INTERPOLATION AND AMALGAMATION IN MODAL CYLINDRIC ALGEBRAS

TAREK SAYED AHMED

PLEASE SUPPLY ADDRESS?
E-Mail rutahmed@gmail.com

Abstract. Let $\alpha$ be an ordinal and $L$ be a unimodal logic (like $S4$ or $S5$). A modal cylindric algebra of dimension $\alpha$, an $LCA_\alpha$, is a cylindric algebra of dimension $\alpha$, expanded with $\alpha$-many $L$ modalities. For a frame $(U; R)$ of $L$, each $k < \alpha$, one defines a diamond box operator on $2^U$: $\diamond_k(X) = \{s \in U : \exists t \in X : t \equiv_k s \wedge (t_k, s_k) \in R\}, X \subseteq U$. This defines the semantics of the $L$ modalities in set algebras, with the rest of the operations defined like in cylindric set algebras of dimension $\alpha$. We study interpolation properties for the corresponding predicate logic having $\alpha$-many variables. Our results are valid for any reflexive $L$ whose frames contain the universal frames $(U, U \times U)$. In particular, they hold for $K5CA_\alpha$, $S4CA_\alpha$ (which is an algebraizable extension of topological predicate logic with semantics induced by Alexandrov topologies).

Mathematics Subject Classification (2010): 03B50, 03B52, 03G15.

Key words: Modal logic, algebraic logic, cylindric algebras, interpolation.

1. Introduction and basic notions.

1.1. History and overview. Daigneault [6] succeeded in stating and proving versions of Beth’s and Craig’s celebrated theorems on definability and interpolation, respectively. Beth’s Theorem says that in first order logic ($L_{\omega, \omega}$) implicit definability implies explicit definability, while Craig’s Theorem says that $L_{\omega, \omega}$ has the interpolation property. This was achieved algebraically by proving that the class of locally finite polyadic algebras of infinite dimension (with and without equality) has the amalgamation property. Later Johnson removed the condition of local finiteness, proving that the class of polyadic algebras of infinite dimension without equality has the strong amalgamation property [15]. In [25] it is proved by the present author that the last class has the so-called super amalgamation property (which is stronger than strong amalgamation as the name suggests), a notion introduced by Maksimova in the context of abstract algebraic logic, cf. [19, 5]. That the super amalgamation property is indeed (strictly) stronger than the strong amalgamation property is proved in [23].

The first systematic use of the link to obtain results about interpolation properties from results of amalgamation, or vice versa, can be found in Pigozzi’s landmark paper [22]. The principal context of [22] is the class of infinite dimensional cylindric algebras, an equational formalism of first order logic. The positive results of Section 2.2 in combination with the negative ones of Section 2.3 of [22] answer most of
the natural questions one could ask about amalgamation for cylindric algebras of infinite dimension. However, most questions concerning the strong amalgamation property for several classes of cylindric algebras were posed as open questions in [22], and other closely related ones appeared after Pigozzi's paper was published. All these questions are answered by the present author and Judit Márkus in [17] and [16]; we refer the reader to [18] for a somewhat comprehensible overview.

1.2. Main results and the methodology used in this paper. In this paper, we construct new predicate logics that can be seen as modal expansions of first order logic given an algebraizable formalism in the sense of [4]. For such logics, we study algebraically various definability properties such as the Craig interpolation property. We show that these properties remain to hold for modal expansions of first order logic as long as there are variables existing outside (atomic) formulas, but these results do not generalize any further. In particular, we show that these definability properties (even in their weakest forms) fail in those infinitary logics when atomic formulas contain all variables available in the signature. Dealing with so called modal cylindric algebras of infinite dimension, our proof methods use the methodology of algebraic logic, which is the natural interface between mathematical logic and universal algebra.

Georgescu [10, 12, 9, 8, 11] studied Chang, modal, topological and tense polyadic algebras of infinite dimension. The work of Georgescu in [11] is substantially generalized in [26] by relaxing the condition of local finiteness and studying besides representability, various amalgamation properties for Heyting polyadic algebras. The work in this paper can be seen as a unification and a far reaching generalization of the work in Georgescu’s remaining aforementioned references which dealt only with representation theorems of locally finite algebras. Here we go further by studying representation as well as amalgamation theorems for various subclasses of modal cylindric algebras of infinite dimension.

The constructions used in our proofs are sophisticated variations on Henkin constructions disguised in an algebraic form of Robinson’s finite forcing in model theory. The addition of constants, added to eliminate quantifier in a sense, is expressed algebraically via forming dilations, cf. Definition 1.8 which is an inverse operation to the operation of forming neat reducts. The notion of complete rich theories in the expanded language is expressed via so-called Henkin Boolean ultrafilters. In the proofs of completeness theorems only one Henkin ultrafilter is used, while in interpolation theorems two Henkin ultrafilter overlapping on the ‘common part (language)’ are used. This technique of proof shows that interpolation theorems proved in this paper can be seen as generalizations of completeness theorems for the logic under scrutiny. Indeed, our results strongly suggest that completeness proved by Georgescu for locally finite algebras in the aforementioned references, and interpolation proved for the more general case of dimension complemented algebras (to be defined below) go hand in hand. From such interpolation theorems, proved for dimension restricted free algebras, we obtain using the methodology of algebraic logic, cf. [24, 17, 18, 7, 19, 22, 5, 21, 16, 27] that several classes of algebras have various forms of the amalgamation property, such as super amalgamation. Conversely, we show that the free algebra on at least 4 generators (when
the dimension is not restricted) fails to have a weak form of interpolation. Using
the same methodology, we infer that any class of modal cylindric algebras, contain-
ing the class of representable ones, does not have the amalgamation property. This
covers infinitely many varieties as will be shown below. We follow the notation of

1.3. Cylindric algebras. A cylindric algebra consists of a Boolean algebra en-
dowed with an additional structure consisting of distinguished elements and opera-
tions, satisfying a certain system of equations. The introduction and study of these
algebras has its motivation in two parts of mathematics: the deductive systems of
first-order logic, and a portion of elementary set theory dealing with spaces of var-
ious dimensions, possibly transfinite, better known as cylindric set algebras. Such
algebras also have a geometric twist, reflected in the terminology ‘cylinder’. If we
are working in 3 dimensions, and we apply the unary operations of cylindrifiers
(algebraizing existential quantifiers) to a ‘circle’, then we are forming the cilinder
based on this circle.

We recall the concrete operations of cylindric set algebras, namely, cylindrifica-
tions and diagonal elements. Let $\alpha$ be an ordinal, $U$ be a non-empty set, $i, j \in \alpha$
and $X \subseteq ^\alpha U$.

$$C_i X = \{ t \in ^\alpha U : \exists s \in X \text{ and } t(j) = s(j) \text{ for all } j \neq i \},$$

$$D_{ij} = \{ s \in ^\alpha U : s_i = s_j \}.$$  

For a set $V$, $B(V)$ is the Boolean set algebra with unit $V$, that is to say, $B(V) =$
$\langle \varnothing(V), \cup, \cap, \setminus, \emptyset, V \rangle$.

**DEFINITION 1.1.** A cylindric set algebra of dimension $\alpha$ is a subalgebra of an algebra
of the form $\langle B(^\alpha U), C_i, D_{ij} \rangle_{i,j<\alpha}$. The class of all cylindric set algebras of dimension
$\alpha$ is denoted by $Cs_\alpha$.

$CA_\alpha$ stands for the class of cylindric algebras of dimension $\alpha$. This last (equa-
tionally defined class) is obtained from $Cs_\alpha$ by a process of abstraction and is
defined by a finite schema of equations given in [13, Definition 1.1.1].

**DEFINITION 1.2.** By a cylindric algebra of dimension $\alpha$, briefly a $CA_\alpha$, we mean
an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{\kappa, \lambda < \alpha},$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra such that $0, 1$, and $d_{ij}$ are distinguished
elements of $A$ (for all $j, i < \alpha$), $-$ and $c_i$ are unary operations on $A$ (for all $i < \alpha$),
$+$ and $\cdot$ are binary operations on $A$, and such that the following postulates are
satisfies for any $x, y \in A$ and any $i, j, \mu < \alpha$:

$$(C_1) \quad c_i 0 = 0,$$

$$(C_2) \quad x \leq c_i x \quad (i.e., x + c_i x = c_i x),$$

$$(C_3) \quad c_i (x \cdot c_i y) = c_i x \cdot c_i y,$$
\((C_4)\) \(c.ic.jx = c.jc.i.x,\)

\((C_5)\) \(d_{ii} = 1,\)

\((C_6)\) if \(i \neq j, \mu,\) then \(d_{j\mu} = c_i(d_{j\mu} \cdot d_{i\mu}),\)

\((C_7)\) if \(i \neq j,\) then \(c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0.\)

Another concrete example of CAs is weak set algebras. Let \(\alpha\) be an ordinal, \(U\) a non-empty set and \(p\) a fixed sequence in \(\alpha U.\) A weak space of dimension \(\alpha,\) denoted by \(\alpha U(p),\) is the set \(\{s \in \alpha U : |i \in \alpha; s_i \neq p_i| < \omega\}.\)

**Definition 1.3.** A weak set algebra of dimension \(\alpha\) is a subalgebra of an algebra of the form \((\mathcal{B}(V), C_i, D_{ij}), i, j \in \alpha\), where \(V\) is a weak space of dimension \(\alpha\) and the \(C_i\)'s and \(D_{ij}\)'s \((i, j \in \alpha)\) are defined like in set algebras but restricted to the top element \(V.\) The class of weak set algebras of dimension \(\alpha\) is denoted by \(\mathcal{W} \mathcal{A}_\alpha[13, \text{Definition 3.1.2}].\)

Obviously for \(\alpha < \omega,\) \(\mathcal{C} S_{\alpha} = \mathcal{W} \mathcal{A}_\alpha.\) For operators on classes of algebras, \(S\) stands for the operation of forming subalgebras, \(P\) stands for that of forming products, and \(H\) stands for the operation of forming homomorphic images. The variety of representable algebras of dimension \(\alpha,\) \(\alpha\) an ordinal, is defined as follows: \(\mathcal{R} \mathcal{C} \mathcal{A}_\alpha = \mathcal{S} \mathcal{P} \mathcal{C} S_{\alpha}.\) It is known that \(\mathcal{R} \mathcal{C} \mathcal{A}_\alpha = \mathcal{S} \mathcal{P} \mathcal{W} \mathcal{A}_\alpha,\) and that \(\mathcal{R} \mathcal{C} \mathcal{A}_\alpha\) is a variety (i.e. closed under \(H\) as well), that cannot be axiomatized by a finite schema of equations for \(\alpha > 2.\)

**Definition 1.4.** An algebra \(\mathfrak{A} \in \mathcal{C} \mathcal{A}_\omega\) is locally finite, if the dimension set of every element \(x \in A\) is finite. The dimension set of \(x,\) or \(\Delta x\) for short, is the set \(\{i \in \omega : c.i.x \neq x\}.\)

Locally finite algebras correspond to Tarski–Lindenbaum algebras of (first order) formulas; in such algebras the dimension set of (an equivalence class of) a formula reflects the number of (finite) set of free variables in this formula. Tarski proved that every locally finite \(\omega\)-dimensional cylindric algebra is representable. The restrictive character of the dimension \(\omega\) and local finiteness were removed in the early course of the development of the subject, and the class \(\mathcal{C} \mathcal{A}_\alpha,\) of cylindric algebras of dimension \(\alpha,\) where \(\alpha\) is any ordinal, finite or transfinite, was introduced. Three pillars in the development of the subject, and even one can say the three pillars in the development of the subject are Tarski’s representability result of locally finite algebras, Henkin’s characterization of the variety of representable algebras of any dimension via neat embeddings, in his celebrated Neat Embedding Theorem [13, Theorem 3.2.10], and Monk’s proof that the variety of representable algebras of dimension \(> 2\) cannot be axiomatized by a finite schema [20]. The last two results involve the central notion of neat reducts:

**Definition 1.5.** Let \(\alpha < \beta\) be ordinals and \(\mathfrak{B} \in \mathcal{C} \mathcal{A}_\beta.\) Then the \(\alpha\)-neat reduct of \(\mathfrak{B},\) in symbols \(\mathfrak{N} \mathcal{R}_\alpha \mathfrak{B},\) is the algebra obtained from \(\mathfrak{B},\) by discarding cylindrifiers and diagonal elements whose indices are in \(\beta \sim \alpha,\) and restricting the universe to the set \(\mathfrak{N} \mathfrak{R}_\alpha \mathfrak{B} = \{x \in \mathfrak{B} : \{i \in \beta : c.i.x \neq x\} \subseteq \alpha\}.\)
Let $\alpha$ be any ordinal. If $\mathfrak{A} \in \text{CA}_\alpha$ and $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathfrak{B}$, with $\mathfrak{B} \in \text{CA}_\beta$ ($\beta > \alpha$), then we say that $\mathfrak{A}$ neatly embeds in $\mathfrak{B}$, and that $\mathfrak{B}$ is a $\beta$-dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context. For $K \subseteq \text{CA}_\beta$, and $\alpha < \beta$, $\text{Nr}_\alpha K = \{ \text{Nr}_\alpha \mathfrak{B} : \mathfrak{B} \in K \} \subseteq \text{CA}_\alpha$. One can show that for any ordinal $\alpha$, $\mathfrak{A} \in \text{RCa} \iff \mathfrak{A} \in \text{SNr}_\alpha \text{CA}_{\alpha+\omega} \iff \mathfrak{A} \in \text{SNr}_\alpha \text{CA}_{\alpha+\kappa}$, $\kappa$ any infinite ordinal, cf. [13, Theorem 2.6.35]. The last pair of equivalences summarize Henkin’s celebrated Neat Embedding Theorem.

