). By [9, Lemma 2.16] \( \mathfrak{A} \) is atomic because \( \mathcal{D} \) is.

For brevity, let \( X = \mathcal{A} \mathcal{E} \mathcal{A} \). Let \( m \) be the local degree of \( \mathcal{D} \), \( c \) its effective cardinality and let \( \beta \) be any cardinal such that \( \beta \geq \epsilon \) and \( \sum_{s} \beta^s = \beta \); such notions are defined in [3].

We can assume that \( \mathcal{D} = \mathcal{N}_r \mathcal{B} \), with \( \mathcal{B} \in \text{PEA}_\beta \) [6, Theorem 5.4.17]. For any ordinal \( \mu \in \beta \), and \( \tau \in \mu \beta \), write \( \tau^+ \) for \( \tau \cup \text{Id}_{\beta \mu} (\in \beta \beta) \). Consider the following family of joins evaluated in \( \mathcal{B} \), where \( p \in \mathcal{D} \), \( \Gamma \subseteq \beta \) and \( \tau \in \alpha \beta \):

\((*) \quad c(\Gamma)p = \sum_{s}^{\mathcal{B}} s_{\tau}p: \quad \tau \in \omega \beta \), \( \tau \uparrow \alpha \setminus \Gamma = \text{Id} \), and \((**) \quad \sum_{\tau} s_{\tau}X = 1 \). The first family of joins exists [3, Proof of Theorem 6.1], and the second exists, because \( \sum_{\tau} X = \sum_{\tau} X = \sum_{\tau} X = 1 \) and \( \tau^+ \) is completely additive, since \( \mathcal{B} \in \text{PEA}_\beta \).

The last equality of suprema follows from the fact that \( \mathcal{D} = \mathcal{N}_r \mathcal{B} \subseteq \mathcal{B} \) and the first from the fact that \( \mathfrak{A} \subseteq \mathcal{D} \). We prove the former, the latter is exactly the same replacing \( \alpha \) and \( \beta \), by \( n \) and \( \alpha \), respectively, proving that \( \mathcal{N}_r \mathcal{D} \subseteq \mathcal{D} \), hence \( \mathfrak{A} \subseteq \mathcal{D} \). We prove that \( \mathcal{N}_r \mathcal{B} \subseteq \mathcal{B} \).

Assume that \( S \subseteq \mathcal{D} \) and \( \sum S = 1 \), and for contradiction, that there exists \( d \in \mathcal{B} \) such that \( s \leq d < 1 \) for all \( s \in S \). Let \( J = \Delta \mathcal{D} \setminus \omega \) and take \( t = -c_{(\beta)}(-d) \in \mathcal{D} \). Then \( c(\beta \alpha) t = c(\beta \alpha)(-c_{(\beta)}(-d)) = c(\beta \alpha) - c_{(\beta)}(-d) = c_{(\beta \alpha)} - c_{(\beta \alpha)} c_{(\beta)}(-d) = c_{(\beta \alpha)}(-d) - c_{(\beta \alpha)} c_{(\beta)}(-d) = -c_{(\beta \alpha)} c_{(\beta)}(-d) = -c_{(\beta \alpha)}(-d) = -c_{(\beta \alpha)}(-d) = -c_{(\beta \alpha)}(-d) = t \). We have proved that \( t \in \mathcal{D} \). We now show that \( s \leq t < 1 \) for all \( s \in S \), which contradicts \( \sum S = 1 \). If \( s \in S \), we show that \( s \leq t \). By \( s \leq d \), we have \( s \cdot -d = 0 \). Hence by \( c(\beta) s = s \), we get \( 0 = c_{(\beta)}(s \cdot -d) = s \cdot c_{(\beta)}(-d) \), so \( s \leq -c_{(\beta)}(-d) \). It follows that \( s \leq t \) as required. Assume for contradiction that \( 1 = c_{(\beta)}(-d) \). Then \( c_{(\beta)}(-d) = 0 \), so \( -d = 0 \) which contradicts that \( d < 1 \). We have proved that \( \sum S = 1 \), so \( \mathcal{D} \subseteq c \mathcal{B} \).
Let $F$ be any Boolean ultrafilter of $\mathcal{B}$ generated by an atom below $a$. We show that $F$ will preserve the family of joins in (*) and (**). We use a topological argument borrowed from [22]. One forms nowhere dense sets in the Stone space of $\mathcal{B}$ corresponding to the aforementioned family of joins as follows: The Stone space of (the Boolean reduct of) $\mathcal{B}$ has underlying set $S$, the set of all Boolean ultrafilters of $\mathcal{B}$. For $b \in \mathcal{B}$, like before, let $N_b$ be the clopen set $\{F \in S : b \in F\}$. The required nowhere dense sets are defined for $\Gamma \subseteq \beta$, $p \in \mathcal{D}$ and $\tau \in \alpha \beta$ via: $A_{\Gamma,p} = N_{e_{\Gamma,p}} \setminus \bigcup_{\tau: \alpha \rightarrow \beta} N_{A_{\tau,p}}$, and $A_{\tau} = S \setminus \bigcup_{x \in X} N_{x_{\alpha}+x}$. The principal ultrafilters are isolated points in the Stone topology, so they lie outside the nowhere dense sets defined above. Hence any such ultrafilter preserve the joins in (*) and (**). Fix a principal ultrafilter $F$ with $a \in F$. Define the equivalence relation $E$ on $\beta$ by setting $iEj \iff d_{ij}^B \in F(i, j \in \beta)$. Define $f: \mathfrak{A} \to \wp(\mathfrak{a}(\beta/E))$, via $x \mapsto \{i \in n(\beta/E) : s_{i}^{\#}(Id_{\beta-x}, x) \in F\}$, where $l(i/E) = t(i)$ ($i < n$) and $t \in n(\beta$).

We show that $f$ is a well-defined homomorphism (from (*)) and that $f$ is complete such that $f(c) \neq 0$. The last follows by observing that $Id \in f(c)$. Let $V = \beta \beta(Id)$. To show that $f$ is well defined, it suffices to show that for all $\sigma, \tau \in V$, if $(\tau(i), \sigma(i)) \in E$ for all $i \in \beta$, then for any $x \in \mathfrak{A}$, $s_{\alpha}x(x \in F \iff s_{\alpha}x \in F)$. We proceed by induction on $|\{i \in \beta : \tau(i) \neq \sigma(i)\}|(\leq \omega)$. If $J = \{i \in \beta : \tau(i) \neq \sigma(i)\}$ is empty, the result is obvious. Otherwise assume that $k \in J$. We introduce a helpful piece of notation. For $x \in V$, let $\eta(k \mapsto l)$ stand for the $\eta'$ that is the same as $\eta$ except that $\eta'(k) = l$. Now take any $\lambda \in \{\eta \in \beta : \sigma^{-1}\{\eta\} = (\tau)^{-1}\{\eta\} = \{\eta\}\} \setminus \Delta x$. Recall that $\Delta x = \{i \in \beta : x, x \neq x\}$ and that $\beta \setminus \Delta x$ is infinite because $\Delta x \subseteq n$, so such a $\lambda$ exists. Now we freely use properties of substitutions for cylindric algebras. We have by [6, 1.11.11(i)(iv)] (a) $s_{\alpha}x = s_{\alpha}^{\lambda}x_{\sigma(\alpha \rightarrow \lambda)}x$, and (b) $s_{\alpha}^{\lambda}(d_{\lambda, \sigma \rightarrow k} \cdot s_{\alpha}x) = d_{\tau \rightarrow k, \sigma \rightarrow k}^x s_{\alpha}x$, and (c) $s_{\alpha}^{\lambda}(d_{\lambda, \sigma \rightarrow k} \cdot s_{\alpha(\lambda \rightarrow k)}x) = d_{\alpha, \sigma \rightarrow k}^x s_{\alpha(\lambda \rightarrow k)}x$, and finally (d) $d_{\lambda, \sigma \rightarrow k}^x s_{\alpha(\lambda \rightarrow k)}x = d_{\alpha, \sigma \rightarrow k}^x s_{\alpha(\lambda \rightarrow k)}x$. Then by (b), (a), (d) and (c), we get,

$$d_{\alpha, \sigma \rightarrow k}^x s_{\alpha}x = s_{\alpha}^{\lambda}(d_{\alpha, \sigma \rightarrow k} \cdot s_{\alpha}x)$$

$$= s_{\alpha}^{\lambda}(d_{\alpha, \sigma \rightarrow k} \cdot s_{\alpha(\lambda \rightarrow k)}x)$$

$$= s_{\alpha}^{\lambda}(d_{\lambda, \sigma \rightarrow k} \cdot s_{\alpha(\lambda \rightarrow k)}x)$$

$$= d_{\alpha, \sigma \rightarrow k}^x s_{\alpha(\lambda \rightarrow k)}x.$$

But $F$ is a filter and $(\tau k, \sigma k) \in E$, we conclude that

$$s_{\alpha}x \in F \iff s_{\alpha(\lambda \rightarrow k)}x \in F.$$

The conclusion follows from the induction hypothesis. We have proved that $f$ is well defined.
We now check that $f$ is a homomorphism, i.e. it preserves the operations. For $\sigma \in \beta\alpha$, recall that $\sigma^+$ denotes $\sigma \cup Id_{\beta \sim n} \in \beta\beta(\text{Id})$.

- Boolean operations: Since $F$ is maximal we have $\bar{\sigma} \in f(x + y) \iff s_{\sigma}(x + y) \in F \iff s_{\sigma}x + s_{\sigma}y \in F \iff s_{\sigma}x$ or $s_{\sigma}y \in F \iff \bar{\sigma} \in f(x) \cup f(y)$.

We now check complementation. $\bar{\sigma} \in f(-x) \iff s_{\sigma}(-x) \in F \iff -s_{\sigma}x \in F \iff s_{\sigma}x \notin F \iff \bar{\sigma} \notin f(x)$.

- Diagonal elements: Let $k, l < n$. Then we have: $\sigma \in f d_{kl} \iff s_{\sigma}d_{kl} \in F \iff d_{sk,sl} \in F \iff (\sigma k, \sigma l) \in E \iff \sigma k / E = \sigma l / E \iff \bar{\sigma}(k) = \bar{\sigma}(l) \iff \bar{\sigma} \in d_{kl}$.

- Cylindrifications: Let $k < n$ and $a \in A$. Let $\bar{\sigma} \in c_k f(a)$. Then for some $\lambda \in \beta$, we have $\bar{\sigma}(k \mapsto \lambda / E) \in f(a)$ hence $s_{\sigma + (k \mapsto \lambda)} a \in F$. It follows from the inclusion $a \leq c_k a$ that $s_{\sigma + (k \mapsto \lambda)} c_k a \in F$, so $s_{\sigma + c_k a} \in F$. Thus $c_k f(a) \subseteq f(c_k a)$.