1.4. Modal cylindric algebras. We recall the concrete versions of algebras dealt with, namely, the set algebras of various dimensions. The dimension is determined by an ordinal (finite or transfinite). Such algebras are expansions of cylindric set algebras by diamond operators whose semantics is induced by a frame for a given unimodal logic:

**Definition 1.6.** Let $U$ be a set and $\alpha$ be a ordinal. Let $i < \alpha$, $L$ a unimodal logic with frame $(U, R)$ and $X \subseteq {}^\alpha U$, then

$\diamond_i(X) = \{ s \in {}^\alpha U : \exists t \in X : t \equiv_i s & (t, s_i) \in R \}, X \subseteq {}^\alpha U,$

The $L$ set algebras of dimension $\alpha$ are subalgebras of algebras of the form $\langle B({}^\alpha U), C_i, D_{ij}, \diamond_i \rangle_{i, j < \alpha}$. Semantics, put in the familiar pattern of modal logic, takes the form

$M, s \models \diamond_i \phi \iff (\exists s)(s \equiv_i t, s_i R t_i) \& M, s \models \phi.$

So this is a restricted quantification along $R$.

Let $\alpha$ be an ordinal. The class of modal cylindric set algebras of dimension $\alpha$ is denoted by $\text{LCs}_\alpha$. The class of subdirect products of those, namely, the class of representable algebras of dimension $\alpha$, is denoted by $\text{RLCa}_\alpha$. If $\mathfrak{A} \in \text{CA}_\alpha$ and $p \in \mathfrak{A}$, $\Delta p$, the dimension set of $p$, is defined like the CA case to be the set $\{ i \in \alpha : c_i p \neq p \}$. For $i < \alpha$, we let $q_i$ denote the dual of $c_i$, that is for $p \in \mathfrak{A}$, $q_i p = -c_i - p$. The unary operation $q_i$ reflects (algebraically) universal quantification. For $i \neq j \in \alpha$, $s_{ij} p = c_i (p \cdot d_{ij})$ and $s_i p = p$. The operation $s_i$ reflects algebraically substituting the variable $x_i$ for the free variable $x_j$ in a formula $\phi$ such that the substitution is free, so that $x_i$ is free in $\phi$. We denote symmetric difference by $\oplus$.

**Definition 1.7.** Let $\alpha$ be an ordinal. Let $L$ be a unimodal logic. An algebra $\mathfrak{A}$ is an $L$ cylindric algebra of dimension $\alpha$, briefly an $\text{LCA}_\alpha$, if $\mathfrak{A}$ has signature of $\text{CA}_\alpha$ expanded with $\alpha$ unary operations (boxes) $\square_i$ such that its cylindric reduct, in symbols $\text{Rd}_\alpha \mathfrak{A}$ is a $\text{CA}_\alpha$ and the following equations hold for $p, q \in \mathfrak{A}$ and $i, j, k \in \alpha$:

1. $q_i(p \oplus q) \leq q_i(\square_i p \oplus \square_i q)$.
2. $\square_i p \leq p$.
3. The finitely many modality axioms for the boxes $\square_i$, $i < n$, where $\square_i$ is the box operator corresponding to the accessibility relations in unifames for $L$. 

4. \( c_k \Box_i p = \Box_i p, k \neq i, k \notin \Delta p \).

5. \( s_j^i \Box_i p = \Box_j s_i^j p, j \notin \Delta p \).

For given two sets \( I, J \), \( I \subseteq J \) denotes that \( I \) is a finite subset of \( J \). For an algebra \( \mathfrak{A} \) and \( X \subseteq \mathfrak{A}, \mathfrak{S}g^\mathfrak{A} X \), or simply \( \mathfrak{S}g X \) if \( \mathfrak{A} \) is clear from context denotes the subalgebra of \( \mathfrak{A} \) generated by \( X \). We try to keep things as general as possible.

From now on, fix a unimodal logic \( L \). We only assume that \( L \) is reflexive. Further conditions might be added whenever needed.

Because of item (3) in Definition 1.7, one can define the notion of neat reducts exactly as in the CA case.

**Definition 1.8.** 1. Given a pair of ordinals \( \alpha < \beta \) and \( \mathfrak{B} \in \text{LCA}_\beta \), then the \( \alpha \)-neat reduct of \( \mathfrak{B} \), in symbols \( \mathfrak{N}r_\alpha \mathfrak{B} \), is the algebra with universe \( \mathfrak{N}r_\alpha B = \{ x \in \mathfrak{B} : \Delta x \subseteq \alpha \} \), where \( \Delta x = \{ i \in \beta : c_i x \neq x \} \) and the operations are those of \( \mathfrak{B} \) up to the index \( \alpha \).

2. Let \( \alpha \) be an infinite ordinal and \( \mathfrak{A} \in \text{LCA}_\alpha \). Then \( \mathfrak{A} \) is said to be dimension complemented if \( \Delta p \neq \alpha \) for all \( p \in \mathfrak{A} \). \( \mathfrak{A} \) is said to be locally finite if \( |\Delta p| < \omega \) for all \( p \in \mathfrak{A} \). The class of (locally finite) dimension complemented \( \text{LCA}_\alpha \)'s, is denoted by \( (\text{LLf}_\alpha)\text{LDc}_\beta \).

It is not hard to show that \( \mathfrak{A} \in \text{LDc}_\alpha \iff \alpha \sim \Delta x \) is infinite for every \( x \in A \), cf. [13, Theorem 1.11.4]. In particular, \( \text{LLf}_\alpha \subseteq \text{LDc}_\alpha \). Extending the cylindric notation, we write \( \mathfrak{N}r_\alpha K \) for the class \( \{ \mathfrak{N}r_\alpha \mathfrak{B} : \mathfrak{B} \in K \} \) for any \( K \subseteq \text{LCA}_\beta \), so that \( \mathfrak{N}r_\alpha K \subseteq \text{LCA}_\alpha \).

**Definition 1.9.** Let \( \mathfrak{A} \in \text{LCA}_\alpha \). Then a filter \( F \) of \( \mathfrak{A} \) is a Boolean filter, that satisfies in addition that whenever \( x \in F \) then \( q_i x \in F \) and every \( x \in A \).

We show that filters so defined correspond to congruences, thus filters and congruences can be treated equally giving quotient algebras.

**Theorem 1.10.** Let \( \mathfrak{A} \in \text{LCA}_\alpha \). Let \( \text{Filt}(\mathfrak{A}) \) be the lattice of filters (with inclusion) on \( \mathfrak{A} \), and \( \text{Co}(\mathfrak{A}) \) be the lattice of congruences on \( \mathfrak{A} \). Then \( \text{Filt}(\mathfrak{A}) \cong \text{Co}(\mathfrak{A}) \). Furthermore, \( \Theta \) restricted to maximal filters is an isomorphism into the set of maximal congruences.

**Proof.** The map \( \Theta : \text{Co}(\mathfrak{A}) \to \text{Filt}(\mathfrak{A}) \), defined via

\[ x \mapsto \{ x \in \mathfrak{A} : x \cong 1 \}, \]

is an isomorphism, with inverse \( \Theta^{-1} : \text{Filt}(\mathfrak{A}) \to \text{Co}(\mathfrak{A}) \) defined by

\[ F \mapsto R = \{ (a, b) \in A \times A : a \oplus b \in F \}, \]

where \( \oplus \) as before denotes the symmetric difference. Indeed, let \( \cong \) be a congruence on \( \mathfrak{A} \). Then we show that \( F = \{ a \in A : a \cong 1 \} \) is a filter. Let \( a, b \in F \). Then \( a \cong 1 \) and \( b \cong 1 \), hence \( a \cdot b \cong 1 \), so that \( a \cdot b \in F \). Let \( a \in F \) and \( a \leq b \). Then \( a + b = b \) and we obtain \( b = a + b \cong 1 + b = 1 \). Hence \( b \in F \). Now finally, assume that \( a \in F \) and
\[ i < \alpha. \text{ Then } a \equiv 1 \text{ so } q_i a \equiv q_i 1 = 1, \text{ and we are done. Conversely, let } F \text{ be a filter and let } \cong_F \text{ be given by } a \oplus b \in F. \text{ Then it is straightforward to see that } \cong_F \text{ is a congruence with respect to the Boolean operations, cylindrifiers and substitutions. It remains to check that } \cong_F \text{ is a congruence with respect to the interior operators. Let } i < \alpha. \text{ If } a \cong_F b \text{ then, by definition, } a \oplus b \in F, \text{ hence } q_i(a \oplus b) \in F \text{ be the definition of } F. \text{ But } q_i(a \oplus b) \leq q_i(\Box_i a \oplus \Box_i b) \leq \Box_i a \oplus \Box_i b \in F \text{ by properties of filters, and the interior operator. Now it remains to show that } F_{\cong_F} = F \text{ and } \cong_{F_{\cong_F}} = \cong. \text{ We prove only the former. Let } a \in F. \text{ Then } a = a \oplus 1, \text{ that is } a \cong_F 1, \text{ and so } a \in F_{\cong_F}. \text{ Conversely, if } a \in F_{\cong_F}, \text{ then } a \cong_F 1, \text{ that is } a \oplus 1 \in F, \text{ hence } a \in F \text{ and we are done. Finally, if } R \text{ is maximal, and } \Theta(R) = F \text{ is not a maximal filter, then there is a proper filter } J \text{ extending } F \text{ properly. Let } x \in J \sim F. \text{ Then } (x, 1) \notin \Theta^{-1}F = R \text{ and } (x, 1) \in R_J, \text{ so that } R \text{ is properly contained in the proper congruence } R_J \text{ which is impossible.} \qed \]

For \( X \subseteq \mathfrak{A} \), \( \mathfrak{F}^\alpha X \) denotes the filter generated by \( X \). If \( \mathfrak{A} \subseteq \text{LCA}_\alpha \) and \( \Gamma \subseteq \omega, \alpha \), \( \Gamma = \{i_0, \ldots, i_{n-1}\} \) say, then \( q_{(\Gamma)} x = q_{i_0}, \ldots, q_{i_{n-1}} x \).

**Lemma 1.11.** Let \( \mathfrak{A}, \mathfrak{B} \subseteq \text{LCA}_\alpha \) with \( \mathfrak{B} \subseteq \mathfrak{A} \). Let \( X \subseteq \mathfrak{A} \) and \( M \) be a filter of \( \mathfrak{B} \). We then have:

1. \( \mathfrak{F}^\alpha X = \{a \in A : \exists n \in \omega, x_0, \ldots, x_n \in X, \text{ and } \Gamma \subseteq \omega, \alpha, q_{(\Gamma)}(x_0 \cdot x_1 \cdot x_n) \leq a\} \).

2. \( \mathfrak{F}^\alpha M = \{x \in A : x \geq b \text{ for some } b \in M\} \).

3. \( M = \mathfrak{F}^\alpha M \cap \mathfrak{B} \).

4. If \( \mathfrak{C} \subseteq \mathfrak{A} \) and \( N \) is a filter of \( \mathfrak{C} \), then \( \mathfrak{F}^\alpha (M \cup N) = \{x \in A : b \cdot c \leq x \text{ for some } b \in M \text{ and } c \in N\} \).

5. For every filter \( N \) of \( \mathfrak{A} \) such that \( N \cap B \subseteq M \), there is a filter \( N' \) in \( \mathfrak{A} \) such that \( N \subseteq N' \) and \( N' \cap B = M \). Furthermore, if \( M \) is a maximal filter of \( \mathfrak{B} \), then \( N' \) can be taken to be a maximal filter of \( \mathfrak{A} \).

**Proof.** Only (iv) might deserve attention. The special case when \( N = \{1\} \) is straightforward. The general case follows from this one, by considering \( \mathfrak{A}/N \), \( \mathfrak{B}/(N \cap \mathfrak{B}) \) and \( M/(N \cap \mathfrak{B}) \), in place of \( \mathfrak{A}, \mathfrak{B} \) and \( M \) respectively. \( \square \)

**Definition 1.12.** The *discrete modalizing* of \( \mathfrak{A} \) is the \( \text{LCA}_\alpha \) obtained from \( \mathfrak{A} \) by expanding \( \mathfrak{A} \) with \( \alpha \) many identity maps as the L modalities.