We prove the other more difficult inclusion that uses the condition (*) of eliminating cylindrifiers. Let $a \in A$ and $k < n$. Let $\bar{\sigma}' \in f c_k a$ and let $\sigma = \sigma' \cup Id_{\beta \sim n}$. Then $s_{\sigma}c_k a = s_{\sigma'}c_k a \in F$. Pick $\lambda \in \{ \eta \in \beta : \sigma^{-1}\{\eta\} = \{\eta\}\} \setminus \Delta a$, such a $\lambda$ exists because $\Delta a$ is finite, and $|\{i \in \beta : \sigma(i) \neq i\}| < \omega$. Let $\tau = \sigma | n \setminus \{k, \lambda\} \cup \{(k, \lambda), (\lambda, k)\}$. Then (in $\mathcal{B}$):

$$c_{\lambda}s_{\tau} a = s_{\tau} c_k a = s_{\tau} c_k a \in F.$$ 

By the construction of $F$, there is some $u(\notin \Delta(s_{\tau} a))$ such that $s_{\tau} a \in F$, so $s_{\sigma + (k \mapsto u)} a \in F$. Hence $\sigma(k \mapsto u) \in f(a)$, from which we get that $\bar{\sigma}' \in c_k f(a)$.

To show that $f$ is an atomic, hence complete representation, one uses (**) as follows: By construction, for every $s \in n(\beta / E)$, there exists $x \in X(= \text{At}\mathfrak{A})$, such that $s_{n \cup Id_{\beta \sim n}} x \in F$, from which we get $\bigcup_{x \in X} f(x) = n(\beta / E)$. \hfill $\square$

Using the same technique in the proof of Theorem 3.1 without the need to check that the defined homomorphism preserves diagonal elements, we get:

**Theorem 3.2.** Let $\alpha$ be an infinite ordinal and $\beta \leq \alpha$. If $\mathfrak{A} \in \text{PA}_\alpha$, is atomic and completely additive, then any complete subalgebra of $\mathfrak{M}_\beta \mathfrak{D}$ is completely representable. In particular, $S_{\alpha} \text{PA}^{\text{ad}} \cap \text{At} = \text{PA}^{\text{ad}} \cap \text{At} = \text{CRPA}_\alpha$ and the class $\text{CRPA}_\alpha$ is elementary [22].

**Proof.** If $\mathfrak{A} \in \text{PA}_\alpha$, we do not need to bother about diagonal elements and so the base of the representation will be simply $\beta$ (as defined above for $\text{PEA}_\alpha$), not $\beta / E$, and the desired homomorphism, with $n \leq \alpha$, is defined via $g: \mathfrak{A} \to \phi(n\beta)$, via $x \mapsto t \in n\beta$: $s_{n \cup Id_{\beta \sim n}} x \in F$. Checking that $g$
preserves the operations and that $g$ is atomic, hence complete, is exactly like the PEA case. For $\text{PA}_\alpha$, atomicity can be expressed by a first order sentence, and complete additivity can be captured by the following continuous many first order formulas, that form a single schema. Let $\text{At}(x)$ be the first order formula expressing that $x$ is an atom. That is $\text{At}(x)$ is the formula $x \neq 0 \land (\forall y)(y \leq x \rightarrow y = 0 \lor y = x)$. For $\tau \in {\alpha}_\alpha$, let $\psi_\tau$ be the formula: $y \neq 0 \rightarrow \exists x(\text{At}(x) \land s_x \cdot y \neq 0)$. Let $\Sigma$ be the set of first order formulas obtained by adding all formulas $\psi_\tau \ (\tau \in {\alpha}_\alpha)$ to the polyadic schema. Then it is not hard to show that $\text{CRPA}_\alpha = \text{Mod}(\Sigma)$. The underlying idea here is that the notion of complete additivity on atomic algebras is definable in $L_{\omega,\omega}$. In more detail: Let $\mathfrak{A} \in \text{CRPA}_\alpha$ with set of atoms $X$. Then, $\sum_{x \in X} s_x x = 1$ for all $\tau \in {\alpha}_\alpha$. Let $\tau \in {\alpha}_\alpha$. Let $a$ be non-zero, then $a \cdot \sum_{x \in X} s_x x = a \neq 0$, hence there exists $x \in X$, such that $a \cdot s_x x \neq 0$, and so $\mathfrak{A} \models \psi_\tau$. Conversely, let $\mathfrak{A} \models \Sigma$. Then for all $\tau \in {\alpha}_\alpha$, $\sum_{x \in X} s_x x = 1$. Indeed, assume that for some $\tau$, $\sum_{x \in X} s_x x \neq 1$. Let $a = 1 - \sum_{x \in X} s_x x$. Then $a \neq 0$. But then, by assumption, there exists $x' \in X$, such that $s_{x'} x' \cdot a = s_{x'} x' \cdot (1 - \sum_{x \in X} s_x x) = s_{x'} x' - \sum_{x \in X} s_{x'} x \neq 0$, which is impossible. \[
abla \]

Let $\text{CPEA}_\alpha$ denote the class of cylindric-polyadic algebras of dimension $\alpha$ as defined in [4, Definition 6.3.7]. $\text{CPEA}_\alpha$ is a variety obtained by restricting the signature and axiomatization of $\text{PEA}_\alpha$ to finite cylindrifiers and keeping all substitution operators. In the next Theorem we address $\text{CPEA}_\alpha$s and $\text{QEA}_\alpha$s in place of $\text{PEA}_\alpha$s ($\alpha \geq \omega$).

**Theorem 3.3.** Let $\alpha$ be an infinite ordinal and $n < \omega$.

1. If $\mathfrak{D} \in \text{CPEA}_\alpha$ is atomic, then any complete subalgebra of $\text{Nr}_n \mathfrak{CmAt}\mathfrak{D}$ is completely representable. In particular, if $\mathfrak{D}$ is complete and atomic, then $\text{Nr}_n \mathfrak{D}$ is completely representable.

2. Let $\mathfrak{D} \in \text{QEA}_\alpha$ be atomic. Assume that for all $x \in \mathfrak{D}$ for all $k < \omega$, $c_k x = \sum_{l \in \omega} s_l^{k}x$. If $\mathfrak{A} \subseteq \text{Nr}_n \mathfrak{D}$ is such that $\mathfrak{A} \subseteq_c \mathfrak{D}$, then $\mathfrak{A}$ is completely representable.

**Proof.** (1): Let $\mathfrak{D} \in \text{CPEA}_\alpha$ be atomic. Let $\mathfrak{D}^* = \text{CmAt}\mathfrak{D}$. Then $\mathfrak{D}^*$ is complete and atomic and $\text{Nr}_n \mathfrak{D}^* \subseteq_c \mathfrak{D}^*$. To prove the last $\subseteq_c$, assume for contradiction that there is some $S \subseteq \text{Nr}_n \mathfrak{D}^*$, $\sum^{\text{Nr}_n \mathfrak{D}^*} S = 1$, and there exists $d \in \mathfrak{D}^*$ such that $s \leq d < 1$ for all $s \in S$. Take $t = -\sum_{i \in \alpha \setminus n} (-c_i - d)$. This infimum is well defined because $\mathfrak{D}^*$ is complete. Like in the previous proof it can be proved that $c_i t = t$ for all $i \in \alpha \setminus n$, hence $t \in \text{Nr}_n \mathfrak{D}^*$ and that $s \leq t < 1$ for all $s \in S$, which contradicts that $\sum^{\text{Nr}_n \mathfrak{D}^*} S = 1$.

Let $\beta$ be a regular cardinal $> |\mathfrak{D}^*|$ and let $\mathfrak{B} \in \text{CPEA}_\beta$ be complete and atomic such that $\mathfrak{D}^* = \text{Nr}_n \mathfrak{B}$. This $\mathfrak{B}$ can be formed exactly like the PEA case, its atomicity and completeness follow from the fact that the Boolean reduct of $\mathfrak{B}$ is a product of the Boolean reduct of $\mathfrak{A}$. Then we have the following chain of complete embeddings: $\text{Nr}_n \mathfrak{D}^* \subseteq_c \mathfrak{D}^* = \text{Nr}_n \mathfrak{B} \subseteq_c \mathfrak{B}$; the last $\subseteq_c$ follows like above using that $\mathfrak{B}$ is complete. From the first $\subseteq_c$, since
D* is atomic, we get that \( \text{At}\mathcal{D}' \) is atomic. Let \( X = \text{At}\mathcal{D}' \). Then also from the first \( \subseteq_c \), we get that \( \sum_{\mathcal{D}'} X = 1 \), so \( \sum_{\mathcal{D}'} X = 1 \) because \( \mathcal{D}^* \subseteq_c \mathcal{B} \). For \( k < \beta \), \( x \in \mathcal{D}^* \) and \( \tau \in \omega^\beta \), the following joins hold in \( \mathcal{B} \): (*) \( c_k x = \sum_{i \in \beta} s^i x \) and (**) \( \sum s_{x, \tau} X = 1 \), where \( \tau^+ = \tau \cup Id_{\beta \setminus \omega}(\in \beta \beta) \). The join (**) holds, because \( s_{x, \tau} \) is completely additive, since \( \mathcal{B} \) is completely additive. To prove (*), fix \( k < \beta \). Then for all \( l \in \beta \), we have \( s^l x \leq c_k x \). Conversely, assume that \( y \in \mathcal{B} \) is an upper bound for \( \{ s^l x : l \in \beta \} \). Let \( l \in \beta \setminus (\Delta \cup \Delta y) \): such an \( l \) exists, because \( |\Delta x| < \beta \), \( |\Delta y| < \beta \) and \( \beta \) is regular. Hence, we get that \( c_k x = x \) and \( c_l y = y \). But then \( c_l s^l x \leq y \), and so \( c_k x \leq y \). We have proved that (*) hold. Let \( \mathcal{A} = \text{At}\mathcal{D}' \). Let \( a \in \mathcal{A} \) be non-zero. Let \( F \) be any Boolean ultrafilter of \( \mathcal{B} \) generated by an atom below \( a \). Then \( F \) will preserve the family of joins in (*) and (**) giving the required complete representation like in the previous item.

Example 3.4. Let \( \mathcal{D} \in \text{QEA}_\omega \) be the weak set algebra with top element \( V = \omega \omega(0) \) (in the sense of [6, Definition 3.1.2(iii)]). Then, it is easy to show that for any \( n < \omega \), \( \text{At}\mathcal{D} \) is a completely representable \( \text{QEA}_\omega \). Let \( X = \{ 0 \} \in \mathcal{D} \). Then for all \( i \in \omega \), we have \( s^i X = X \). But \( (1, 0, \ldots) \in c_0 X \), so that \( \sum_{i \in \omega} s^i X = X \neq c_0 X \). Hence the joins in item (3) of Theorem 3.1 do not hold. If we take \( \mathcal{D}' = \mathcal{D} \circ \text{At}\mathcal{D} \), then \( \text{At}\mathcal{D}' \) will be a completely representable \( \text{QEA}_n \) for all \( n < \omega \) and \( \mathcal{D}' \) of course will still be a weak set algebra, and it will be locally finite, so that \( c_i x = \sum \mathcal{D}' s^i j \) for all \( i < j < \omega \) by [6, Theorem 1.11.6]. Furthermore, \( \mathcal{D}' \) is atomless. as we proceed to show. Assume for contradiction that it is not, and let \( x \in \mathcal{D}' \) be an atom. Choose \( k, l \in \omega \) with \( k \neq l \) and \( c_k x = x \), this is possible since \( \omega \sim \Delta x \) is infinite. Then \( c_k (x \cdot d_{kl}) = x \), so \( x \cdot d_{kl} \neq 0 \). But \( x \) is an atom, so \( x \leq d_{kl} \). This gives that \( \Delta x = 0 \), and by [6, Theorem 1.3.19] \( x \leq -c_k - d_{kl} \). It is also easy to see that \( (c_k - d_{kl}) \mathcal{D}' = \omega \omega(0) \), from which we conclude that \( x = 0 \), which is a contradiction.