Exactly like the proof of [13, Theorem 2.6.49] defining the modal operator via discrete modalizing we get:

**Lemma 1.13.** Let \( \alpha \geq \omega. \text{ If } \mathfrak{A} \subseteq \text{LDc}_\alpha \) and \( \beta > \alpha \), then there exists \( \mathfrak{B} \subseteq \text{LCA}_\beta \) such that \( \mathfrak{A} \subseteq \mathfrak{N}_{\alpha} \mathfrak{B} \) and for all \( X \subseteq \mathfrak{A} \), \( \mathfrak{S} g^X = \mathfrak{N}_{\alpha} \mathfrak{S} g^\mathfrak{B} X \).
2. Interpolation. We need a few definitions and a technical computational lemma. When dealing with infinitary extensions of a predicate modal logic, the Craig interpolation property ramifies into several forms of interpolation properties that are not necessarily equivalent.

**Definition 2.1.** Let $\alpha$ be any ordinal (finite included) and $\mathfrak{A} \in \mathrm{LCA}_\alpha$.

1. $\mathfrak{A}$ has the weak interpolation property, $\text{WIP}$ for short, if for all $X_1, X_2 \subseteq \mathfrak{A}$, for all $x \in \mathfrak{A}^X X_1$, $z \in \mathfrak{A}^X X_2$ if $x \leq z$, then there exist $\Gamma \subseteq \omega \alpha$ and $y \in \mathfrak{A}^X (X_1 \cap X_2)$ such that $q_{\Gamma} x \leq y \leq c_{\Gamma} z$.

2. $\mathfrak{A}$ has the universal interpolation property, $\text{UIP}$ for short, if the conclusion in $\text{WIP}$ is replaced by ‘there exist $\Gamma \subseteq \omega \alpha$ and $y \in \mathfrak{A}^X (X_1 \cap X_2)$ such that $q_{\Gamma} x \leq y \leq z.’$.

3. $\mathfrak{A}$ has the interpolation property, $\text{IP}$ for short, if the conclusion in $\text{WIP}$ is replaced by ‘there exists $y \in \mathfrak{A}^X (X_1 \cap X_2)$ such that $x \leq y \leq z.’$

**Lemma 2.2.** Let $\alpha \geq \omega$ and let $\beta$ be a cardinal $> 3$. Assume that $\mathfrak{B} \in \mathrm{LCs}_\alpha$ has universe $\phi(\alpha)$ and let $\mathfrak{D} = \mathfrak{Fr}_\beta \mathrm{RLCA}_\alpha$ with $\{x, y, z, w\}$ the first four generators of $\mathfrak{D}$. Let $a, b$ and $c$ be the elements of $\mathfrak{D}$ defined as follows:

$$a = c_0 (x \cdot c_1 y) \cdot c_0 (x \cdot -c_1 y),$$

$$b = c_0 c_1 (c_1 z \cdot s_1 c_1 z \cdot -d_0) + c_0 (x \cdot -c_1 z),$$

and $c$ is obtained from $b$ by replacing $z$ by $w$. Let $\lnj = \{f \in \alpha \cdot \alpha : f \text{ is injective}\} (\in \mathfrak{B})$ and let $\text{Id} : \alpha \rightarrow \alpha$ be the identity map. Then the following hold:

1. There is a $Y \in \mathfrak{B}$ such that $\lnj \cap f (a) \neq \emptyset$ for every homomorphism $f : \mathfrak{D} \rightarrow \mathfrak{B}$ satisfying $f (x) = \lnj$ and $f (y) = Y$.

2. For every finite $\Gamma \subseteq \alpha$, there are $Z, W \in \mathfrak{B}$ such that $\text{Id} \notin c_{\Gamma} h (b \cdot c)$ for every homomorphism $h : \mathfrak{D} \rightarrow \mathfrak{B}$ satisfying $h (x) = \lnj$, $h (z) = Z$ and $h (w) = W$.

**Proof.** For the first part. Let $\sigma_1 \in \alpha \cdot \alpha$ be such that $\sigma_1 (0) = 0$, and $\sigma_1 (\kappa) = \kappa + 1$ for every non-zero $\kappa < \omega$ and $\sigma_1 (\eta) = \eta$ for all $\eta$, $\omega \leq \eta < \alpha$, if $\omega < \alpha$. Let $\sigma_2 : \alpha \rightarrow \alpha$ be the map such that $\sigma_2 \upharpoonright \alpha \sim \{0\} = \sigma$ and $\sigma_2 (0) = 1$. Then $\sigma_1, \sigma_2 \in \lnj$. Take $Y = \{\sigma_1\}$. Then $\sigma_1 \in \lnj \cap c_1 Y$ and $\sigma_2 \in \lnj \sim c_1 Y$ and hence $\sigma_1 \in c_0 (\lnj \cap c_1 Y) \cap c_0 (\lnj \sim c_1 Y)$. Therefore, we have $\sigma_1 \in f (a)$ for every homomorphism $f : \mathfrak{D} \rightarrow \mathfrak{B}$ such that $f (x) = \lnj$ and $f (y) = Y$.

For the second part. No generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{g \in \lnj : g (0) < g (1)\} \cap c_{\Gamma} \{\text{Id}\}$$

and

$$W = \{g \in \lnj : g (0) > g (1)\} \cap c_{\Gamma} \{\text{Id}\}.$$ 

We use the following notation: If $\Delta$ is any non-empty set of ordinals, then $\bigcap \Delta$ is the smallest ordinal in $\Delta$, and if, in addition, $\Delta$ is finite, then $\bigcup \Delta$ is the largest element ordinal in $\Delta$. Let $g \in \alpha \cdot \alpha$. Then, setting

$$\Delta g = \Gamma \sim g [\Gamma \sim \{0, 1\}]$$
for every $g$, we successively compute:

$$c_1Z = \{g : |\Delta g| = 2, g(0) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\},$$

$$(\text{lnj} \sim c_1Z) \cap c_{(\Gamma)}\{Id\} = \{g : |\Delta g| = 2, g(0) = \bigcup \Delta g, g(1) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\},$$

and, finally,

$$c_0(\text{lnj} \sim c_1Z) \cap c_{(\Gamma)}\{Id\} = \{g : |\Delta g| = 2, g(1) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\}.$$

Similarly, we obtain

$$c_0(\text{lnj} \sim c_1W) \cap c_{(\Gamma)}\{Id\} = \{g : |\Delta g| = 2, g(1) = \bigcup \Delta g \cap c_{(\Gamma)}\{Id\}.$$

The last two formulas together give

$$c_0(\text{lnj} \sim c_1Z) \cap c_0(\text{lnj} \sim c_1W) \cap c_{(\Gamma)}\{Id\} = 0.$$

Continuing the computation, we successively obtain:

$$c_1Z \cap d_{01} = \{g : |\Delta g| = 2, g(0) = g(1) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\},$$

$$s_1^0c_1Z = \{g : |\Delta g| = 2, g(1) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\},$$

$$c_1Z \cap s_1^0c_1Z = \{g : |\Delta g| = 2, g(0) = g(1) = \bigcap \Delta g \cap c_{(\Gamma)}\{Id\};$$

hence we finally get

$$c_0c_1(c_1Z \cap s_1^0c_1Z \sim d_{01}) = c_0c_10 = 0,$$

and similarly we get

$$c_0c_1(c_1W \cap s_1^0c_1W \sim d_{01}) = 0.$$

Now take $h$ to be any homomorphism from $\mathfrak{D}$ into $\mathfrak{B}$ such that $h(x) = \text{lnj}$, $h(z) = Z$ and $h(w) = W$. Then from the above $h(a \cdot b) \cap c_{(\Gamma)}\{Id\} = \emptyset$. Then applying $c_{(\Gamma)}$ to both sides of this last equation, we get $c_{(\Gamma)}h(a \cdot b) \cap c_{(\Gamma)}\{Id\} = \emptyset$, so $Id \notin c_{(\Gamma)}h(a \cdot b).$

Next we recall a generalized notion of free algebras:

**Definition 2.3.** Let $K \subseteq \text{LCA}_{\alpha}$, $\beta$ be a non-zero cardinal, and $\rho : \beta \to \varphi(\alpha)$. Let $\mathfrak{A} \subseteq \text{LCA}_{\alpha}$ have a set of generators $\text{rng}(x)$ where $x = \langle x_i : i < \beta \rangle$ such that $\Delta x_i \subseteq \rho(i)$. We say that $x$ $K$-freely generates $\mathfrak{A}$ under the dimension restricting function $\rho$ if for any algebra $\mathfrak{B} \in K$, for any $y = \langle y_i : i < \beta \rangle \in \beta \mathfrak{B}$ whenever $\Delta y_i \subseteq \rho(i)$ for all $i < \beta$, then there is a unique homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h \circ x = y$. 
Theorem 2.4. For any $K \subseteq \text{LCA}_\alpha$, $\beta$ be a non-zero cardinal, and $\rho : \beta \rightarrow \wp(\alpha)$, there is a unique (up to isomorphism) algebra $\mathfrak{A} \in \text{LCA}_\alpha$ with a set of generators $x \in \beta \mathfrak{A}$, such that $x$ $K$–freely generates $\mathfrak{A}$ under the dimension restricting function $\rho$. We denote this algebra by $\mathfrak{Fr}_\beta^K$. If $\rho = \beta \times \{\alpha\}$, then $\mathfrak{Fr}_\beta^K$ is the $K$ free algebra on $\beta$ generators. In this case we suppress $\rho$, and denote it using the the standard notation by $\mathfrak{Fr}_\beta^K$.

Proof. \[13, \text{Theorems 2.5.36–2.5.37}.\]

The next Theorem is the heart and soul of the present paper:

Theorem 2.5. As usual let $L$ be a reflexive uni-modal logic. Let $\alpha$ be an infinite ordinal. Let $\beta$ be a non-zero cardinal and let $\rho : \beta \rightarrow \wp(\alpha)$.

1. If $\alpha \sim \rho(i)$ is infinite for all $i \in \beta$, then $\mathfrak{Fr}_\beta^L \text{LCA}_\alpha$ has the IP.

2. If $\rho(i) = \alpha$ for all $i \in \beta$, then $\mathfrak{Fr}_\beta^L \text{LCA}_\alpha = \mathfrak{Fr}_\beta^\text{RLCA}_\alpha$ and $\mathfrak{Fr}_\beta^\text{RLCA}_\alpha = \mathfrak{Fr}_\beta^\text{RLCA}_\alpha$ have WIP.

3. If $L$ has universal frames, then for any $K$ between $\text{LCA}_\alpha$ and $\text{RLCA}_\alpha$, the (free) algebra $\mathfrak{Fr}_\alpha^K$ does not have UIP, a priori they do not have IP.

Proof. (1) To dilute the proof we first give the general idea: Let $\beta$ be a non-zero cardinal. For the sake of brevity we denote $\mathfrak{Fr}_\beta^L \text{LCA}_\alpha$ by $\mathfrak{A}$. Let $X_1, X_2 \subseteq \beta$. We assume without loss that $\beta = X_1 \cup X_2$. Let $a \in \mathfrak{Sg}^\mathfrak{A} X_1$ and $c \in \mathfrak{Sg}^\mathfrak{A} X_2$ be such that $\mathfrak{A} \models a \leq c$. We must show that there exists an interpolant of $a$ and $c$. Like the standard proof of the the interpolation theorem for first order logic, the proof is by contradiction. We assume no such $c$ exists. Then we construct a model of $(\mathfrak{A}, a-c)$, that is to say, we construct a homomorphism from $\mathfrak{A}$ into a set algebra such that $f(a-c) \neq 0$. This contradicts $\mathfrak{A} \models a \leq c$. We divide the proof into three parts (i), (ii) and (iii). Using model-theoretic jargon and terminology in parts (i) and (ii) we force with $\mathfrak{A}$, preparing for constructing the model. This is done by adding enough supply of witnesses, in preparation of defining a notion of forcing. Algebraically we neatly embed $\mathfrak{A}$ into “enough” spare dimensions, or expressed differently, we take a suitable (minimal) $\kappa$–dilation of $\mathfrak{A}$, where $\kappa$ is a regular cardinal $> \max\{|\alpha|, |\mathfrak{A}|\}$.

The neat embedding part is done in part (i). In part (ii) we define a notion of forcing and construct two appropriate maximal filters, that are in some sense potential representations. In part (iii) we construct the desired model from these maximal filters using a third maximal filter that agree with both maximal filters on the algebra $\mathfrak{Sg}^\mathfrak{A}(X_1 \cap X_2)$.