It remains to be seen whether if \( \mathcal{B} \in \text{CA}_\omega \) is atomic and \( 2 < n < \omega \), then \( \text{At}\mathcal{B} \) is completely representable.
4. Non first order definability of the notion of complete representations

This section is devoted to showing that several classes of completely representable algebras (of relations) are not elementary. We need some preparing to do. From now on, unless otherwise indicated, \( n \) is fixed to be a finite ordinal \( > 2 \). Let \( i < n \). For \( n \)-ary sequences \( \bar{x} \) and \( \bar{y} \), we write \( \bar{x} \equiv_i \bar{y} \iff \bar{y}(j) = \bar{x}(j) \) for all \( j \neq i \). For \( i, j < n \) the replacement \([i/j]\) is the map that is like the identity on \( n \), except that \( i \) is mapped to \( j \) and the transposition \([i,j]\) is the like the identity on \( n \), except that \( i \) is swapped with \( j \).

**Definition 4.1.** 1. An \( n \)-dimensional atomic network on an atomic algebra \( \mathfrak{A} \in \text{QA}_n \) is a map \( N: \overset{n}{\Delta} \rightarrow \text{At}\mathfrak{A} \), where \( \Delta \) is a non-empty finite set of nodes, denoted by \( \text{nodes}(N) \), satisfying the following consistency conditions for all \( i < j < n \):

(i) If \( \bar{x} \in \overset{n}{\text{nodes}}(N) \) then \( N(\bar{x}) \leq d_{ij} \iff \bar{x}_i = \bar{x}_j \),

(ii) If \( \bar{x}, \bar{y} \in \overset{n}{\text{nodes}}(N) \), \( i < n \) and \( \bar{x} \equiv_i \bar{y} \), then \( N(\bar{x}) \leq c_i N(\bar{y}) \),

(iii) (Symmetry): if \( \bar{x} \in \overset{n}{\text{nodes}}(N) \), then \( s_{[i,j]} N(\bar{x}) = N(\bar{x} \circ [i,j]) \).

If \( \mathfrak{A} \in \text{CA}_n \), then an \( \mathfrak{A} \) network is a map defined like above satisfying only (i) and (ii). If \( \mathfrak{A} \in \text{QA}_n \), then an \( \mathfrak{A} \) network satisfies (ii) and (iii) together with the condition that if \( \bar{x} \in \overset{n}{\text{nodes}}(N) \), then \( s_{[i,j]} N(\bar{x}) = N(\bar{x} \circ [i,j]) \) (instead of (i)). Finally, if \( \mathfrak{A} \in \text{Sc}_n \), then an \( \mathfrak{A} \) network satisfies the last condition together with (ii).

Let \( i < n \). For \( n \)-dimensional atomic networks \( M \) and \( N \), we write \( M \equiv_i N \iff M(\bar{y}) = N(\bar{y}) \) for all \( \bar{y} \in \overset{n}{\text{nodes}}(n \sim \{i\}) \).

2. Let \( \mathcal{K} \) be any class between \( \text{Sc} \) and \( \text{QEA} \). Assume that \( \mathfrak{A} \in \mathcal{K}_n \) is atomic and that \( m, k \leq \omega \). The atomic game \( G^m_k(\mathfrak{A}) \), or simply \( G^m_k \), is the game played on atomic networks of \( \mathfrak{A} \) using \( m \) nodes and having \( k \) rounds [10, Definition 3.3.2], where \( \forall \) is offered only one move, namely, a cylindrifier move:

Suppose that we are at round \( t > 0 \). Then \( \forall \) picks a previously played network \( N_i \) (\( \text{nodes}(N_i) \subseteq m \), \( i < n \), \( a \in \text{At}\mathfrak{A} \), \( \bar{x} \in \overset{n}{\text{nodes}}(N_i) \), such that \( N_i(\bar{x}) \leq c_i a \). For her response, \( \exists \) has to deliver a network \( M \) such that \( \text{nodes}(M) \subseteq m \), \( M \equiv_i N \), and there is \( \bar{y} \in \overset{n}{\text{nodes}}(M) \) that satisfies \( \bar{y} \equiv_i \bar{x} \) and \( M(\bar{y}) = a \).

We write \( G_k(\text{At}\mathfrak{A}) \), or simply \( G_k \), for \( G^m_k(\text{At}\mathfrak{A}) \) if \( m \geq \omega \).

3. The \( \omega \)-rounded game \( G^m(\text{At}\mathfrak{A}) \) or simply \( G^m \) is like the game \( G^m_\omega(\text{At}\mathfrak{A}) \) except that \( \forall \) has the option to reuse the \( m \) nodes in play.

Observe that for \( k, m \leq \omega \), the games \( G^m_k(\text{At}\mathfrak{A}) \) and \( G^m(\text{At}\mathfrak{A}) \) depend on the signature of \( \mathfrak{A} \).
Proposition 4.2. Let $\mathcal{K}$ be any class between $\mathcal{Sc}$ and QEA. Suppose that $\mathfrak{A} \in \mathcal{K}_n$ is atomic having countably many atoms. Then $\exists$ has a winning strategy in $G_\omega(\mathfrak{A}) \iff \exists$ has a winning strategy in $G^\omega(\mathfrak{A})$ if $\mathfrak{A} \in \mathcal{K}_n$. In particular, if $\mathfrak{A}$ is finite, then $\exists$ has a winning strategy in $G_\omega(\mathfrak{A}) \iff \mathfrak{A}$ is representable.

Proof. [10, Theorem 3.3.3] (for the CA case), together with observing that the game $G_\omega(\mathfrak{A})$ is equivalent to the game $G^\omega(\mathfrak{A})$, in the sense that $\exists$ has a winning strategy in $G^\omega(\mathfrak{A}) \iff \exists$ has a winning strategy in $G_\omega(\mathfrak{A})$ whenever $\mathfrak{A}$ is atomic with countably many atoms (the converse implication is trivial). The rest of the cases are analogous.

For a class $\mathcal{K}$ of BAOs, we denote by $\mathcal{K}^{ad}$ the class of completely additive algebras in $\mathcal{K}$.

Lemma 4.3. Let $\mathcal{K}$ be any class between $\mathcal{Sc}$ and QEA. If $\mathfrak{A} \in \mathcal{S}_c \mathcal{N}_n \mathcal{K}^{ad}_m$ is atomic, then $\exists$ has a winning strategy in $G^m(\mathfrak{A})$.

Proof. We give the proof for CAs italicizing the part where additivity is used. The stipulated additivity condition when considering only CAs is superfluous since it holds anyway. First a piece of notation. Let $m$ be a finite ordinal $> 0$. An $s$ word is a finite string of substitutions $(s^j_i)$ $(i, j < m)$, a $c$ word is a finite string of cylindrifications $(c^i)$, $i < m$; an $sc$ word $w$, is a finite string of both, namely, of substitutions and cylindrifications. An $sc$ word induces a partial map $\hat{w} : m \to m$: $\hat{c} = Id$, $w^j_i = \hat{w} \circ [i, j]$ and $\bar{w} c^i = \hat{w} \restriction (m \setminus \{i\})$. If $\bar{a} \in < m$, we write $s_{\bar{a}}$ or $s_{\bar{a}_0...\bar{a}_{k-1}}$, where $k = |\bar{a}|$, for an arbitrary chosen $sc$ word $w$ such that $\hat{w} = \bar{a}$. Such a $w$ exists by [9, Definition 5.23 Lemma 13.29].

Fix $2 < n < m$. Assume that $\mathcal{C} \in \mathcal{C}A_m$, $\mathfrak{A} \subseteq \mathcal{N}_n \mathcal{C}$ is an atomic $\mathcal{C}A_n$ and $N$ is an $\mathfrak{A}$-network with $\text{nodes}(N) \subseteq m$. Define $N^+ \in \mathcal{C}$ by

$$N^+ = \prod_{i_0, \ldots, i_{n-1} \in \text{nodes}(N)} s_{i_0, \ldots, i_{n-1}} N(i_0, \ldots, i_{n-1}).$$

For a network $N$ and function $\theta$, the network $N \theta$ is the complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N)\}$, and labelling defined by

$$(N \theta)(i_0, \ldots, i_{n-1}) = N(\theta(i_0), \theta(i_1), \ldots, \theta(i_{n-1})), \text{ for } i_0, \ldots, i_{n-1} \in \theta^{-1}(\text{nodes}(N)).$$

Then the following hold:

1. for all $x \in \mathcal{C} \setminus \{0\}$ and all $i_0, \ldots, i_{n-1} < m$, there is $a \in \mathfrak{A}$, such that $s_{i_0, \ldots, i_{n-1}} a \cdot x \neq 0$.

2. for any $x \in \mathcal{C} \setminus \{0\}$ and any finite set $I \subseteq m$, there is a network $N$ such that $\text{nodes}(N) = I$ and $x \cdot N^+ \neq 0$. Furthermore, for any networks $M, N$ if $M^+ \cdot N^+ \neq 0$, then $M |_{\text{nodes}(M) \cap \text{nodes}(N)} = N |_{\text{nodes}(M) \cap \text{nodes}(N)}$. 


(3): if $\theta$ is any partial, finite map $m \to m$ and if $\text{nodes}(N)$ is a proper subset of $m$, then $N^\theta + 0 \to (N\theta)^+ \neq 0$. If $i \not\in \text{nodes}(N)$, then $c_i N^+ = N^+$.

Since $\mathfrak{A} \subseteq \mathfrak{At}_{E}$, then $\sum E \mathfrak{At}\mathfrak{A} = 1$. For $(1)$, $s_j$ is a completely additive operator (any $i, j < m$, hence $s_{i_0, \ldots, i_{n-1}}$ is, too). So $\sum E \{s_{i_0, \ldots, i_{n-1}} \cdot a : a \in \mathfrak{At}\mathfrak{A}\} = s_{i_0, \ldots, i_{n-1}} 1 = 1$ for any $i_0, \ldots, i_{n-1} < m$. Let $x \in \mathfrak{C} \setminus \{0\}$. Assume for contradiction that $s_{i_0, \ldots, i_{n-1}} a \cdot x = 0$ for all $a \in \mathfrak{At}\mathfrak{A}$. Then $1 - x$ will be an upper bound for $\{s_{i_0, \ldots, i_{n-1}} a : a \in \mathfrak{At}\mathfrak{A}\}$. But this is impossible because $\sum E \{s_{i_0, \ldots, i_{n-1}} a : a \in \mathfrak{At}\mathfrak{A}\} = 1$.

To prove the first part of (2), we repeatedly use $(1)$. We define the edge labelling of $N$ one edge at a time. Initially, no hyperedges are labelled. Suppose $E \subseteq \text{nodes}(N) \times \text{nodes}(N) \cdots \times \text{nodes}(N)$ is the set of labelled hyperedges of $N$ (initially $E = \emptyset$) and $x \cdot \prod_{e \in E} s_e N(e) \neq 0$. Pick $d$ such that $d \not\in E$. Then by $(1)$ there is $a \in \mathfrak{At\mathfrak{A}}$ such that $x \cdot \prod_{e \in E} s_e N(e) \cdot s_d a \neq 0$. Include the hyperedge $d$ in $E$. We keep on doing this until eventually all hyperedges will be labelled, so we obtain a completely labelled graph $N$ with $N^+ \neq 0$. It is easily checked that $N$ is a network.

For the second part of (2), we proceed contrapositively. Assume that there is $e \in \text{nodes}(M) \cap \text{nodes}(N)$ such that $M(e) \neq N(e)$. Since edges are labelled by atoms, we have $M(e) \cdot N(e) = 0$, so $0 = s_e M(e) \cdot s_e N(e) \geq M^+ \cdot N^+$. A notion of piece. For $i < m$, let $\text{Id}_{-i}$ be the partial map $\{(k, k) : k \in m \setminus \{i\}\}$. For the first part of $(3)$ (cf. [9, Lemma 13.29] using the notation in op.cit.), since there is $k \in m \setminus \text{nodes}(N)$, $\theta$ can be expressed as a product $\sigma_0 \sigma_1 \ldots \sigma_t$ of maps such that, for $s \leq t$, we have either $\sigma_s = \text{Id}_{-i}$ for some $i < m$ or $\sigma_s = [i/j]$ for some $i, j < m$ and where $i \not\in \text{nodes}(N\sigma_0 \ldots \sigma_{s-1})$. But clearly $(N\text{Id}_{-j})^+ \geq N^+$ and if $i \not\in \text{nodes}(N)$ and $j \in \text{nodes}(N)$, then $N^+ \neq 0 \to (N[i/j])^+ \neq 0$. The required now follows. The last part is straightforward.

Using the above proven facts, we are now ready to show that $\exists$ has a winning strategy in $\mathbb{G}^m$. She can always play a network $N$ with $\text{nodes}(N) \subseteq m$, such that $N^+ \neq 0$.

In the initial round, let $\forall$ play $a \in \mathfrak{At\mathfrak{A}}$. $\exists$ plays a network $N$ with $N(0, \ldots, n - 1) = a$. Then $N^+ = a \neq 0$. Recall that here $\forall$ is offered only one (cylindrifier) move. At a later stage, suppose $\forall$ plays the cylindrifier move, which we denote by $(N, \langle f_0, \ldots, f_{n-2}, k, b, l \rangle)$. He picks a previously played network $N$, $f_i \in \text{nodes}(N)$, $l < n$, $k \not\in \{f_i : i < n - 2\}$, such that $b \leq c_i N(f_0, \ldots, f_{l-1}, x, f_{l+1}, \ldots, f_{n-2})$ and $N^+ \neq 0$. Let $\bar{a} = \langle f_0, \ldots, f_{l-1}, k, f_{l+1}, \ldots, f_{n-2} \rangle$.

Then by second part of $(3)$ we have that $c_i N^+ \cdot s_b \bar{a} \neq 0$ and so by first part of $(2)$, there is a network $M$ such that $M^+ \cdot c_i N^+ \cdot s_b \bar{a} \neq 0$. Hence $M(f_0, \ldots, f_{l-1}, k, f_{l+1}, \ldots, f_{n-2}) = b$, $\text{nodes}(M) = \text{nodes}(N) \cup \{k\}$, and $M^+ \neq 0$, so this property is maintained. □
We let $LCA_n$ denote the elementary class of $RCA_n$s satisfying the Lyndon conditions [10, Definition 3.5.1].

**Theorem 4.4.** Let $2 < n < m \leq \omega$. $\text{ElNr}_nCA_ω ∩ At \subseteq LCA_n$. Furthermore, for any elementary class $K$ between $\text{ElNr}_nCA_ω ∩ At$ and $LCA_n$, $RCA_n$ is generated by $AtK$.

**Proof.** Throughout the proof fix $2 < n < \omega$.

It suffices to show that $\text{Nr}_nCA_ω ∩ At \subseteq LCA_n$, since the last class is elementary. This follows from Lemma 4.3, since if $\mathfrak{A} \in \text{Nr}_nCA_ω$ is atomic, then $\exists$ has a winning strategy in $G^ω(At\mathfrak{A})$, hence in $G_ω(At\mathfrak{A})$, a fortiori, $\exists$ has a winning strategy in $G_k(At\mathfrak{A})$ for all $k < \omega$, so (by definition) $\mathfrak{A} \in LCA_n$.

To show strictness of the last inclusion, let $V = ^nQ$ and let $\mathfrak{A} \in Cs_n$ have universe $\phi(V)$. Then $\mathfrak{A} \in \text{Nr}_nCA_ω$. Let $y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}$ and $\mathcal{E} = \bigcap \{\{y\} ∪ X)\}$, where $X = \{\{s\} : s \in V\}$. Now $\mathcal{E}$ and $\mathfrak{A}$ having same top element $V$, share the same atom structure, namely, the singletons, so $\text{cmAtE} = \mathfrak{A}$. Furthermore, plainly $\mathfrak{A}, \mathcal{E} \in CRCA_n$. So $\mathcal{E} \in CRCA_n \subseteq LCA_n$, and as proved in [20], $\mathcal{E} \notin \text{ElNr}_nCA_{n+1}$, hence $\mathcal{E}$ witnesses the required strict inclusion.

Now we show that $At\text{ElNr}_nCA_ω$ generates $RCA_n$. Let $\text{FCs}_n$ denote the class of full $Cs_n$s, that is $Cs_n$s having universe $\phi(\omega U)$ ($U$ non-empty set). First we show that $\text{FCs}_n \subseteq \text{cmAt}\text{Nr}_nCA_ω$. Let $\mathfrak{A} \in \text{FCs}_n$. Then $\mathfrak{A} \in \text{Nr}_nCA_ω ∩ At$, hence $At\mathfrak{A} \in At\text{Nr}_nCA_ω$ and $\mathfrak{A} = \text{cmAt}\mathfrak{A} \in \text{cmAtNr}_nCA_ω$. The required now follows from the following chain of inclusions:

$$RCA_n = \text{SPFCs}_n \subseteq \text{SPcmAt}(\text{Nr}_nCA_ω) \subseteq \text{SPcmAt}(\text{ElNr}_nCA_ω)$$
$$\subseteq \text{SPcmAtK} \subseteq \text{SPcmLCA}_n \subseteq RCA_n,$$

where $K$ is given above. □

**Theorem 4.5.** Let $\kappa$ be an infinite cardinal. Then there exists an atomless $\mathcal{C} \in CA_ω$ such that for all $2 < n < \omega$, $\text{Nr}_n\mathcal{C}$ is atomic, with $|At(\text{Nr}_n\mathcal{C})| = 2^n$, $\text{Nr}_n\mathcal{C} \in LCA_n$, but $\text{Nr}_n\mathcal{C}$ is not completely representable.

**Proof.** We use the following uncountable version of Ramsey’s theorem due to Erdős and Rado: If $r \geq 2$ is finite, $k$ an infinite cardinal, then $\exp_r(k^+) \to (k^+)\exp_{r+1}(k)$ where $\exp_0(k) = k$ and inductively $\exp_{r+1}(k) = 2^{\exp_r(k)}$. The above partition symbol describes the following statement. If $f$ is a coloring of the $r+1$ element subsets of a set of cardinality $\exp_r(k^+)$ in $k$ many colors, then there is a homogeneous set of cardinality $k^+$ (a set, all whose $r+1$ element subsets get the same $f$-value). Let $\kappa$ be the given cardinal. We use a variation on the construction in [15] which is a simplified more basic version of a rainbow construction where only the two predominant colours, namely, the reds and blues are available. The algebra $\mathcal{C}$ will be constructed from a relation algebra possessing an $\omega$-dimensional cylindric basis. To define the relation algebra we specify its atoms and the forbidden
triples of atoms. The atoms are \( \text{Id}, g_0^i : i < 2^\kappa \) and \( r_j : 1 \leq j < \kappa \), all symmetric. The forbidden triples of atoms are all permutations of \((\text{Id}, x, y)\) for \( x \neq y \), \((r_j, r_j, r_j)\) for \( 1 \leq j < \kappa \) and \((g_0^i, g_0^{i'}, g_0^{i''})\) for \( i, i', i'' < 2^\kappa \). Write \( g_0 \) for \( \{g_0^i : i < 2^\kappa\} \) and \( r_+ \) for \( \{r_j : 1 \leq j < \kappa\} \). Call this atom structure \( \alpha \). Consider the term algebra \( \mathfrak{A} \) defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that \( \mathfrak{A} \) is a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if \( \mathfrak{A} \) and \( \mathfrak{B} \) are atomic relation algebras sharing the same atom structure, so that \( \text{At} \mathfrak{A} = \text{At} \mathfrak{B} \), then \( \mathfrak{A} \) is completely representable \( \iff \mathfrak{B} \) is completely representable.

Assume for contradiction that \( \mathfrak{A} \) has a complete representation \( M \). Let \( x, y \) be points in the representation with \( M \models r_1(x, y) \). For each \( i < 2^\kappa \), there is a point \( z_i \in M \) such that \( M \models g_0^i(x, z_i) \land r_1(z_i, y) \). Let \( Z = \{z_i : i < 2^\kappa\} \). Within \( Z \), each edge is labelled by one of the \( \kappa \) atoms in \( r_+ \). The Erdős–Rado theorem forces the existence of three points \( z^1, z^2, z^3 \in Z \) such that \( M \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z_1) \), for some single \( j < \kappa \). This contradicts the definition of composition in \( \mathfrak{A} \) (since we avoided monochromatic triangles).

Let \( S \) be the set of all atomic \( \mathfrak{A} \)-networks \( N \) with nodes \( \omega \) such that \( \{r_i : 1 \leq i < \kappa; r_i \text{ is the label of an edge in } N\} \) is finite. Then it is straightforward to show \( S \) is an amalgamation class, that is for all \( M, N \in S \) if \( M \equiv N \) then there is \( L \in S \) with \( M \equiv L \equiv N \). Let \( X \) be the set of finite \( \mathfrak{A} \)-networks \( N \) with nodes \( \leq \kappa \) such that:

1. each edge of \( N \) is either (a) an atom of \( \mathfrak{A} \) or (b) a cofinite subset of \( r_+ = \{r_j : 1 \leq j < \kappa\} \) or (c) a cofinite subset of \( g_0 = \{g_0^i : i < 2^\kappa\} \) and
2. \( N \) is ‘triangle-closed’, i.e. for all \( l, m, n \in \text{nodes}(N) \) we have \( N(l, n) \leq N(l, m) ; N(m, n) \). That means if an edge \( (l, m) \) is labelled by \( \text{Id} \) then \( N(l, n) = N(m, n) \) and if \( N(l, m), N(m, n) \leq g_0 \) then \( N(l, n) \cdot g_0 = 0 \). and if \( N(l, m) = N(m, n) = r_j \) (some \( 1 \leq j < \omega \)) then \( N(l, n) \cdot r_j = 0 \).