Now for the detailed technical proof:

Part (i): Let $\mathfrak{A} = \mathfrak{Fr}_\beta^L \text{LCA}_\alpha$. Let $a \in \mathfrak{Sg} X_1$ and $c \in \mathfrak{Sg} X_2$ be such that $a \leq c$. We want to find an interpolant in $\mathfrak{Sg}^\mathfrak{A}(X_1 \cap X_2)$. By lemma 1.13, take $\mathfrak{B} \in \text{LCA}_\kappa, \kappa$ a regular cardinal $> \max\{|\alpha|, |\mathfrak{B}|\}$, such that $\mathfrak{A} \models \mathfrak{Fr}_\alpha^\mathfrak{B} \mathfrak{B}$. Assume that no such interpolant exists in $\mathfrak{A}$, then no interpolant exists in $\mathfrak{B}$, because if $b$ is an interpolant in $\mathfrak{Sg}^\mathfrak{B}(X_1 \cap X_2)$, then there exists a finite set $\Gamma \subseteq \kappa \sim \alpha$, such that
\(c^{(\Gamma)b} \in \mathfrak{N}_\alpha \mathcal{S}g^{3^\alpha}(X_1 \cap X_2) = \mathcal{S}g^{3^\alpha}(X_1 \cap X_2) = \mathcal{S}g^{3\alpha}(X_1 \cap X_2)\); which is clearly an interpolant in \(\mathfrak{A}\).

Part (ii): Arrange \(\kappa \times \mathcal{S}g^{3\alpha} X_1 \) and \(\kappa \times \mathcal{S}g^{3\alpha} X_2\) into \(\kappa\)-termed sequences:

\[\langle (k_i, x_i) : i \in \kappa \rangle \quad \text{and} \quad \langle (l_i, y_i) : i \in \kappa \rangle\]

respectively.

Since \(\kappa\) is regular, we can define by recursion \(\kappa\)-termed sequences of witnesses:

\[\langle u_i : i \in \kappa \rangle \quad \text{and} \quad \langle v_i : i \in \kappa \rangle\]

such that for all \(i \in \kappa\) we have:

\[u_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}\]

and

\[v_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}\]

For a Boolean algebra \(\mathcal{C}\) and \(Y \subseteq \mathcal{C}\), we write \(fl^\mathcal{C} Y\) to denote the Boolean filter generated by \(Y\) in \(\mathcal{C}\). Now let

\[Y_1 = \{a\} \cup \{-c, x_i + s_{x_i}^i x_i : i \in \kappa\},\]

\[Y_2 = \{-c\} \cup \{-c, y_i + s_{y_i}^i y_i : i \in \kappa\},\]

\[H_1 = fl^{Bl\mathcal{S}g^{3\alpha}}(X_1) Y_1, \quad H_2 = fl^{Bl\mathcal{S}g^{3\alpha}}(X_2) Y_2,\]

\[H = fl^{Bl\mathcal{S}g^{3\alpha}}(X_1 \cap X_2) [(H_1 \cap \mathcal{S}g^{3\alpha}(X_1 \cap X_2)) \cup (H_2 \cap \mathcal{S}g^{3\alpha}(X_1 \cap X_2))].\]

Then \(H\) is a proper filter of \(\mathcal{S}g^{3\alpha}(X_1 \cap X_2)\). To prove this claim it is sufficient to consider any pair of finite, strictly increasing sequences of natural numbers

\[\eta(0) < \eta(1) \cdots < \eta(n-1) < \alpha \quad \text{and} \quad \xi(0) < \xi(1) < \cdots < \xi(m-1) < \alpha,\]

and to prove that the following condition (+) holds:

For any \(b_0, b_1 \in \mathcal{S}g^{3\alpha}(X_1 \cap X_2)\) such that

\[a \cdot \prod_{i \leq n} (-s_{\eta(i)}^i c_{\eta(i)} x_{\eta(i)} + s_{\eta(i)}^i s_{\eta(i)}^i x_{\eta(i)}) \leq b_0\]

and

\[(-c) \cdot \prod_{i \leq m} (-s_{\xi(i)}^i c_{\xi(i)} y_{\xi(i)} + s_{\xi(i)}^i s_{\xi(i)}^i y_{\xi(i)}) \leq b_1\]

we have

\[b_0 \cdot b_1 \neq 0.\]

This can be proved by a tedious induction on \(n + m\). We only give the base of the induction. If \(n + m = 0\), then (+) simply expresses the fact that no interpolant of \(a\) and \(c\) exists in \(\mathcal{S}g^{3^\alpha}(X_1 \cap X_2)\). In more detail: if \(n + m = 0\), then \(a_0 \leq b_0\) and \(-c \leq b_1\). So if \(b_0 \cdot b_1 = 0\), we get \(a \leq b_0 \leq -b_1 \leq c\). Proving that \(H\) is a proper filter of \(\mathcal{S}g^{3\alpha}(X_1 \cap X_2)\), let \(H^*\) be a (proper Boolean) ultrafilter of \(\mathcal{S}g^{3\alpha}(X_1 \cap X_2)\)
containing $H$. We obtain ultrafilters $F_1$ and $F_2$ of $\mathfrak{S}g^{\mathfrak{A}_1}$ and $\mathfrak{S}g^{\mathfrak{A}_2}$, respectively, such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathfrak{S}g^{\mathfrak{A}_1}(X_1 \cap X_2) = H^* = F_2 \cap \mathfrak{S}g^{\mathfrak{A}_2}(X_1 \cap X_2).$$

Now for all $x \in \mathfrak{S}g^{\mathfrak{A}_1}(X_1 \cap X_2)$ we have

$x \in F_1$ if and only if $x \in F_2$.

Also from how we defined our ultrafilters, $F_i$ for $i \in \{1, 2\}$ are Henkin, that is, they satisfy the following condition:

(*) For all $k < \mu$, for all $x \in \mathfrak{S}g^{\mathfrak{A}_i}$ if $c_kx \in F_i$ then $s^k_i x$ is in $F_i$ for some $l \notin \Delta x$. We obtain ultrafilters $F_1$ and $F_2$ of $\mathfrak{S}g^{\mathfrak{A}_1}$ and $\mathfrak{S}g^{\mathfrak{A}_2}$, respectively, such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathfrak{S}g^{\mathfrak{A}_1}(X_1 \cap X_2) = H^* = F_2 \cap \mathfrak{S}g^{\mathfrak{A}_2}(X_1 \cap X_2).$$

Now for all $x \in \mathfrak{S}g^{\mathfrak{A}_1}(X_1 \cap X_2)$ we have

$x \in F_1$ if and only if $x \in F_2$.

Part (iii): Fix $m \in \{1, 2\}$. We denote $\mathfrak{S}g^{\mathfrak{A}_m}X_1$ by $\mathfrak{A}_m$. For $i, j \in \beta$, define $iE_j \iff d^m_i \in F_m$. Then $E$ is an equivalence relation on $\beta$. Recall that a weak space of dimension $\alpha$ is a set of the form $V = \{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ for a given fixed in advance $p \in {}^\alpha U$ (for some non-empty set $U$); we denote it by $\alpha U(p)$. Let $V = \alpha \beta(\text{Id})$ where $\text{Id}$ is the identity map on $\alpha$. Since $\mathfrak{B} \in \text{LDC}_\beta$, each $\tau \in \beta \beta(\text{Id})$, induces a unary substitution operator $s_\tau$ on $\mathfrak{B}$ which is (among many other things) a Boolean endomorphism defined exactly like in [13, Definition 1.11.11] with properties summarized in [13, Theorem 1.11.12]. This operation reflects algebraically the metalogical operation of simultaneous substitution of finitely many variables for variables, such that the substitution is free. For $\tau, \sigma \in V$, if $\sigma(i)E\tau(i)$ for all $i \in \beta$, then for any $a \in A_m$, one can show that $s_\tau^{-1}a \in F \iff s_\sigma^{-1}a \in F$. Let $W = \alpha [\beta/E](\text{Id}) = \{s : \alpha \rightarrow \beta/E : \{s(i) \neq i/E\} < \omega\}$. Define $f : \mathfrak{A}_m \rightarrow \varphi(W)$ via

$$p \mapsto \{\bar{x} \in W : s_\tau^{-1}\bar{x} \subseteq \text{Id}_{\tau, \beta_{\alpha}} \} \in F,$$

where $\bar{x}(i/E) = x(i)/E$ for $i < \alpha$. It can be checked that $f$ is a well defined cylindric homomorphism. Next we represent the boxes. For $i \in \alpha$ and $p \in \mathfrak{A}$, define $O_{p,i} = \{k/E \in \alpha/E : s^k_i \Delta_i p \in F\}$. Let $\mathcal{B} = \{O_{p,i} : i \in \alpha, p \in A\}$.

For a sequence $s$, we write $s^i_k$ for the sequence which has the same domain as $s$ and agrees with $s$ at $\text{dom}(s) \sim \{i\}$, but $s^i_k(i) = u$. For $x \in {}^\alpha \beta(\text{Id})$, we write $[x]$ for the map $\bar{x} \in W$ defined by $\bar{x}(i/E) = x(i)/E$ for all $i \in \alpha$. For each $i < \alpha$, define $\square_i : \varphi(W) \rightarrow \varphi(W)$ by

$$[x] \equiv \square_i X \iff \exists U \in \mathcal{B}(x_i/E \in U \subseteq \{u/E \in \alpha/E : [x]u_i/E \in X\}),$$
where $X \subseteq W$. Note that $[x]^a_{u/E} = [x]^a_u$.

We need to show that for all $i < \alpha$, $f(\Box_i p) = \Box_i (f(p))$. The reasoning is analogous to that in [10]. Let $\sup \langle x \rangle = \{ k \in \alpha : x_k \neq k \}$. Then, $|\sup \langle x \rangle| < \omega$ and by definition, $s_x \Box_i p \in F$. Hence $s_x^i \Box_i \ldots \Box_i n p \in F$, where $\sup \langle x \rangle \setminus \{ i \} = \{ j_1, \ldots, j_n \}$. Let $y = [j_1|x_1] \ldots [j_n|x_n]$. Then $x_i/E \in \{ u/E : s_x^i \Box_i s_y p \in F \}$. But by reductivity, we have $\Box_i s_y p \leq s_y p$, hence $U = \{ u/E : s_x^i \Box_i s_y p \in F \} \subseteq \{ u/E : s_x^i s_y p \in F \}$. It follows that $x_i/E \in U \subseteq \{ u/E : x_i^u \in \Psi(p) \}$. Thus $[x] \in \Box_i f(p)$.

Now we prove the other direction. Let $[x] \in \Box_i f(p)$. Let $U \in \mathfrak{B}$ be such that $x_i/E \in U \subseteq \{ u/E \in \alpha : [x]^a_{u/E} \in \Box_i f(p) \}$. Then $x_i/E \in U \subseteq \{ u/E \in \alpha : s_x^i s_y p \in F \}$. Assume that $U = O_{r,j}$, where $r \in \mathfrak{A}$ and $j \in \alpha$. Let $u \in \alpha \sim [\Delta p \cup \Delta r \cup \{ i, j \}]$. By dimension complementedness such a $u$ exists. Then we have:

$$s_x^i \Box_j r \in F \iff s_x^i s_x p \in F,$$

$$s_x^j \Box_j r \in F \iff s_x^i \Box_j r \in F.$$

But $s_x^j \Box_j r = s_x^i \Box_i s_x^j r$, so letting $\oplus$ denote symmetric difference, we have

$$s_x^i \Box_i s_x^j r \cdot s_x^i s_x p + s_x^i \Box_i s_x^j r \in F \iff s_x^i \Box_i s_x^j r \cdot s_x p + s_x^j r \in F,$$

$$\iff q_i \Box_i s_x^j r \cdot s_x p + s_x^i \Box i s_x^j r \in F,$$

$$\iff s_x^i \Box i s_x^j r \cdot s_x p + s_x^j r \in F,$$

$$\iff s_x^j \Box_j r \cdot s_x^i s_x p + s_x^i \Box_j r \in F,$$

$$\iff s_x^i \Box_j r \cdot s_x p + s_x^i \Box_j r \in F.$$

But $s_x^i \Box_j r \in F$, hence $s_x \Box_i p \in F$, and so $x \in f(\Box_i p)$ as required.

Without loss of generality, we can assume that $X_1 \cup X_2 = X$. We have $f_1$ and $f_2$ agree on $X_1 \cap X_2$. So that $f_1 \cup f_2$ defines a function on $X_1 \cup X_2$. By (dimension restricted) freeness, it follows that there is a homomorphism $f$ from $\mathfrak{A}$ to $(\mathfrak{G}(W), \Box_i)_{i<\alpha}$ such that $f_1 \cup f_2 \subseteq f$. Then $Id \in f(a) \cap f(-c) = f(a \cdot -c)$. This is so because $s_{Id} a = a \in F_1$ $s_{Id} (-c) = -c \in F_2$. But this contradicts the premise that $a \leq c$.