For \( N \in X \) let \( \hat{N} \in \mathfrak{C}(S) \) be defined by

\[
\{L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N)\}.
\]

For \( i \in \omega \), let \( N \mid_{-i} \) be the subgraph of \( N \) obtained by deleting the node \( i \). Then if \( N \in X \), \( i < \omega \) then \( \text{c}_{\hat{N}}(N) \subseteq (N \mid_{-i}) \). The inclusion \( \text{c}_{\hat{N}}(N) \subseteq (N \mid_{-i}) \) is clear. Conversely, let \( L \in (N \mid_{-i}) \). We seek \( M \equiv L \) with \( M \in \hat{N} \). This will prove that \( L \in \text{c}_{\hat{N}}(N) \), as required. Since \( L \in S \) the set \( T = \{r_i \notin L\} \) is infinite. Let \( T \) be the disjoint union of two infinite sets \( Y \cup Y' \), say. To define the \( \omega \)-network \( M \) we must define the labels of all edges involving the node \( i \) (other labels are given by \( M \equiv L \)). We define these labels by enumerating the edges and labeling them one at a time. So let \( j \neq i < \kappa \). Suppose \( j \in \text{nodes}(N) \). We must choose \( M(i, j) \leq N(i, j) \). If \( N(i, j) \) is an atom then of
course \( M(i,j) = N(i,j) \). Since \( N \) is finite, this defines only finitely many labels of \( M \). If \( N(i,j) \) is a cofinite subset of \( g_0 \) then we let \( M(i,j) \) be an arbitrary atom in \( N(i,j) \). And if \( N(i,j) \) is a cofinite subset of \( r_+ \) then let \( M(i,j) \) be an element of \( N(i,j) \cap Y \) which has not been used as the label of any edge of \( M \) which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If \( j \notin \text{nodes}(N) \) then we can let \( M(i,j) = r_k \in Y \) some \( 1 \leq k < \kappa \) such that no edge of \( M \) has already been labelled by \( r_k \). It is not hard to check that each triangle of \( M \) is consistent (we have avoided all monochromatic triangles) and clearly \( M \in \tilde{N} \) and \( M \equiv_i L \). The labeling avoided all but finitely many elements of \( Y' \), so \( M \in S \). So

\[
(N \setminus \{ i \}) \subseteq C_i N.
\]

Now let \( \tilde{X} = \{ \tilde{N} : N \in X \} \subseteq \mathfrak{A}(S) \). Then we claim that the subalgebra of \( \mathfrak{A}(S) \) generated by \( \tilde{X} \) is simply obtained from \( \tilde{X} \) by closing under finite unions. Clearly all these finite unions are generated by \( \tilde{X} \). We must show that the set of finite unions of \( \tilde{X} \) is closed under all cylindric operations. Closure under unions is given. For \( \tilde{N} \in X \) we have

\[
\tilde{N} = \bigcup_{m,n \in \text{nodes}(N)} \tilde{N}_{mn} \text{ where } N_{mn} \text{ is a network with nodes } \{m,n\} \text{ and labeling } N_{mn}(m,n) = -N(m,n). \text{ } N_{mn} \text{ may not belong to } X \text{ but it is equivalent to a union of at most finitely many members of } \tilde{X}. \text{ The diagonal } d_{i,j} \in \mathfrak{A}(S) \text{ is equal to } \tilde{N} \text{ where } N \text{ is a network with nodes } \{i,j\} \text{ and labeling } N(i,j) = \text{Id}. \text{ Closure under cylindrification is given. Let } \mathfrak{C} \text{ be the subalgebra of } \mathfrak{A}(S) \text{ generated by } \tilde{X}. \text{ Then } \mathfrak{A} = \mathfrak{R}(\mathfrak{C}). \text{ To see why, each element of } \mathfrak{A} \text{ is a union of a finite number of atoms, possibly a co-finite subset of } g_0 \text{ and possibly a co-finite subset of } r_+. \text{ Clearly } \mathfrak{A} \subseteq \mathfrak{R}(\mathfrak{C}). \text{ Conversely, each element } z \in \mathfrak{R}(\mathfrak{C}) \text{ is a finite union } \bigcup_{N \in F} \tilde{N}, \text{ for some finite subset } F \text{ of } X, \text{ satisfying } c_i z = z, \text{ for } i > 1. \text{ Let } i_0, \ldots, i_k \text{ be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in } F. \text{ Then, }
\]

\[
c_i z = \bigcup_{N \in F} c_i \tilde{N} = \bigcup_{N \in F} (\tilde{N} \setminus \{0,1\}) \in \mathfrak{A}. \text{ So } \mathfrak{R}(\mathfrak{C}) \subseteq \mathfrak{A}. \mathfrak{A} \text{ is relation algebra reduct of } \mathfrak{C} \subseteq \mathfrak{A}_\omega \text{ but has no complete representation. Let } n > 2. \text{ Let } \mathfrak{B} = \mathfrak{R}_n \mathfrak{C}. \text{ Then } \mathfrak{B} \in \mathfrak{N} \mathfrak{R}_n \mathfrak{A}_\omega \text{, is atomic, but has no complete representation for plainly a complete representation of } \mathfrak{B} \text{ induces one of } \mathfrak{A}. \text{ In fact, because } \mathfrak{B} \text{ is generated by its two dimensional elements, and its dimension is at least three, its DF reduct is not completely representable. It remains to show that the } \omega \text{-dilation } \mathfrak{C} \text{ is atomless. For any } N \in X, \text{ we can add an extra node extending } N \text{ to } M \text{ such that } \emptyset \subseteq M' \subseteq N', \text{ so that } N' \text{ cannot be an atom in } \mathfrak{C}. \text{ } \Box
\]

Let \( \text{LQEA}_n \) be the class of algebras satisfying the Lyndon condition in the signature of \( \text{QEA}_n \). Then like the CA case, \( \text{LQEA}_n = \text{ELCRQEA}_n \).

**Theorem 4.6.** For any infinite cardinal \( \kappa \), there exists an atomless \( \mathfrak{C} \in \text{QEA}_\omega \) such that for all \( 2 < n < \omega \), \( \mathfrak{N} \mathfrak{C} \) is atomic, with \( |\text{At}(\mathfrak{N} \mathfrak{C})| = 2^n \).
\(\mathfrak{R}_n \in \mathbb{LQEA}_n\), but \(\mathfrak{R}_0 \not\in \mathbb{LQEA}_n\) is not completely representable. In particular, for any class \(K\) between \(\mathfrak{Sc}\) and \(\mathbb{QEA}\), \(\mathbb{CRK}_n\) is not elementary.

**Proof.** Using the notation in the proof of Theorem 4.5, we have \(S\) defined in opcit is symmetric, that is, if \(N \in S\) and \(\theta : \omega \to \omega\) is a finitary function, in the sense that \(\{i \in \omega : \theta(i) \neq i\}\) is finite, then \(N \theta\) is in \(S\). It follows that the complex algebra \(\mathfrak{Ca}(S) \in \mathbb{QEA}_\omega\). The required now follows. \(\square\)

In the previous construction used in Proposition 4.5, \(\mathfrak{A}\) also satisfies the Lyndon conditions by [7, Theorem 33] but is not completely representable. Thus:

**Corollary 4.7 ([8]).** Let \(2 < n < \omega\). Then the classes \(\mathbb{CRRA}\) and \(\mathbb{CRCA}_n\) are not elementary.

**Theorem 4.8.** The class \(\mathbb{CRRA}\), are not closed under \(\equiv_{\infty, \omega}\).

**Proof.** Take \(\mathfrak{A}\) to be a symmetric, atomic relation algebra with atoms

\([l, r(i), y(i), b(i) : i < \omega]\).

Non-identity atoms have colours, \(r\) is red, \(b\) is blue, and \(y\) is yellow. All atoms are self-converse. Composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of \([l, x, y]\) for \(x \neq y\), and

\((r(i), r(i), r(j)), (y(i), y(i), y(j)), (b(i), b(i), b(j))\) \(i \leq j < \omega\).

\(\mathfrak{A}\) is the complex algebra over this atom structure. Let \(\alpha\) be an ordinal. \(\mathfrak{R}_\alpha\) is obtained from \(\mathfrak{R}\) by splitting the atom \(r(0)\) into \(\alpha\) parts \(r^k(0) : k < \alpha\) and then taking the full complex algebra. In more detail, we put red atoms \(r^k(0)\) for \(k < \alpha\). In the altered algebra the forbidden triples are

\((y(i), y(i), y(j)), (b(i), b(i), b(j))\), \(i \leq j < \omega\), \((r(i), r(i), r(j)), 0 < i \leq j < \omega\),

\((r^k(0), r^l(0), r^m(0))\), \(0 < j < \omega, k, l < \alpha\), \((r^k(0), r^l(0), r^m(0))\), \(k, l, m < \alpha\).

In \(\mathfrak{R}_\alpha\), we use the following abbreviations: \(r(0) = \sum_{k < \alpha} r^k(0)\), \(r = \sum_{i < \omega} r(i), y = \sum_{i < \omega} y(i)\), \(b = \sum_{i < \omega} b(i)\). These suprema exist because they are taken in the complex algebras which are complete. The index of \(r(i), y(i)\) and \(b(i)\) is \(i\) and the index of \(r^k(0)\) is also 0. Now let \(\mathfrak{B} = \mathfrak{R}_\omega\) and \(\mathfrak{A} = \mathfrak{R}_n\) with \(n \geq 2^{\aleph_0}\) We claim that \(\mathfrak{B} \in \mathfrak{RaCA}_\omega\). For the first required, we show that \(\mathfrak{B}\) has a cylindric bases by exhibiting a winning strategy for \(\exists\) in the cylindric-basis game, which is a simpler version of the hyperbasis game [9, Definition 12.26]. At some stage of the game, let the play so far be \(N_0, N_1, \ldots, N_t\) for some \(t < \omega\). We say that an edge \((m, n)\) of an atomic network \(N\) is a diversity edge if \(N(m, n) \cdot \text{ld} = 0\). Each diversity edge of each atomic network in the play has an owner – either \(\exists\) or \(\forall\), which we will allocate as we define \(\exists\)’s strategy. If an edge \((m, n)\) belongs to player \(p\) then so does the reverse edge \((n, m)\) and we will only specify one of them. Since our
algebra is symmetric, so the label of the reverse edge is equal to the label of the edge, so again need to specify only one. For the next round ∃ must define N_t in response to ∀'s move. If there is an already played network N_t (some i < t) and a finitary map σ: ω → ω such that N_tσ ‘answers’ his move, then she lets N_t = N_tσ. From now on we assume that there is no such N_t and σ. We consider the three types of ∀ can make. If he plays an atom move by picking an atom a, ∃ plays an atomic network N with N(0, 1) = a and for all x ∈ ω \ {1}, N(0, x) = Id.

If ∀ plays a triangle move by picking a previously played N_x (some x < t), nodes i, j, k with k / ∈ {i, j} and atoms a, b with a; b ≥ N_x(i, j), we know that a, b / ∈ 1', as we are assuming the ∃ cannot play an embedding move (if a = Id, consider N_x and the map [k/i]). ∃ must play a network N_t ≡ k N_x such that N_t(i, k) = a, N_t(k, j) = b. These edges, (i, k) and (k, j), belong to ∀ in N_t. All diversity edges not involving k have the same owner in N_t as they did in N_x. And all edges (l, k) for k / ∈ {i, j} belong to ∃ in N_x. To label these edges ∃ chooses a colour c different than the colours of a, b (we have three colours so this is possible). Then, once at a time, she labels each edge (l, k) by an atom with colour c and a non-zero index which has not yet been used to label any edge of any network played in the game. She does this one edge at a time, each with a new index. There are infinitely many indices to choose, so this can be done.

Finally, ∀ can play an amalgamation move by picking M, N ∈ {N_x: s < t}, nodes i, j such that M ≡_{ij} N. If there is N_s (some s < t) and a map σ: nodes(N_x) → nodes(M) ∪ nodes(N) such that M ≡_s N_sσ ≡_s N then ∃ lets N_t = N_sσ. Ownership of edges is inherited from N_s. If there is no such N_s and σ then there are two cases. If there are three nodes x, y, z in the ‘amalgam’ such that M(j, x) and N(x, i) are both red and of the same index, M(j, y), N(y, i) are both yellow and of the same index and M(j, z), N(z, i) are both blue and of the same index, then the new edge (i, j) belongs to ∀ in N_t. It will be labelled by either r^0(0), b(0) or y(0) and it is easy to show that at least one of these will be a consistent choice. Otherwise, if there is no such x, y, z then the new edge (i, j) belongs to ∃ in N_t. She chooses a colour c such that there is no x with M(j, x) and N(x, i) both having colour c and the same index. And she chooses a non-zero index for N_t(i, j) which is new to the game (as with triangle moves). If k / ∈ k' ∈ M ∩ N then (j, k) has the same owner in N_t as it does in M, (k, i) has the same owner in N_t as it does in N and (k, k') belongs to ∃ in N_t if it belongs to ∃ in either M or N, otherwise it belongs to ∀ in N_t. Now the only way ∃ could lose, is if ∀ played an amalgamation move (M, N, i, j) such that there are x, y, z ∈ M ∩ N such that M(j, x) = r^k(0), N(x, i) = r^k'(0), M(j, y) = N(y, i) = b(0) and M(j, z) = N(z, i) = y(0). But according to ∃’s strategy, she never chooses atoms with index 0, so all these edges must have been chosen by ∀. This contradiction proves the required.
Now, let $\mathcal{H}$ be an $\omega$-dimensional cylindric basis for $\mathcal{B}$. Then $\mathcal{C} \mathcal{A} \mathcal{H} \in \mathcal{C} \mathcal{A}_\omega$. Consider the cylindric algebra $\mathcal{C} = \mathcal{S}_G \mathcal{C} \mathcal{A}_B$, the subalgebra of $\mathcal{C} \mathcal{A} \mathcal{H}$ generated by $\mathcal{B}$. In principal, new two dimensional elements that were not originally in $\mathcal{B}$, can be created in $\mathcal{C}$ using the spare dimensions in $\mathcal{C} \mathcal{A} \mathcal{H}$. But next we exclude this possibility. We show that $\mathcal{B}$ exhausts the 2-dimensional elements of $\mathcal{R} \mathcal{A} \mathcal{C}$, more concisely, we show that $\mathcal{B} = \mathcal{R} \mathcal{A} \mathcal{C}$. For this purpose we want to find out what are the elements of $\mathcal{C} \mathcal{A} \mathcal{H}$ that are generated by $\mathcal{B}$. Let $M$ be a (not necessarily atomic) finite network over $\mathcal{B}$ whose nodes are a finite subset of $\omega$.

- Define (using the same notation in the proof of Theorem 4.5) $\widehat{M} = \{ N \in \mathcal{H} : N \leq M \} \in \mathcal{C} \mathcal{A} \mathcal{H}$. ($N \leq M$ means that for all $i, j \in M$ we have $N(i, j) \leq M(i, j)$.)

- A block is an element of the form $\widehat{M}$ for some finite network $M$ such that

1. $M$ is triangle-closed, i.e. for all $i, j, k \in M$ we have $M(i, k) \leq M(i, j); M(j, k)$
2. If $x$ is the label of an irreflexive edge of $M$ then $x = \text{Id}$ or $x \leq r$ or $x \leq y$ or $x \leq b$ (we say $x$ is ‘monochromatic’), and $\{|i: x \cdot (r(i) + y(i) + b(i)) \neq 0|\}$ is either 0, 1 or infinite (we say that the number of indices of $x$ is either 0, 1 or infinite).

We prove:

1. For any block $\widehat{M}$ and $i < \omega$ we have

$$c_i \widehat{M} = (M \mid_{\text{dom}(M) \setminus \{i\}})$$

2. The domain of $\mathcal{C}$ consists of finite sums of blocks.

$c_i \widehat{M} \subseteq (M \mid_{\text{dom}(M) \setminus \{i\}})$ is obvious. If $i \notin M$ the equality is trivial. Let $N \in (M \mid_{\text{dom}(M) \setminus \{i\}})$, i.e. $N \leq M \mid_{\text{dom}(M) \setminus \{i\}}$. We must show that $N \in c_i \widehat{M}$ and for this we must find $L \equiv_i N$ with $L \in \widehat{M}$. $L \equiv_i N$ determines every edge of $L$ except those involving $i$. For each $j \in M$, if the number of indices in $M(i, j)$ is just one, say $M(i, j) = r(k)$, then let $L(i, j)$ be an arbitrary atom below $r(k)$. There should be no inconsistencies in the labelling so far defined for $L$, by triangle-closure for $M$. For all the other edges $(i, j)$ if $j \in M$ there are infinitely many indices in $M(i, j)$ and if $j \notin M$ then we have an unrestricted choice of atoms for the label. These edges are labelled one at a time and each label is given an atom with a new index, thus avoiding any inconsistencies. This defines $L \equiv_i N$ with $L \in \widehat{M}$. For the second part, we already have seen that the set of finite sums of blocks is closed under cylindrification. We'll show that this set is closed under all the cylindric
operations and includes \( \mathfrak{B} \). For any \( x \in \mathfrak{B} \) and \( i, j < \omega \), let \( N_{x}^{ij} \) be the \( \mathfrak{B} \)-network with two nodes \( \{i, j\} \) and labelling \( N_{x}^{ij}(i, i) = N_{x}^{ij}(j, j) = \text{id} \), and \( N^{ij}(i, j) = x \), \( N_{x}^{ij}(j, i) = \tilde{x} \). Clearly \( N_{x}^{ij} \) is triangle closed. And \( N_{x}^{01} = x \). For any \( x \in \mathfrak{B} \), we have \( x = x \cdot \text{id} + x \cdot r + x \cdot y + x \cdot b \), so \( x = N_{x}^{01} + N_{x}^{01} + N_{x}^{01} + N_{x}^{01} \) and the labels of these four networks are monochromatic. The first network defines a block and for each of the last three, if the number if indices is infinite then it is a block. If the number of indices is finite then it is a finite union of blocks. So every element of \( \mathfrak{B} \) is a finite union of blocks.

For the diagonal elements, \( d_{ij} = N_{\text{id}}^{ij} \). Closure under sums is obvious. For negation, take a block \( \overline{M} \). Then \( \overline{M} = \sum_{i,j \in M} N_{\overline{N}(i,j)}^{ij} \). As before we can replace \( N_{\overline{N}(i,j)}^{ij} \) by a finite union of blocks. Thus the set of finite sums of blocks includes \( \mathfrak{B} \) and the diagonals and is closed under all the cylindric operations. Since every block is clearly generated from \( \mathfrak{B} \) using substitutions and intersection only. It remains to show that \( \mathfrak{B} = \mathfrak{RaC} \). Take a block \( \overline{M} \in \mathfrak{RaC} \). Then \( c_{i} \overline{M} = \overline{M} \) for \( 2 \leq i < \omega \). By the first part of the lemma, \( \overline{M} = \overline{M} \big|_{\{0,1\}} \in \mathfrak{B} \).

We finally show that \( \exists \) has a winning strategy in an Ehrenfeucht–Fraïssé-game over \((\mathfrak{A}, \mathfrak{B})\) concluding that \( \mathfrak{A} \equiv_{\omega} \mathfrak{B} \). At any stage of the game, if \( \forall \) places a pebble on one of \( \mathfrak{A} \) or \( \mathfrak{B} \), \( \exists \) must place a matching pebble, on the other algebra. Let \( \bar{a} = (a_{0}, a_{1}, \ldots, a_{n-1}) \) be the position of the pebbles played so far (by either player) on \( \mathfrak{A} \) and let \( \bar{b} = (b_{0}, \ldots, b_{n-1}) \) be the position of the pebbles played on \( \mathfrak{B} \). \( \exists \) maintains the following properties throughout the game.

- For any atom \( x \) (of either algebra) with \( x \cdot r(0) = 0 \) then \( x \in a_{i} \iff x \in b_{i} \).
- \( \bar{a} \) induces a finite partition of \( r(0) \) in \( \mathfrak{A} \) of \( 2^{n} \) (possibly empty) parts \( p_{i}: i < 2^{n} \) and \( b \) induces a partition of \( r(0) \) in \( \mathfrak{B} \) of parts \( q_{i}: i < 2^{n} \). \( p_{i} \) is finite iff \( q_{i} \) is finite and, in this case, \( |p_{i}| = |q_{i}| \).

Now we show that \( \text{CRRA} \) is not closed under \( \equiv_{\omega} \). Since \( \mathfrak{B} \in \mathfrak{RaCA}_{\omega} \) has countably many atoms, then \( \mathfrak{B} \) is completely representable [7, Theorem 29]. For this purpose, we show that \( \mathfrak{A} \) is not completely representable. We work with the term algebra, \( \mathfrak{TmAtA} \), since the latter is completely representable \( \iff \) the complex algebra is. Let \( r = \{ r(i): 1 \leq i < \omega \} \cup \{ r^{k}(0): k < 2^{R_{0}} \} \), \( y = \{ y(i): i \in \omega \} \), \( b^{+} = \{ b(i): i \in \omega \} \). It is not hard to check every element of \( \mathfrak{TmAtA} \subseteq \mathfrak{c}(\mathfrak{A}) \) has the form \( F \cup R_{0} \cup B_{0} \cup Y_{0} \), where \( F \) is a finite set of atoms, \( R_{0} \) is either empty or a co-finite subset of \( r \), \( B_{0} \) is either empty or a co-finite subset of \( b \), and \( Y_{0} \) is either empty or a co-finite subset of \( y \). Using an argument similar to that used in the proof of Theorem 4.5, we show that the existence of a complete rep-
presentation necessarily forces a monochromatic triangle, that we avoided at the start when defining $\mathfrak{A}$. Let $x, y$ be points in the representation with $M \models y(0)(x, y)$. For each $i < 2^{80}$, there is a point $z_i \in M$ such that $M \models \text{red}(x, z_i) \wedge y(0)(z_i, y)$ (some red red $\in \mathfrak{r}$). Let $Z = \{ z_i : i < 2^{80} \}$. Within $Z$ each edge is labelled by one of the $\omega$ atoms in $y^+$ or $b^+$. The Erdős–Rado theorem forces the existence of three points $z_1, z_2, z_3 \in Z$ such that $M \models y(j)(z_1, z_2) \wedge y(j)(z_2, z_3) \wedge y(j)(z_3, z_1)$, for some single $j < \omega$ or three points $z_1, z_2, z_3 \in Z$ such that $M \models b(l)(z_1, z_2) \wedge b(l)(z_2, z_3) \wedge b(l)(z_3, z_1)$, for some single $l < \omega$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles). We have proved that CRRA is not closed under $\equiv_{\infty}$, since $\mathfrak{A} \equiv_{\infty} \mathfrak{B}$, $\mathfrak{A}$ is not completely representable, but $\mathfrak{B}$ is completely representable.