(2) For a term $t$ in the language of LCA$_\alpha$, $\beta$ an infinite ordinal, we write $\text{ind}(t)$ for the set of indices in $t$ and $\text{var}(t)$ for the set of variables in $t$. For example if $t$ is $c_i(x_0 \cdot x_1)$ $(i \in \beta)$, then $\text{ind}(t) = \{ i \}$ and $\text{var}(t) = \{ x_0, x_1 \}$. Now let $\alpha \geq \omega$. Let $K$ be a class of algebras such that RLCA$_\alpha \subseteq K \subseteq \text{LCA}_\alpha$. Then, it suffices to show that for any terms of the language of LCA$_\alpha$, $\sigma, \tau$ say, if $K \models \sigma \leq \tau$, then there exist a term $\pi$ with $\text{Var}(\pi) \subseteq \text{Var}(\sigma) \cap \text{Var}(\tau)$ and a finite $\Delta \subseteq \alpha$ such that

$$K \models q(\Delta) \sigma \leq \tau \leq c(\Delta) \tau.$$

In particular, for any non-zero cardinal $\beta$, $\exists \tau K$, has the WIP. Assume that $K \models \sigma \leq \tau$. We want to find an interpolant, i.e a $\pi$ as in the conclusion. Let $\mathfrak{L}_\alpha$ be the language of LCA$_\alpha$ and for $n \leq \omega$, let $\mathfrak{L}_n^\alpha$ or simply $\mathfrak{L}^n$ be the language of
\( LCA_{\alpha+n} \). We write \( \mathcal{L} \) for \( L^{(0)} \). For an assignment \( s : \omega \rightarrow B, B \in LCA_\alpha \), we write \( s \) for its extension, done recursively the usual way, to all terms of \( L_\alpha \). Now since \( \mathcal{N}_\alpha LCA_{\alpha+\omega} \subseteq K \), then for every \( B \in LCA_{\alpha+\omega} \), for every \( s : \omega \rightarrow B \), such that \( \text{rngs} \subseteq \mathcal{N}_\alpha B \), \( B \models (\sigma \leq \tau)[s] \).

Hence, by the previous item, since \( L \) is (still) reflexive, there is a term \( \pi \) of \( L^{(\omega)} \) which contains only occurrences of variables which occur in both \( \sigma \) and \( \tau \), which satisfies that for all \( B \in LCA_{\alpha+\omega} \), for every \( s : \omega \rightarrow B \), such that \( \text{rngs} \subseteq \mathcal{N}_\alpha B \),

\[
B \models (\sigma \leq \tau)[s] \text{ and } (\pi \leq \tau)[s].
\]

Expand each language \( L^{(n)} \), \( n < \omega \), by adjoining to its signature a fixed \( \omega \)-termed sequence \( a = \langle a_0, a_1, a_2, \ldots \rangle \) of distinct individual constants symbols. Let \( \sigma' \), \( \tau' \) and \( \pi' \) be the terms of the language extending \( L^{(\omega)} \) that are obtained respectively from \( \sigma \), \( \tau \) and \( \pi \) by replacing each variable \( v_k \) in all of its occurrences by the constant symbol \( a_k \). For each \( k < \omega \) let \( \Pi^{(k)} \) be the set of all identities of the form \( c_\mu a_v = a_v \), where \( \alpha < \mu < \alpha + \beta \) and \( v < \omega \). Now we have

\[
\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)} \models (\sigma' \leq \pi') \land (\pi' \leq \tau')
\]

where \( \Sigma^{(\omega)} = \text{Eq}(LCA_{\alpha+\omega}) \). Therefore, by the compactness theorem there is a finite subset \( \theta \) of \( \Sigma^{(\omega)} \) union \( \bigcup_{k<\omega} \Pi^{(k)} \) such that

\[
\theta \models (\sigma' \leq \pi') \land (\pi' \leq \tau').
\]

Then there there is a finite ordinal, \( \delta \), say such that \( \theta \subseteq \Sigma^{(\delta)} \cup \Pi^{(\delta)} \). Now \( \text{ind} (\pi) \subseteq \alpha + \beta \). Choose two pairwise disjoint sets \( \Gamma, \Delta \subseteq \alpha \) such that \( |\Gamma| = |\Delta| = \delta \) and such that neither \( \Gamma \) nor \( \Delta \) contains any index occurring in \( \sigma \), \( \tau \), or \( \pi \). Let \( \mu, \nu \) be two sequences of length \( \delta \) which enumerate the elements of \( \Gamma \), and \( \Delta \), respectively (and hence are necessarily injective). Let \( \bar{\sigma} \) and \( \bar{\tau} \) be the terms of \( L \) that are obtained from \( \sigma \) and \( \tau \), respectively, by replacing each variable \( v_k \) in all its occurrences by the term

\[
\sigma^{\mu_0}_{v_0} \sigma^{\mu_1}_{v_1} \ldots \sigma^{\mu_{\delta-1}}_{v_{\delta-1}} v_k.
\]

Let \( \bar{\pi} \) be obtained from \( \pi \) by making these same replacements and also by replacing every index \( \lambda \) in \( \pi \) such that \( \lambda \in (\alpha + \beta) \) \( \sim \alpha \) to an ordinal in \( \Gamma \); so that different ordinals are substituted for different ordinals. Notice that \( \bar{\pi} \), as well as \( \bar{\sigma} \) and \( \bar{\tau} \), is a term of \( L \). Let \( \rho \) be any injective map from \( \alpha + \beta \) onto \( \alpha \) such that \( \beta \xi = \xi \) for every \( \xi < \alpha, \xi \in \text{ind} \sigma \cup \text{ind} \tau \cup \text{ind} \pi \) such that

\[
\rho(k + k) = \mu_k \text{ for every } k < \delta.
\]

Here we are using that \( \alpha \) is infinite. Then for every \( C \in LCA_\alpha \) we have, \( R\sigma^{(\rho)} C \) as defined in [13, Definition 2.6.1] is in \( LCA_{\alpha+\delta} \) and by hypothesis, we get

\[
\sigma^{\mu_0}_{v_0} \sigma^{\mu_1}_{v_1} \ldots \sigma^{\mu_{\delta-1}}_{v_{\delta-1}} x \in \mathcal{N}_\alpha R\sigma^{(\rho)} C \text{ for every } x \in C.
\]

We can now readily conclude that

\[
LCA_\alpha \models \bar{\sigma} \leq \bar{\pi} \text{ and } LCA_\alpha \models \bar{\pi} \leq \bar{\tau}.
\]
Now neither the $\Gamma$ nor $\Delta$ contains an index occurring in $\sigma$, $\tau$, or $\pi$ then
\[
\text{LCA}_\alpha \models \bar{\sigma} = s_{v_0}^\mu s_{v_1}^\mu \ldots s_{v_{\delta-1}}^\mu \sigma \quad \text{and} \quad \text{LCA}_\alpha \models \bar{\tau} = s_{v_0}^\mu s_{v_1}^\mu \ldots s_{v_{\delta-1}}^\mu \tau.
\]
Combining these results we get that
\[
\text{LCA}_\alpha \models s_{v_0}^\mu s_{v_1}^\mu \ldots s_{v_{\delta-1}}^\mu \sigma \leq \bar{\pi} \leq s_{v_0}^\mu s_{v_1}^\mu \ldots s_{v_{\delta-1}}^\mu \tau;
\]
in particular, in every member of $K$; the same is true of
\[
q_{\mu_0} q_{\mu_1} \ldots q_{\mu_{\delta-1}} \sigma \leq \bar{\pi} \leq c_{\mu_0} c_{\mu_1} \ldots c_{\mu_{\delta-1}} \tau.
\]
Therefore, since $\bar{\pi}$ is a term of $\mathcal{L}$ and it contains like $\pi$ only occurrences of variables which occur at the same time in both $\sigma$ and $\tau$ we have shown that the inclusion $\sigma \leq \tau$ can indeed be interpolated relative to $K$.

(3) Let $K$ be as in the hypothesis and assume for contradiction that $\mathfrak{F}_\omega K$ has UIP. We first show that $K$ has the amalgamation property with respect to countable algebras. That is, we show that if $A, B, C \in \text{RLCA}_\alpha$ are countable, $C \subseteq A$ and $C \subseteq B$, then they have an amalgam in $K$: There is $D \in K$ and injective homomorphisms $m : A \to D$, $n : B \to D$ such that $m \upharpoonright C = n \upharpoonright C$. For brevity, let $\mathfrak{F} = \mathfrak{F}_\omega K$. Take $\beta_1, \beta_2 \subseteq \omega$, such that $\omega = \beta_1 \cup \beta_2$, and such that there are surjective homomorphisms $h : \mathfrak{F} \to C$, $h_1 : \mathfrak{F} \to A$ and $h_2 : \mathfrak{F} \to B$, with $h_1 \upharpoonright \beta_1$ is a bijection from $\beta_1$ to $A$, $h_2 \upharpoonright \beta_2$ is a bijection from $\beta_2$ to $B$ and $h \upharpoonright \beta_1 \cap \beta_2$ is a bijection from $\beta_1 \cap \beta_2$ to $C$. Such homomorphisms exist since $\text{RLCA}_\alpha \subseteq K$, all algebras considered are countable, and by the freeness of $\mathfrak{F}$. Let $\mathfrak{F}_1 = \mathfrak{F}^\beta_1$ and $\mathfrak{F}_2 = \mathfrak{F}^\beta_2$. Let $M = \{d \in \mathfrak{F}_1 : h_1(d) = 1\}$ and $N = \{d \in \mathfrak{F}_2 : h_2(d) = 1\}$. Then $M$ and $N$ are filters. We show using UIP that $P = \mathfrak{F}_0^\beta (M \cup N)$ (the filter generated by $M \cup N$ in $\mathfrak{F}$) is a proper filter of $\mathfrak{F}$. Then we show that $\mathfrak{F}/P$ is the required amalgam. Let $x \in P \cap \mathfrak{F}_1$. Then $b \in M$ and $c \in N$ such that $b + c \leq x$. Thus $c \leq x + -b$. But $x + -b \in \mathfrak{F}_1$ and $c \in \mathfrak{F}_0$, it follows from the assumption that $\mathfrak{F}$ has UIP, that there exist $J \subseteq \omega \alpha$ and $d \in \mathfrak{F}_0$, such that $q_{(J)c} \leq d \leq x + -b$. Notice that $c \in N$, so $q_{(J)c} \in N$ by properties of filters, thus $d \in N$, hence $d \in M$, so $x \in M$ and $x + -b \in M$. But $b \in M$ so $x + -b + b = x \in M$. We have shown that $P \cap \mathfrak{F}_1 = M$. Similarly $P \cap \mathfrak{F}_2 = N$. In particular, $P$ is proper, and $\mathfrak{F}/P$ is the required amalgam. To see why, the algebra $\mathfrak{F}_1/M \cong B$ embeds into $\mathfrak{F}/P$ via $f/M \mapsto f/P$ ($f \in \mathfrak{F}_1$) and similarly the algebra $\mathfrak{F}_2/N \cong A$ embeds into $\mathfrak{F}/P$ via $f/N \mapsto f/P$ ($f \in \mathfrak{F}_2$). These embeddings, as can be easily checked, coincide on $\mathfrak{F}/M \cap \mathfrak{F}_0 = \mathfrak{F}_0/N \cap \mathfrak{F}_0 \cong A$.

Now we construct three countable algebras $A_0, A_1, A_2 \in \text{RLCA}_\alpha$ such that $A_0$ embeds into both $A_1$ and $A_2$, but such algebras have no amalgam in $\text{LCA}_\alpha$ over $A_0$ which contradicts what we just proved. Let $\mathfrak{A} = \mathfrak{F}_4 \text{LCA}_\alpha$ with $\{x, y, z, w\}$ its free generators. Let $X_1 = \{x, y\}$ and $X_2 = \{x, z, w\}$. To dilute the rest of the proof we start by giving the gist of the idea. The technical detailed proof will follow. Again the proof can be divided into three parts.

(i) First we define three elements $r \in \mathfrak{F}_4 X_1$ and $s, t \in \mathfrak{F}_4 X_2$ such that $t + s \leq r$. This part is highly computational, depending essentially on the interaction of substitutions corresponding to replacements with cylindrifications. Here
the properties of substitutions and cylindrifications summarized in [13, Theorem 1.11.14] are used repeatedly.

(ii) Now let \( \{x', y', z', w'\} \) be the 4 generators of \( \mathfrak{D} = \mathfrak{F}_4 \mathfrak{RLCA}_\alpha \). Then we let \( \Psi \) be the homomorphism \( \Psi : \mathfrak{F}_4 \mathfrak{RLCA}_\alpha \to \mathfrak{F}_4 \mathfrak{RLCA}_\alpha \) specified by \( \Psi(i) = i' \) for \( i \in \{x, y, w, z\} \). Let \( r' = \psi(r), u' = \psi(u), s' = \psi(s) \) and \( t' = \psi(t) \). Then \( r', u', s', t' \in \mathfrak{D} \) and using properties of filters we show that \( q_{(\gamma)}(s' + t') \leq u' \leq r' \). This basically says that \( r' \leq s' \cdot t' \) can be interpolated in \( \mathfrak{D} \). Here many manipulations of the properties of filters summarized in Lemma 1.11 are carried out.

(iii) Denoting \( \Psi(a) \) by \( a' \), we next define a set algebra \( \mathfrak{B} \), and an element \( X_{Id} \in B \) (denoted by \( \text{Inj} \) in Lemma 2.2), and using Lemma 2.2, we show that there is a subset \( Y \) of \( \alpha \alpha \) such that \( X_{Id} \cap f(r') \neq 0 \) for every homomorphism \( f : \mathfrak{D} \to \mathfrak{B} \) such that \( f(x') = X_{Id} \) and \( f(y') = Y \) and also that for every \( \Gamma \subseteq \omega \alpha \), there are subsets \( Z, W \) of \( \alpha \alpha \) such that \( X_{Id} \sim C_\Gamma g(s' \cdot t') \neq 0 \) for every homomorphism \( g : \mathfrak{D} \to \mathfrak{B} \) such that \( g(x') = X_{Id}, g(z') = Z \) and \( g(w') = W \). Finally, we define an atomic subalgebra \( \mathfrak{C} \) (whose elements are unions of atoms of the form \( X_R, R \) an equivalence relation) of \( \mathfrak{B} \) in which \( X_{Id} \) is an atom. Take any \( k : \mathfrak{D} \to \mathfrak{B} \) a homomorphism such that \( k(x') = X_{Id}, k(y') = Y, k(z') = Z, \) and \( k(w') = W \). This is possible by the freeness of \( \mathfrak{D} \). Using the fact that \( X_{Id} \cap k(r') \) is non-empty we get \( X_{Id} \cap k(u') = k(x' \cdot u') \ni k(x' \cdot r') \neq 0 \). Then using the fact that \( X_{Id} \sim C_\Gamma k(s' \cdot t') \) is non-empty, we get \( X_{Id} \sim k(u') = k(x' \cdot u') \ni k(x' \cdot c_\Gamma(s' \cdot t')) \neq 0 \). However, it is impossible for \( X_{Id} \) to intersect both \( k(u') \) and its complement since \( k(u') \in C \) and \( X_{Id} \) is an atom.

**The details of the above sketch:**

Part (i): Let \( r, s \) and \( t \) be defined as follows:

\[
\begin{align*}
r &= q_0(-x - c_1y) + q_0(-x + c_1y), \\
s &= q_0 q_1(-c_1z + s_0 c_1z + d_{01}) + q_0(-x + c_1z), \\
t &= q_0 q_1(-c_1w + s_0 c_1w + d_{01}) + q_0(-x + c_1w),
\end{align*}
\]

where \( x, y, z, \) and \( w \) are the first four free generators of \( \mathfrak{A} \). Then we claim that \( s + t \leq r \). To prove the claim, let \( e = x \cdot c_1y \cdot -c_0(x \cdot -c_1z) \), and \( f = x \cdot -c_1y \cdot -c_0(x \cdot -c_1z) \). We have:

\[
c_1 e \cdot c_1 f \leq c_1(x \cdot c_1y) \cdot c_1(x \cdot -c_1y) \quad \text{by} \quad [13, \text{Theorem 1.2.7}] \\
= c_1 x \cdot c_1 y \cdot c_1 x \cdot -c_1 y \quad \text{by} \quad [13, \text{Theorem 1.2.11}]
\]

and so

\[
c_1 e \cdot c_1 f = 0. \tag{1}
\]

From the inclusion \( x \cdot -c_1z \leq c_0(x \cdot -c_1z) \), we get \( x \cdot -c_0(x \cdot -c_1z) \leq c_1z \). Thus \( e, f \leq c_1z \) and hence, by [13, Theorem 1.2.9],

\[
c_1 e \leq c_1 z \quad \text{and} \quad c_1 f \leq c_1 z. \tag{2}
\]
We now compute:
\[
\begin{align*}
c_0e \cdot c_0f & \leq c_0c_1e \cdot c_0c_1f \\
& = c_0c_1e \cdot c_0s_1^0c_1f \quad \text{by [13, Theorems 1.5.8(i), 1.5.9(i)]} \\
& = c_1(c_0c_1e \cdot s_1^0c_1f) \\
& = c_0c_1(c_1e \cdot s_1^0c_1f) \\
& = c_0c_1[c_1e \cdot s_1^0c_1f \cdot(-d_{01} + d_{01})] \\
& = c_0c_1[(c_1e \cdot s_1^0c_1f \cdot -d_{01}) + (c_1e \cdot s_1^0c_1f \cdot d_{01})] \\
& = c_0c_1[(c_1e \cdot s_1^0c_1f \cdot -d_{01}) + (c_1e \cdot c_1f \cdot d_{01})] \quad \text{by [13, Theorem 1.5.5]} \\
& = c_0c_1(c_1e \cdot s_1^0c_1f \cdot -d_{01}) \quad \text{by (1)} \\
& \leq c_0c_1(c_1z \cdot s_1^0c_1z \cdot -d_{01}) \quad \text{by (2), [13, Theorem 1.2.7].}
\end{align*}
\]

We have proved that
\[
c_0[x \cdot c_1y \cdot -c_0(x \cdot -c_1z)] \cdot c_0[x \cdot -c_1y \cdot -c_0(x \cdot -c_1z)] \leq c_0c_1(c_1z \cdot s_1^0c_1z \cdot -d_{01}).
\]

In view of [13, 1.2.11] and axiom \((C_3)\) in [13, Definition 1.1.1] this gives
\[
c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y) \cdot -c_0(x \cdot -c_1z) \leq c_0c_1(c_1z \cdot s_1^0c_1z \cdot -d_{01}).
\]

Similarly
\[
c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y) \cdot -c_0(x \cdot -c_1w) \leq c_0c_1(c_1w \cdot s_1^0c_1w \cdot -d_{01}).
\]

By setting
\[
\begin{align*}
-r &= c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y) \\
-t &= c_0c_1(c_1z \cdot s_1^0c_1z \cdot -d_{01}) + c_0(x \cdot -c_1z), \\
-s &= c_0c_1(c_1w \cdot s_1^0c_1w \cdot -d_{01}) + c_0(x \cdot -c_1w),
\end{align*}
\]
we get from the above that \(-r \leq -t\) and \(-r \leq -s\), hence \(-r \leq -t \cdot -s\). Thus \(t + s \leq r\), and we are done with the claim.

Part (ii): Let \(\mathcal{D} = \mathfrak{fr}_{t} LRCA_{\alpha}\) with free generators \(\{x', y', z', w'\}\). Let \(\psi : \mathfrak{A} \to \mathcal{D}\) be defined by the extension of the map \(t \mapsto t'\), for \(t \in \{x, y, x, w\}\). For \(i \in \mathfrak{A}\), we denote \(\psi(i) \in \mathcal{D}\) by \(i'\). Let \(I = \mathfrak{fr} \mathcal{G}^D X_1 \{r'\}\) and \(J = \mathfrak{fr} \mathcal{G}^D X_2 \{s' + t'\}\), and let
\[
L = I \cap \mathcal{G}^D (X_1 \cap X_2) \quad \text{and} \\
K = J \cap \mathcal{G}^D (X_1 \cap X_2).
\]

Then \(L = K\), and \(\mathfrak{A}_0 = \mathcal{G}^D (X_1 \cap X_2)/L\) can be embedded into \(\mathfrak{A}_1 = \mathcal{G}^D X_1/I\) and \(\mathfrak{A}_2 = \mathcal{G}^D X_2/J\). The embedding of \(\mathfrak{A}_0\) into \(\mathfrak{A}_1\) is defined by \(d/L \mapsto d/I\), and the embedding of \(\mathfrak{A}_0\) into \(\mathfrak{A}_2\) is defined by \(d/L \mapsto d/J\). We show that \(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2\) are as required. Clearly \(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in RTCA_{\alpha}\). We show \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) do not have an amalgam in \(\text{LCA}_{\alpha}\) over \(\mathfrak{A}_0\) which will finish the proof.

Assume for contradiction that there is an (abstract) amalgam of such algebras. Let \(\mathfrak{A} = \mathfrak{fr}_{t} \text{LCA}_{\omega}\), and let \(\mathfrak{R}\) be the kernel of the map \(\psi : \mathfrak{A} \to \mathcal{D}\) defined above. Let
\[
M = \mathfrak{fr} \mathcal{G}^A X_2 \{s + t\} \cup (\mathfrak{R} \cap \mathcal{G}^A X_2);\]
\[ N = \bigcap g^\alpha \{ [M \cap S g^\alpha (X_1 \cap X_2)] \cup (R \cap S g^\alpha (X_1)) \}. \]

Then
\[ N \cap S g^\alpha (X_1 \cap X_2) = M \cap S g^\alpha (X_1 \cap X_2). \]

For \( R \) a filter of \( \mathfrak{A} \), and \( X \subseteq A \), by \((\mathfrak{A}/R)(X)\) we understand the subalgebra of \( \mathfrak{A}/R \) generated by \( \{ x/R : x \in X \} \). Then
\[ \phi : (\mathfrak{A}(X_1)/N)(X_1 \cap X_2) \to (\mathfrak{A}(X_2)/M)(X_1 \cap X_2) \]
defined by
\[ a/N \mapsto a/M \]
is a well defined isomorphism. Now \((S g^\alpha X_1/N)(X_1 \cap X_2)\) embeds into \( S g^\alpha X_1/N \) via the inclusion map; it also embeds in \( S g^\alpha X_2/M \) via \( i \circ \phi \) where \( i \) is also the inclusion map. But \( \mathfrak{A}_1 \cong S g^\alpha X_1/N \) via the map \( \Psi_1 : a/L \mapsto a'/N \) and \( \mathfrak{A}_2 \cong S g^\alpha X_2/M \) via the map \( \Psi_2 : a/K \mapsto a'/M \), and
\[ \Psi_1 \upharpoonright \mathfrak{A}_0 = \Psi_2 \upharpoonright \mathfrak{A}_0, \]
is an isomorphism from
\[ \mathfrak{A}_0 \to (S g^\alpha X_1/N)(X_1 \cap X_2) \cong (S g^\alpha X_2)/M)(X_1 \cap X_2). \]

So we can assume (identifying isomorphic algebras) that if \( j = i \circ \phi \), then \( \mathfrak{A}_0 \) embeds in \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) via \( i \) and \( j \), respectively.

So by assumption, there exists an amalgam, i.e there exists \( \mathfrak{P} \in \text{LCA}_\alpha \) and injective homomorphisms \( f \) and \( g \) from \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) respectively to \( \mathfrak{P} \) such that \( f \circ i = g \circ j \). Let \( \bar{f} : S g^\alpha X_1 \to \mathfrak{P} \) be defined by \( a \mapsto f(a/N) \) and \( \bar{g} : S g^\alpha X_2 \to \mathfrak{P} \) be defined by \( a \mapsto g(a/M) \). Let \( \mathfrak{P}' \) be the subalgebra of \( \mathfrak{P} \) generated by \( \text{rng} f \cup \text{rng} g \). Then \( \bar{f} \cup \bar{g} \upharpoonright X_1 \cup X_2 \to \mathfrak{P}' \) is a function since \( \bar{f} \) and \( \bar{g} \) coincide on \( X_1 \cap X_2 \). By freeness of \( \mathfrak{A} \), there exists \( h : \mathfrak{A} \to \mathfrak{P}' \) such that \( h \upharpoonright X_1 \cup X_2 = \bar{f} \cup \bar{g} \). Let \( P = \text{ker} h \).