5. Other notions of representability

Although all our coming results are true of $\text{Scs}$, $\text{QAs}$ and $\text{QEA}s$, we formulate (and prove) our results only for $\text{CA}s$ so that our presentation remains, shorter, more focused and more streamlined. We emphasize that the arguments used proving the $\text{CA}$ case can be modified without much ado to adapt the aforementioned algebras. In the following $\text{Up}$, $\text{Ur}$, (and recall that) $\text{P}$ and $\text{H}$ denote the operations of forming ultraproducts, ultraroots, products and homomorphic images, respectively.

**Theorem 5.1.** For $2 < n < \omega$ the following hold:
1. $\text{CRCA}_n \subseteq \text{Sc}_{\text{Nr}_n}(\text{CA}_\omega \cap \text{At}) \cap \text{At} \subseteq \text{Sc}_{\text{Nr}_n} \text{CA}_\omega \cap \text{At}$. At least two of these three classes are distinct,
2. All reverse inclusions in the previous item hold, if algebras considered have countably many atoms,
3. All classes in the first item are closed under $\text{Sc}$ (a fortiori under $\text{Sd}$), $\text{P}$, but are not closed under $\text{S}$, nor $\text{H}$ nor $\text{Ur}$. Their elementary closure coincides with $\text{LCA}_n$,
4. $\text{Nr}_n \text{CA}_\omega \subseteq \text{Sc}_{\text{Nr}_n} \text{CA}_\omega \subseteq \text{Sc}_{\text{Nr}_n} \text{CA}_\omega \subseteq \text{ElS}_{\text{Nr}_n} \text{CA}_\omega \subseteq \text{RCA}_n$. Furthermore, the strictness of inclusions are witnessed by atomic algebras.

**Proof.** Throughout the proof, fix $2 < n < \omega$.
(1): The proof of the first inclusion is similar to the proof for (the analogous result on) relation algebras proved in [7, Theorem 29]. The construction in Proposition 4.5 shows that the first and last classes are distinct.
(2): By [18, Theorem 5.3.6] the class $\text{CRCA}_n$ coincides with the class $\text{Sc}_{\text{Nr}_n} \text{CA}_\omega$ on atomic algebras with countably many atoms. Taken together with [10, Theorem 3.3.3], we are done.
(3): We start with $\text{CRCA}_n$. Closure under $\mathbf{P}$ is straightforward. We show that $\text{S}_0\text{CRCA}_n = \text{CRCA}_n$. Assume that $\mathfrak{D} \in \text{CRCA}_n$ and $\mathfrak{A} \subseteq \mathfrak{D}$. Identifying set algebras with their domain let $f: \mathfrak{D} \rightarrow \wp(V)$ be a complete representation of $\mathfrak{D}$, where $V$ is a $\mathfrak{G}_n$ unit. We claim that $g = f \upharpoonright \mathfrak{A}$ is a complete representation of $\mathfrak{A}$. Let $X \subseteq \mathfrak{A}$ be such that $\sum^\mathfrak{A} X = 1$. Then by $\mathfrak{A} \subseteq \mathfrak{D}$, we have $\sum^\mathfrak{D} X = 1$. Furthermore, for all $x \in X (\subseteq \mathfrak{A})$ we have $f(x) = g(x)$, so that $\bigcup_{x \in X} f(x) = V$, since $f$ is a complete representation, and we are done.

Let $\mathbb{C}$ be any of the two remaining classes. Closure under $\mathbf{S}_{\mathbb{U}}$. Theorem they are not closed under $\mathbf{S}_{\mathbb{U}}$. For analogous cases), by the Keisler-Shelah ultrapower §21 and [9, §9.3], since they are pseudo-elementary classes (cf. [7, Theorem 4.5]), they are not closed under $\mathbf{S}_{\mathbb{U}}$. Then $\mathbb{U}$ will be inside them all proving that none of the three is elementary, so being closed under $\mathbf{S}_{\mathbb{U}}$, since they are pseudo-elementary classes (cf. [7, Theorem 21] and [9, §9.3] for analogous cases), by the Keisler-Shelah ultrapower Theorem they are not closed under $\mathbf{U}_{\mathbb{R}}$. For the last required, we show that $\text{LCA}_n = \text{EICRCA}_n = \text{EI}(\text{S}_n\text{S}_0\text{CA}_\omega \cap \text{At})$. Assume that $\mathfrak{A} \in \text{LCA}_n$. Then, by definition, for all $k < \omega$, $\exists$ has a winning strategy in $G_k(\text{At}\mathfrak{A})$. Using ultrapowers followed by an elementary chain argument like in [10, Theorem 3.3.5], $\exists$ has a winning strategy in $G_\omega(\text{At}\mathfrak{A})$ for some countable $\mathfrak{B} \equiv \mathfrak{A}$, and so by [10, Theorem 3.3.3] $\mathfrak{B}$ is completely representable. Thus $\mathfrak{A} \in \text{EICRCA}_n$. One shows that $\text{EI}(\text{S}_n\text{S}_0\text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n$ exactly like in item (1) of Theorem 4.4. So $\text{LCA}_n = \text{EICRCA}_n \subseteq \text{EI}(\text{S}_n\text{S}_0\text{CA}_\omega \cap \text{At}) \subseteq \text{LCA}_n$, and we are done.

(4): The algebra $\mathfrak{C}$ used in the first item of Theorem 4.4 witnesses that $\text{Sr}_n\text{CA}_\omega \subseteq \text{S}_n\text{Sr}_n\text{CA}_\omega$, because, as proved in [20], $\mathfrak{C} \notin \text{ELI}\text{Sr}_n\text{CA}_\omega (\supseteq \text{Sr}_n\text{CA}_\omega)$ and $\mathfrak{C} \subseteq \mathfrak{A}$ where $\mathfrak{A} \in \text{Sr}_n\text{CA}_\omega$ is the full $\mathfrak{G}_n$ with top element $n^\mathfrak{A}$ (and universe $\wp(n^\mathfrak{A})$). The atomic countable $\mathfrak{C} \in \text{RCA}_n$ used in the previous item is in $\text{EI}\text{Sr}_n\text{CA}_\omega \sim \text{Sr}_n\text{Sr}_n\text{CA}_\omega$. Let $\mathfrak{A} \in \text{RCA}_n$ be countable and atomic such that such $\mathfrak{C} \in \text{RCA}_n$. Such algebras exist [23]. Then $\mathfrak{A} \notin \text{LCA}_n$, because $\mathfrak{A}$ does not satisfy the Lyndon conditions, lest $\mathfrak{C} \mathfrak{A} \in \text{LCA}_n (\subseteq \text{RCA}_n)$. We conclude that $\mathfrak{A} \notin \text{EI}\text{Sr}_n\text{CA}_\omega$ proving the strictness of the last inclusion. Since $\mathfrak{C}, \mathfrak{E}$ and $\mathfrak{A}$ are all atomic, we are done.

In the next theorem we collect and prove some properties of CRRA in terms of taking the $\text{Ra}$ reducts of (ω-dilations in the class) $\text{CA}_\omega$ analogously to the results proved for $\text{CA}_\omega$s in Theorems 4.4 and 5.1.
Theorem 5.2. 1. \( \text{CRRA} \subseteq S_c \text{Ra}(CA_\omega \cap \text{At}) \cap \text{At} \subseteq S_c \text{RaCA}_\omega \cap \text{At} \). All reverse inclusions hold, if algebras considered have countably many atoms.

2. All classes in the first item are closed under \( S_c \) and \( P \), but are not closed under \( S \), nor \( H \) nor \( U_r \).

3. \( \text{RaCA}_\omega \cap \text{At} \not\subseteq \text{CRRA} \), \( \text{RaCA}_\omega \cap \text{At} \not\subseteq S_c \text{RaCA}_\omega \cap \text{At} \) and \( \text{CRRA} \not\subseteq S_c \text{RaCA}_\omega \cap \text{At} \).

4. Neither of the classes \( \text{CRRA} \) and \( S_c \text{RaCA}_\omega \cap \text{At} \) are contained in each other.

5. At least two of the three classes in the first item are distinct. The elementary closure of all these classes coincides with LRRA (class of RAs satisfying the Lyndon conditions). Furthermore, \( \text{RaCA}_\omega \subseteq LRRA \) and for any class \( L \) of atom structures between \( \text{AtRaCA}_\omega \) and \( \text{AtLRRA} \), \( \text{SPc} \text{mL} = \text{RRA} \). For LRRA we can remove the \( P \).

Proof. (1): [7, Theorem 29 (1) \( \Rightarrow \) (2)]. The second part follows by observing that the class \( \text{CRRA} \) coincides with the class \( S_c \text{RaCA}_\omega \) on atomic algebras having countably many atoms [7, Theorem 29].

(2): One proceeds like the CA case. In particular, for non-closure under \( U_r \), proved for CAs in item (2) of Theorem 5.4, one uses an atomic RRA with countably many atoms that is not completely representable, but is elementary equivalent to some \( \mathfrak{B} \in \text{CRRA} \), cf [7, 8].

(3): First \( \not\subseteq \) follows from the construction in Theorem 4.5. Second \( \not\subseteq \) follows from [7, Theorem 36]. Last \( \not\subseteq \) follows from the first two parts in this item together with the inclusions in the first item.

(4): That \( S_c \text{RaCA}_\omega \not\subseteq \text{CRRA} \) follows from the construction in Theorem 4.5. That \( \text{CRRA} \not\subseteq S_c \text{RaCA}_\omega \) follows from [7, Theorem 36] where a finite, hence a completely representable RRA, was constructed outside \( S_c \text{RaCA}_\omega \).

(5) is proved like the CA case, cf item (1) of Theorem 4.4 and item (3) of Theorem 5.1, respectively.

Fix \( m > 2 \). Then the varieties \( S\text{RaCA}_m \) and \( \text{RA}_m \) are ‘\( m \)-approximations’ to RRA that form a strict hierarchy (by varying \( m \)). Now we define an ‘\( m \) approximation’ to the elementary class LRRA(\( \subseteq \) RRA) using games. Let \( \text{CRCA}_{m,s} \) denote the class of RAs having \( m \)-square representations. Define \( \mathfrak{R} \in \text{LC}_{m,s} \iff \mathfrak{R} \) is atomic and \( \exists \) has a winning strategy in \( G^n_k(\text{At}\mathfrak{R}) \) for all \( k < \omega \). So here we are restricting the number of nodes used in the play to \( m \). It is easy to show that \( \text{LC}_{m,s} = \text{ElCRCA}_{m,s} \); in particular, \( \text{LC}_{m,s} \) is an elementary class.\(^1\) We thereby obtain a strict hierarchy of elementary classes of relation algebras for which the ‘gaps’ are not finitely axiomatizable.

Theorem 5.3. For \( m \geq 4 \), \( \text{LC}_{m+1,s} \) is not finitely axiomatizable over \( \text{LC}_{m,s} \).