Then
\[ P \cap S g^\alpha X_1 = N, \]
and
\[ P \cap S g^\alpha X_2 = M. \]

(Here we are showing that the amalgamation property implies the congruence extension property.) Thus we can infer that \( s + t \in P \), and hence, since \( s + t \leq r \), we have, by properties of filters, that \( r \in P \), hence \( r \in N \), since \( r \in S g^\alpha X_1 \) and \( P \cap S g^\alpha X_1 = N \). By the definition of \( N \), and properties of filters, there exist elements
\[ u \in M \cap S g^\alpha X_1 \cap X_2 \]
and \( b \in R \) such that
\[ u \cdot b \leq r. \]
Since \( u \in M \), we get by the definition of \( M \) and again by properties of filters that there is a \( \Gamma \subseteq \omega \\alpha \) and \( c \in \mathbb{R} \) such that
\[
q_{(\Gamma)}(s + t) \cdot c \leq u.
\]
Recall that \( \psi \) is the homomorphism from \( \mathfrak{A} \) to \( \mathfrak{D} \) such that \( \psi(i) = i' \) for \( i \in \{x, y, w, z\} \). It follows that
\[
q_{(\Gamma)}(\psi(s) + \psi(t)) \cdot \psi(c) \leq \psi(u) \cdot \psi(b) \leq \psi(r).
\]
Since \( \ker \psi = \mathbb{R} \), we have \( \psi(b) = \psi(c) = 1 \), and so:
\[
q_{(\Gamma)}(\psi(s) + \psi(t)) \leq \psi(u) \leq \psi(r).
\]
Let \( r' = \psi(r) \), \( u' = \psi(u) \), \( s' = \psi(s) \) and \( t' = \psi(t) \). Then \( r', u', s', t' \in \mathfrak{D} \) and (*)
\[
q_{(\Gamma)}(s' + t') \leq u' \leq r'.
\]
Part (iii): Let \( \mathfrak{B}_{ca} \) is the full cylindric set algebra with base \( \alpha \) and top element \( ^\alpha \alpha \). Let \( \mathfrak{B}_2 \in \text{LCs}_\alpha \) be the algebra expanded with diamonds defined with respect to the universal frame \( (\alpha; \alpha \times \alpha) \).

For a map \( f \), let \( \ker f = \{(x, y) \in 2^{\text{dom}(f)} : f(x) = f(y)\} \). Let \( E \) be the set of all equivalence relations on \( \alpha \), and for each \( R \in E \) set
\[
X_R = \{s \in ^\alpha \alpha : kers = R\}.
\]
Let
\[
C = \{ \bigcup_{R \in L} X_R : L \subseteq E \}.
\]
\( C \) is clearly closed under the formation of arbitrary unions, and since \( \sim \bigcup_{R \in L} X_L = \bigcup_{R \in E \sim L} X_R \) for every \( L \subseteq E \), we see that \( C \) is closed under the formation of complements with respect to \( ^\alpha \alpha \). Thus \( C \) is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of \( \mathfrak{B} \); moreover, it is obvious that
\[
X_R \text{ is an atom of } (C, \cup, \cap, \sim, 0, ^\alpha \alpha) \text{ for each } R \in E. \quad (3)
\]
For all \( i, j \in \alpha \), \( d_{ij} = \bigcup \{ X_R : (i, j) \in R \} \) and hence \( d_{ij} \in C \). Also,
\[
c_i X_R = \bigcup \{ X_S : S \in E, 2(\alpha \sim \{i\}) \cap S = 2(\alpha \sim \{i\}) \cap R \}
\]
for any \( i \in \alpha \) and \( R \in E \). Thus, because \( c_i \) is completely additive, \( C \) is closed under the operation \( c_i \). We consider \( \mathfrak{C} \) with domain based on the universal frame \( (\alpha; \alpha \times \alpha) \), so \( \mathfrak{C} \subseteq \mathfrak{B} \). Note that \( X_{Id} \) was denoted by \( \text{Inj} \) in Lemma 2.2. By Lemma 2.2, replacing \( a, b, c \) by \( -r', -s', -t' \), and \( x, y, z, w \) by \( x', y', z', w' \), respectively, there is a subset \( Y \) of \( ^\alpha \alpha \) such that
\[
X_{Id} \sim f(r') \neq 0 \text{ for every homomorphism } f : \mathfrak{D} \to \mathfrak{B} \text{ such that } f(x') = X_{Id} \text{ and } f(y') = Y, \quad (4)
\]
and also that for every $\Gamma \subseteq \omega$, $\alpha$, there are subsets $Z, W$ of $\alpha \alpha$ such that

$$X_{\text{id}} \cap q(\Gamma) g(s' + t') \neq 0 \quad \text{for every homomorphism} \quad h : \mathcal{D} \to \mathfrak{B}$$

such that $h(x') = X_{\text{id}}, h(z') = Z$ and $h(w') = W$. \hfill (5)

By (*) there exists a finite $\Gamma \subseteq \alpha$ and $u' \in \mathfrak{S}g^\mathcal{D}\{x'\}$, such that

$$q(\Gamma)(s' + t') \leq u' \leq r'.$$

There also exist $Y, Z, W \subseteq \alpha \alpha$ such that (4) and (5) hold.

Part (iv): Recall that $\mathcal{D} = \mathfrak{RLCA}_\alpha$. Take any homomorphism $k : \mathcal{D} \to \mathfrak{B}$ such that $k(x') = X_{\text{id}}, k(y') = Y, k(z') = Z$, and $k(w') = W$. This is possible by the freeness of $\mathcal{D}$. Then using the fact that $X_{\text{id}} \sim k(r')$ is non-empty by (4) we get

$$X_{\text{id}} \sim k(u') = k(x' \cdot -u') \supseteq k(x' \cdot r') \neq 0.$$

And using the fact that $X_{\text{id}} \cap q(\Gamma) k(s' + t')$ is non-empty by (5) we get

$$X_{\text{id}} \cap k(u') = k(x' \cdot u') \supseteq k(x' \cdot q(\Gamma)(s' + t')) \neq 0.$$

However, in view of (3), it is impossible for $X_{\text{id}}$ to intersect both $k(u')$ and its complement since $k(u') \in C$ and $X_{\text{id}}$ is an atom; to see that $k(u')$ is indeed contained in $C$ recall that $h(u) = u' \in \mathfrak{S}g^\mathcal{D}\{x'\}$, and then observe that because $X_{\text{id}} \in C$, we must have $k[\mathfrak{S}g^\mathcal{D}\{x'\}] \subseteq C$. \hfill $\square$

3. Amalgamation. We recall some variations on the amalgamation property to be addressed in what follows. Such properties correspond to the interpolation properties defined earlier.

**Definition 3.1.** Let $\mathbf{L}$ be a class of algebras.

1. $\mathfrak{A}_0 \in \mathbf{L}$ is in the amalgamation base of $\mathbf{L}$ if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathbf{L}$ and injective homomorphisms $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist $\mathcal{D} \in \mathbf{L}$ and injective homomorphisms $m_1 : \mathfrak{A}_1 \to \mathcal{D}$ and $m_2 : \mathfrak{A}_2 \to \mathcal{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. In this case, we say that $\mathcal{D}$ is an amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, via $m_1$ and $m_2$, or simply an amalgam.

2. Let everything be as in (i). If, in addition, we have $m_1 \circ i_1(\mathfrak{A}_0) = m_1(\mathfrak{A}_1) \cap m_2(\mathfrak{A}_2)$ then $\mathfrak{A}_0$ is said to be in the strong amalgamation base of $\mathbf{L}$, and like in the previous item $\mathcal{D}$ is called a strong amalgam.

3. Let everything be as in (i) and assume that the algebras considered are endowed with a partial order. If in addition, $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$, then we say that $\mathfrak{A}_0$ lies in the super amalgamation base of $\mathbf{L}$, and $\mathcal{D}$ is called a super amalgam.
4. \textbf{L} has the \textit{amalgamation property}, if the amalgam base of \textbf{L} coincides with \textbf{L}. The definition extends verbatim to the strong and super amalgamation properties.

We write \textit{AP}, \textit{SAP} and \textit{SUPAP} for the amalgamation, strong amalgamation and super amalgamation properties, respectively. We write \textit{APbase(K)}, \textit{SAPbase(K)}, and \textit{SUPAPbase(K)}, for the amalgamation, strong amalgamation, and super amalgamation base of the class \textbf{K}, respectively. Notice that \textit{SUPAP} also implies \textit{SAP} by writing the extra condition for \textit{SAP} as follows:

\[(\forall x \in A_1)(\forall y \in A_2)[m(x) = n(y) \implies (\exists z \in A_0)(x = f(z) \land y = h(z))].\]

We will sometimes seek amalgams, and for that matter strong or super amalgams, for algebras in a certain class in a possibly bigger one. However, in the definition we only require that the amalgam can be found in another class, which makes the scope of the definition bigger.

\textbf{Theorem 3.2.} Any class \textbf{K}, such that \textit{RLCA}_\alpha \subseteq \textbf{K} \subseteq \textit{LCA}_\alpha fails to have the \textit{AP}. Furthermore, there are infinitely many subvarieties of \textit{LCA}_\alpha containing \textit{RLCA}_\alpha that do not have \textit{AP}.

\textbf{Proof.} The first part follows from the proof of the third item of Theorem 2.5. For the second part, using discrete modalizing and the techniques in [14], it can be proved that \textit{SNr}_\alpha \textit{LCA}_\alpha + j : 1 \leq j < \omega) is a strictly decreasing sequence of distinct varieties that do not have \textit{AP}. Indeed, for each such \(j\), we have \(A_0, A_1, A_2 \in \textit{RLCA}_\alpha(\subseteq \textit{SNr}_\alpha \textit{LCA}_\alpha + j)\), and there is no amalgam in \textit{SNr}_\alpha \textit{LCA}_\alpha + j, lest there will be one in \textit{LCA}_\alpha which we know is not the case.

Let \(\alpha\) be an ordinal. Then \(\textbf{K} = \{A \in \textit{LCA}_\alpha + \omega : \exists g^A \textit{SNr}_\alpha A = A\}\). Notice that \(\textbf{K} \subseteq \textit{LDC}_\alpha + \omega\), hence \(\textbf{K} \subseteq \textit{RLCA}_\alpha + \omega\).

\textbf{Theorem 3.3.} If \textbf{L} is reflexive, then for any ordinal \(\alpha\), the class \textbf{K} has \textit{SUPAP}.

\textbf{Proof.} Let \(\textbf{A}, \textbf{B}, \textbf{C} \in \textbf{K}\) be such that \(\textbf{C} \subseteq \textbf{A}\) and \(\textbf{C} \subseteq \textbf{B}\). We first find an amalgam. That is, we want to find an algebra \(\textbf{D} \in \textbf{K}\) and injective homomorphisms \(f : \textbf{A} \rightarrow \textbf{D}\) and \(g : \textbf{B} \rightarrow \textbf{D}\), such that \(f \upharpoonright \textbf{C} = g \upharpoonright \textbf{C}\). We proceed like as we did in the proof of the last item of theorem 2.5, working with dimension restricted free algebras. Let \(\beta\) be a cardinal \(> \max|\textbf{B}|, |\textbf{A}|\). Let \(\textbf{F} = \textbf{F}^{\alpha + \omega}_\alpha \textit{LCA}_\alpha + \omega\), where \(\rho : \beta \rightarrow \varphi(\alpha + \omega)\) is defined by \(\rho(i) = \alpha\) for all \(i \in \beta\). Take \(\beta_1, \beta_2 \subseteq \beta\), such that \(\beta_1 \cup \beta_2\), and such that there are surjective homomorphisms \(h_1 : \textbf{F} \rightarrow \textbf{A}\) and \(h_2 : \textbf{F} \rightarrow \textbf{B}\), with \(h_1 \upharpoonright \beta_1\) is a bijection from \(\beta_1\) to \(\textbf{A}\), \(h_2 \upharpoonright \beta_2\) is a bijection from \(\beta_2\) to \(\textbf{B}\) and \(h \upharpoonright \beta_1 \cap \beta_2\) is a bijection from \(\beta_1 \cap \beta_2\) to \(\textbf{C}\). Such homomorphisms exist by the dimension restricted freeness (and cardinality) of \(\textbf{F}\), and by the fact that \(\textbf{A}, \textbf{B}, \textbf{C} \in \textbf{K}\), so each has a generating set of elements whose dimension set is included in \(\alpha\). Take \(\textbf{F}_1 = \textbf{F}^{\beta_1}_\beta\) and \(\textbf{F}_2 = \textbf{F}^{\beta_2}_\beta\). Let \(M = \{d \in \textbf{F}_1 : h_1(d) = 1\}\) and \(N = \{d \in \textbf{F}_2 : h_2(d) = 1\}\). Then \(M\) and \(N\) are filters. Let \(\tilde{h}_1 : \textbf{F}_1/M \rightarrow \textbf{A}, h_2 : \textbf{F}_2/N \rightarrow \textbf{B}\) be the isomorphisms induced
by \( h_1 \) and \( h_2 \) (by the first isomorphism theorem), that is, \( \bar{h}_1(d/M) = h_1(d) \) and similarly for \( \bar{h}_2 \). Let \( \bar{\mathfrak{f}}_0 = \mathfrak{g}^\beta(\beta_1 \cap \beta_2) \). Let \( l_1 : \bar{\mathfrak{f}}_0/\mathfrak{f}_0 \cap M \to \mathfrak{c} \) be defined via \( d/\mathfrak{f}_0 \cap M \mapsto h_1(d) \), and \( l_2 : \bar{\mathfrak{f}}_0/\mathfrak{f}_0 \cap N \to \mathfrak{c} \) be defined via \( d/\mathfrak{f}_0 \cap N \mapsto h_2(d) \). Then \( l_1 \) and \( l_2 \) are well defined, so \( \mathfrak{f}_0 \cap N = \mathfrak{f}_0 \cap M \).