\(^1\)It is known that \( \text{CRCA}_{m,s} \) is not elementary for \( m \geq 5 \) [8].
Proof. Fix $m \geq 4$. For $r \geq 4$, define $\mathfrak{A}_r^m$ as in [9, p. 521] replacing $n$ by $m$. Then by [9, Lemma 7.15], for all $r \geq 4$, $\exists$ has a winning strategy in $G_k^m(\mathfrak{A}_r^m)$ for all $k < \omega$, hence $\mathfrak{A}_r^m \in \mathcal{LC}_{m,s}$. But $\exists$ has a winning strategy in $G_{\omega}^{m+1}(\Pi_{4 \leq i < \omega} \mathfrak{A}_i^m / D)$, cf. [9, Theorem 17.18], a fortiori, $\exists$ has a winning strategy in $G_k^{m+1}(\Pi_{4 \leq i < \omega} \mathfrak{A}_i^m / D)$ for all $k < \omega$, so $\Pi_{4 \leq i < \omega} \mathfrak{A}_i^m / D \in \mathcal{LC}_{m+1,s}$. But for each $r \geq 4$, $\forall$ has a winning strategy in $G_{\omega}^{m+1}\mathfrak{A}_r^m$, and $\mathfrak{A}_r^m$ is finite, so $\mathfrak{A}_r^m \notin \mathcal{LC}_{m+1,s}$, proving the required using a standard Los’ argument. \qed

Fix $2 < n < \omega$. Call an atomic $\mathfrak{A} \in \mathcal{CA}_n$ weakly (strongly) representable $\iff \text{At}\mathfrak{A}$ is weakly (strongly) representable. Let $\mathfrak{WRCA}_n$ (SrCA$_n$) denote the class of all such $\mathcal{CA}_n$s, respectively. Then the class $\mathcal{SRCA}_n$ is not elementary and $\mathcal{LCA}_n \subseteq \mathcal{SA}_{CA} \subseteq \mathcal{WRCA}_n$ [10]; the strictness of the two inclusions follow from the fact that the classes $\mathcal{LCA}_n$ and $\mathcal{WRCA}_n$ are elementary. For an atom structure $\text{At}$, let $\mathfrak{A}(\text{At})$ be the subalgebra of $\mathfrak{CA}$ consisting of all sets of atoms in $\text{At}$ of the form $\{a \in \text{At}: \text{At} \models \phi(a, \vec{b})\}$ ($\in \mathfrak{CA}$), for some first order formula $\phi(x, \vec{y})$ of the signature of $\text{At}$ and some tuple $\vec{b}$ of atoms, cf.[9, item (3), p. 456] for the analogous definition for relation algebras. Let $\mathcal{FCA}_n$ be the class of subalgebras as just described, cf. [9, item (3), p. 456]. Then it can be proved, similarly to the RA case that $\mathcal{SRCA}_n \subseteq \mathcal{FCA}_n$ and that $\mathcal{FCA}_n$ is elementary, cf. [9, Theorem 14.17], hence the inclusion is strict.

Corollary 5.4. Let $2 < n < \omega$. Then the following hold:

1. $\mathcal{N}_n \mathcal{CA}_n \cap \text{At} \subseteq \mathcal{S}_n \mathcal{N}_n \mathcal{CA}_n \cap \text{At} \subseteq \mathcal{S}_n \mathcal{N}_n \mathcal{CA}_n \cap \text{At} \subseteq \mathcal{EL} \mathcal{S}_n \mathcal{N}_n \mathcal{CA}_n \cap \text{At} = \mathcal{LCA}_n \subseteq \mathcal{SRCA}_n \subseteq \mathcal{EL} \mathcal{SRCA}_n \subseteq \mathcal{FCA}_n \subseteq \mathcal{WRCA}_n$,

2. $\mathcal{CRC}_n \subseteq \mathcal{S}_n \mathcal{N}_n \mathcal{CA}_n \cap \text{At} \subseteq \mathcal{EL} \mathcal{CRC}_n = \mathcal{EL} \mathcal{S}_n \mathcal{N}_n \mathcal{CA}_n \cap \text{At} = \mathcal{LCA}_n$,

3. For any class $K(\subseteq \mathcal{RA}_n \cap \text{At})$ occurring in the previous two items, $\mathcal{EL} \mathcal{K} = \mathcal{SRCA}_n$ is not finitely axiomatizable, and $\mathcal{SK} = \mathcal{RA}_n$.

Proof. We slightly modify the construction in [18, Lemma 5.1.3, Theorem 5.1.4]. Using the same notation, the algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in op.cit satisfy $\mathfrak{A} \in \mathcal{N}_n \mathcal{CA}_n$, $\mathfrak{B} \notin \mathcal{N}_n \mathcal{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but it can be fixed that they are atomic, giving the same result with the rest of the proof unaltered. This is done by interpreting the uncountably many ternary relations in the signature of $\mathfrak{M}$ defined in [18, Lemma 5.1.3], which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be disjoint in $\mathfrak{M}$, not just distinct. The construction is presented this way in [16], where (the equivalent of) $\mathfrak{M}$ is built in a more basic step-by-step fashion. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in op.cit lift verbatim to any such $n$. Let $u \in n$. Write $1_u$ for $\chi^M_u$ (denoted by $1_u$ (for $n = 3$) in [18, Theorem 5.1.4]). We denote by $\mathfrak{A}_u$ the Boolean algebra $\mathfrak{R}^u_1 \mathfrak{A} = \{x \in \mathfrak{A}: x \leq 1_u\}$ and similarly for $\mathfrak{B}_u$, writing $\mathfrak{B}_n$ short hand for the Boolean algebra $\mathfrak{R}^n_1 \mathfrak{B} = \{x \in \mathfrak{B}: x \leq 1_u\}$. Using that $\mathfrak{M}$ has quanti-
fier elimination we get, using the same argument in op.cit that $\mathfrak{A} \in \text{Nr}_n\text{CA}_\omega$. The property that $\mathfrak{B} \notin \text{Nr}_n\text{CA}_{n+1}$ is also still maintained. To see why, consider the substitution operator $s(0,1)$ (using one spare dimension) as defined in the proof of [18, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \text{Nr}_n\mathfrak{C}$, with $\mathfrak{C} \in \text{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots, n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term $s(0,1)$ acts like a substitution operator corresponding to the transposition $[0,1]$; it 'swaps' the first two co-ordinates. Now one can show that $s(0,1)\mathfrak{B}_u \subseteq \mathfrak{B}_u|_u = \mathfrak{B}_{Id}$, so $|s(0,1)\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{Id}$ was forced by construction to be countable. But $s(0,1)$ is a Boolean automorphism with inverse $s(1,0)$, so that $|\mathfrak{B}_u| = |s(0,1)\mathfrak{B}_u| > \omega$, contradiction.

We show that $\mathfrak{B}$ is in fact outside $S_d\text{Nr}_n\text{CA}_{n+1} \cap \text{At}$. Take $\kappa$ the signature of $M$; more specifically, the number of $n$-ary relation symbols to be $2^{2^n}$, and assume for contradiction that $\mathfrak{B} \in S_d\text{Nr}_n\text{CA}_{n+1} \cap \text{At}$. Then $\mathfrak{B} \subseteq_d \text{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \text{CA}_{n+1}$ and $\text{Nr}_n\mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \text{Nr}_n\mathfrak{D}$, then $\text{Ri}_{Id}\mathfrak{B} \subseteq_d \text{Ri}_{Id}\mathfrak{C}$. Since $\mathfrak{C}$ is atomic, then $\text{Ri}_{Id}\mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|\text{Ri}_{Id}\mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} \in \text{Nr}_n\text{CA}_{n+1}$). By the choice of $\kappa$, we get that $|\text{At}\text{Ri}_{Id}\mathfrak{C}| > \omega$. By density, $\text{At}\text{Ri}_{Id}\mathfrak{C} \subseteq \text{At}\text{Ri}_{Id}\mathfrak{B}$. Hence $|\text{At}\text{Ri}_{Id}\mathfrak{B}| \geq |\text{At}\text{Ri}_{Id}\mathfrak{C}| > \omega$. But by the construction of $\mathfrak{B}$, $|\text{Ri}_{Id}\mathfrak{B}| = |\text{At}\text{Ri}_{Id}\mathfrak{B}| = \omega$, which is a contradiction and we are done. We have show that the algebra $\mathfrak{B}$ is in $S_d\text{Nr}_n\text{CA}_\omega \cap \text{At} \sim S_d\text{Nr}_n\text{CA}_\omega$.

For the strictness of the last inclusion in the first item, we refer the reader to [9, Theorem 14.17] for the relation algebra analogue.

From [6, Construction 3.2.76, p.94] the elementary closure of any class $\mathfrak{K}$, such that $\text{Nr}_n\text{CA}_\omega \cap \text{At} \subseteq \mathfrak{K} \subseteq \text{RCA}_n$, $\mathfrak{K}$ is not finitely axiomatizable. In the aforementioned construction, non-representable finite (Monk) algebras outside $\text{RCA}_n$ are constructed, such that any (atomic) non-trivial ultraprouct of such algebras is in $\text{Nr}_n\text{CA}_\omega \cap \text{At}$. We now prove (3): Let $\mathfrak{A} \in \text{RCA}_n$. Then $\mathfrak{A} \cong \mathfrak{B}$, $\mathfrak{B} \in \text{Gs}_n$ with top element $V$ say. Let $\mathfrak{C}$ be the full $\text{Gs}_n$ with top element $V$ (and universe $\phi(V)$). Then $\mathfrak{C} \in \text{Nr}_n\text{CA}_\omega \cap \text{At}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. Thus $\mathfrak{A} \in S(\text{Nr}_n\text{CA}_\omega \cap \text{At})$. For classes in the last item one has another option; one can take canonical extensions instead of generalized full set algebras upon observing that $\mathfrak{A} \in \text{RCA}_n$ $\iff$ $\mathfrak{A}^+ \in \text{CRCA}_n$. □

We relate the various notions of representability defined above to classes of algebras having special neat embedding properties.

**Theorem 5.5.** Let $2 < n < \omega$. Then the following hold:
1. $S_c\text{Nr}_n\text{CA}_\omega \cap \text{Count} = \text{CRCA}_n \cap \text{Count}$, and $\text{EIS}_c\text{Nr}_n\text{CA}_\omega \cap \text{At} = \text{LCA}_n$,
2. $\text{SNr}_n\text{CA}_\omega \cap \text{At} = \text{WRCA}_n$,
3. $\text{PEIS}_c\text{Nr}_n\text{CA}_\omega \cap \text{At} \subseteq \text{SRC}_n$, and $\text{ElPEIS}_c\text{Nr}_n\text{CA}_\omega \cap \text{At} \subseteq \text{FCA}_n$. 


Proof. Item (2) follows by definition taking into account that \( \text{RCA}_n = \text{SNR}_n \text{CA}_n \). Item (3) follows from that \( \text{LCA}_n \subseteq \text{SRCA}_n \), that (it is easy to check that) \( \text{SRCA}_n \) is closed under \( \mathbf{P} \), that \( \text{SRCA}_n \subseteq \text{FCA}_n \) and finally that the last class is elementary.

An entirely analogous result holds for RAs upon replacing \( \text{Nr}_n \) by \( \text{Ra} \).

REFERENCES