We show, like before, that \( P = \mathfrak{f}/(M \cup N) \) (the filter generated by \( M \cup N \) in \( \mathfrak{f} \)) is a proper filter of \( \mathfrak{f} \) and that \( \mathfrak{f}/P \) is the required amalgam. Let \( x \in P \cap \mathfrak{f}_1 \). Then \( b \in M \) and \( c \in N \) such that \( b \cdot c \leq x \). Thus \( c \leq x + b \). But \( x + -b \in \mathfrak{f}_1 \) and \( c \in \mathfrak{f}_2 \), it follows from the assumption that \( \mathfrak{f} \) has \( IP \), that there exist \( d \in \mathfrak{f}_0 \), such that \( c \leq d \leq x + b \). Notice that \( c \in N \) so \( d \in N \), hence \( d \in M \), so \( x \in M \) and \( x + b \in M \). But \( b \in M \) so \( x + b + b = x \in M \). We have shown that \( P \cap \mathfrak{f}_1 = M \), so that \( P \) is proper. Also like before, the algebra \( \mathfrak{f}/P \) is the required amalgam via the maps \( k \) and \( h \) defined from \( \mathfrak{f}_1/M \) and \( \mathfrak{f}_2/N \) to \( \mathfrak{f}/P \) by \( d/M \mapsto d/P \) (\( d \in \mathfrak{f}_1 \)) and \( d/N \mapsto d/P \) (\( d \in \mathfrak{f}_2 \)), respectively. (Recall that \( \mathfrak{f}_1/M \simeq \mathfrak{A} \), \( \mathfrak{f}_2/N \simeq \mathfrak{B} \), and \( \mathfrak{f}_0/M \cap \mathfrak{f}_0 = \mathfrak{f}_0/N \cap F_0 \simeq \mathfrak{C} \).)

We now prove that \( \mathfrak{f}/P \) is actually a super amalgam implying that \( \mathfrak{K} \) has the super amalgamation property. Assume that \( a \in \mathfrak{f}_1/M \) and \( b \in \mathfrak{f}_2/N \) and that \( k(a) \leq h(b) \). By symmetry it suffices to show that there is a \( t \in \mathfrak{C} \) such that \( t \geq a \) and \( t < b \). There exists \( x \in \mathfrak{f}_1 \) such that \( x/P = k(a) \) and there exists \( z \in \mathfrak{f}_2 \) such that \( z/P = h(b) \). Since \( x/P \leq z/P \), we get \( -x + z \in P \). Therefore there are \( r \in M \), \( s \in N \), such that \( r + s \leq -x + z \). Hence \( r + x \leq z + -s \). Now \( r + x \in \mathfrak{f}_1 \) and \( z + -s \in \mathfrak{f}_2 \). It follows using theorem 2.5 (again) that there are \( u \in \mathfrak{f}_0 \), such that \( r + x \leq u \leq z + -s \). Therefore \( -u + u \in P \), so we conclude that (*) \( u/P \geq x/P \). Similarly, \( -u + z \in P \), hence we have (**): \( u/P \leq z/P \). Since \( \mathfrak{f}_0 \cap M = \mathfrak{f}_0 \cap N \) and \( \mathfrak{C} \simeq \mathfrak{f}_0/M \cap \mathfrak{f}_0 \) and \( \mathfrak{C} \simeq \mathfrak{f}_0/N \cap \mathfrak{F} \), then there is a \( t \in \mathfrak{C} \) such that \( t = u/M \) and \( t = u/N \). Recall that \( x/P = k(a) \). So by (*) we have

\[
    k(t) = k(u/M) = u/P \geq x/P = k(a).
\]

Thus \( k(t) \geq k(a) \). Since \( k \) is injective, we get that \( t \geq a \). Similarly, from (**), we get that \( h(t) \leq h(b) \), hence again by injectivity we get that \( t \geq b \), and we are done.

**Definition 3.4.** Let \( \mathfrak{A} \) be an algebra, and \( C \subseteq \bigcup_{\mathfrak{B} \subseteq \mathfrak{A}} \mathfrak{C} \). \( \mathfrak{A} \) is said to have the congruence extension property relative to \( C \) if for any \( X_1, X_1 \subseteq \mathfrak{A} \) such that \( X_1 \cap X_2 = X \), if \( R \in \mathfrak{C}(\mathfrak{g}^\mathfrak{A}X_1) \cap C \) and \( S \in \mathfrak{C}(\mathfrak{g}^\mathfrak{A}X_2) \cap C \), such that \( R \cap \mathfrak{g}^\mathfrak{A}(X_1 \cap X_2) = S \cap \mathfrak{g}^\mathfrak{A}(X_1 \cap X_2) \), then there is a \( T \in (\mathfrak{C}(\mathfrak{A})) \cap C \) such that \( T \cap \mathfrak{g}^\mathfrak{A}X_1 = R \) and \( T \cap \mathfrak{g}^\mathfrak{A}X_2 = S \). If \( C = \bigcup_{\mathfrak{B} \subseteq \mathfrak{A}} \mathfrak{C} \), we say that \( \mathfrak{A} \) has the congruence extension property, or \( CP \) for short.

**Theorem 3.5.** The following conditions are equivalent for a subvariety \( V \) of \( LCA_\alpha \), \( \alpha > 1 \):

1. Semisimple algebras in \( V \) have \( AP \).
2. Simple algebras in \( V \) have \( AP \).
3. Free \( V \) algebras have \( WIP \).
4. Free \( V \) algebras have \( CP \) with respect to maximal congruences.

In particular, by the second item of Theorem 2.5, the class of semisimple algebras have \( AP \).
Proof. (1) implies (2) is trivial. We show (2) implies (1). Assume that $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ are semi-simple and let $h : \mathbf{C} \to \mathbf{A}$ and $g : \mathbf{C} \to \mathbf{B}$ be injective homomorphisms. We want to find a semi-simple amalgam. By semisimplicity, there exist a system $\langle \mathbf{C}_l : l \in I \rangle$ of simple algebras and $k : \mathbf{A} \to \prod_{l \in I} \mathbf{C}_l$ such that $\pi_l \circ k(\mathbf{A}) = \mathbf{C}_l$ for each $l$ where $\pi_l$ is the projection map. Now fix $l \in I$. Let $t_l = \pi_l \circ h \circ g$. Then $t_l : g(\mathbf{C}) \to t_l(g(\mathbf{C}))$ is a surjective homomorphism and $i_{g(\mathbf{C})} : g(\mathbf{C}) \to \mathbf{B}$. We claim that there exist simple $\mathbf{D}_l$, surjective $j_l : \mathbf{B} \to \mathbf{D}_l$ and injective $i_l : t_l(g(\mathbf{C})) \to \mathbf{D}_l$, such that $j_l \circ i_{g(\mathbf{C})} = i_l \circ t_l$. To see why, let $M$ be a maximal filter of $g(\mathbf{C})$ such that $g(\mathbf{C})/M \cong t_l(g(\mathbf{C}))$ via $i_l(a) = t_l(a)/M$. Then $i_{g(\mathbf{C})}(M)$ is a filter in $\mathbf{B}$. By lemma 1.11, let $N$ be a maximal filter of $\mathbf{B}$ such that $N \cap i_{g(\mathbf{C})}(g(\mathbf{C})) = i_{g(\mathbf{C})}(M)$. Let $\mathbf{D}_l = \mathbf{B}/N$ and define $j_l(b) = b/N$. Then $j_l : \mathbf{B} \to \mathbf{D}_l$. Let $a \in t_l(g(\mathbf{C}))$. Choose $x \in t_l^{-1}(a)$ and define $i_l(a) = j_l \circ i_{g(\mathbf{C})}(x)$. Then $i_l$ is well defined, injective, and as is required. Now $\mathbf{C}_l$ and $\mathbf{D}_l$ amalgamate over $t_l(g(\mathbf{C}))$, hence there exist $\mathbf{F}_l$ simple and $m_l, n_l$ such that $i_l \circ n_l = m_l$. Define $m(a) = (m_l \circ \pi_l \circ k(a) : l \in I)$ and $n(b) = (n_l \circ j_l(b) : l \in I)$ Then for $l \in I$, we have $m(h(c)) = m_l \circ \pi_l \circ k(h(c)) = m_l \circ \pi_l \circ k \circ h \circ g^{-1} \circ g(c) = m_l \circ t_l \circ g(c) = n_l \circ i_l \circ t_l \circ g(c) = n_l \circ j_l \circ g(c) = n(c)$. Therefore $\mathbf{F} \circ h = n \circ g$, and we are done.

We prove (3) $\implies$ (2). Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ be simple algebras such that $\mathbf{C} \subseteq \mathbf{A}$ and $\mathbf{C} \subseteq \mathbf{B}$. We want to find an amalgam. Let $\mathfrak{F}$ be the free algebra on $\beta$ generators, where $\beta$ is a cardinal $> \text{max} \{ |B|, |A| \}$. As before, take $\beta_1, \beta_2 \subseteq \beta$, such that $\beta = \beta_1 \cup \beta_2$, and such that there are surjective homomorphisms $h : \mathfrak{F} \to \mathbf{C}$, $h_1 : \mathfrak{F} \to \mathbf{A}$ and $h_2 : \mathfrak{F} \to \mathbf{B}$, with $h_1 \upharpoonright \beta_1$ is a bijection from $\beta_1$ to $\mathbf{A}$, $h_2 \upharpoonright \beta_2$ is a bijection from $\beta_2$ to $\mathbf{B}$, and $h \upharpoonright \beta_1 \cap \beta_2$ is a bijection from $\beta_1 \cap \beta_2$ to $\mathbf{C}$. Such homomorphisms exist by the freeness (and cardinality) of $\mathfrak{F}$. Let $\mathfrak{F}_1 = \mathfrak{S}g^\beta \beta_1$ and $\mathfrak{F}_2 = \mathfrak{S}g^\beta \beta_2$. Let $M = \{ d \in \mathfrak{F}_1 : h_1(d) = 1 \}$ and $N = \{ d \in \mathfrak{F}_2 : h_2(d) = 1 \}$. Now here $M$ and $N$ are maximal filters because $\mathbf{B}$ and $\mathbf{C}$ are simple. We show using WIP that $P = \mathfrak{F}_1^\beta (M \cup N)$ (the filter generated by $M \cup N$ in $\mathfrak{F}$) is a proper filter of $\mathfrak{F}$. Let $x \in P \cup \mathfrak{F}_1$. It suffices to show that $x \neq 0$. There are $b \in M$ and $c \in N$ such that $b + c = x$. Thus $c \leq x + \mathfrak{F}_1$. But $x + b \in \mathfrak{F}_1$, and $c \in \mathfrak{F}_2$, it follows from the assumption that $\mathfrak{F}$ has WIP, that there exist $J \subseteq \omega \alpha$ and $d \in \mathfrak{F}_1$, such that $d \leq c_{(J)}(x + b)$. Notice that $c \in N$, so $c_{(J)}(x + b) \in N$, thus $d \in N$. But $M \cap \mathfrak{F}_1 = N \cap \mathfrak{F}_1$, so $d \in M$. Assume for contradiction that $x = 0$, then by $d \leq c_{(J)}(-b)$, we get $c_{(J)}(-b) \in M$. But from $b \in M$, we get that $q_{(\Delta)}b = -c_{(\Delta)}(-b) \in M$, hence $0 \in M$ which is impossible.

Now we prove that (3) $\iff$ (4). Assume CP relative to $U$, where $U$ is the set of proper maximal filters in subalgebras of $\mathfrak{A}$. Let $X_1, X_2 \subseteq \mathfrak{A}$, and $x \in \mathfrak{S}g^\alpha X_1$ and $z \in \mathfrak{S}g^\alpha X_2$, such that $x \leq z$ and assume for contradiction that there is no $y$ and no finite $\Gamma \subseteq \alpha$, such that $q_{(\Gamma)}x \leq y \leq c_{(\Gamma)}z$. Then $q_{(\Gamma)}x - y > 0$ or $q_{(\Gamma)}[-z] \cdot y > 0$ whenever $y \in \mathfrak{S}g^\alpha (X_1 \cap X_2)$. Hence for any finite subsets $\Delta, \theta$ of $\alpha$, we have $u \cdot w > 0$ for all $u, w \in \mathfrak{S}g^\alpha (X_1 \cap X_2)$ such that $\sum \geq q_{(\Delta)}x$ and $w \geq q_{(\theta)}[-z]$. Let

$$P = \mathfrak{F}^\mathfrak{S}g^\alpha (X_1 \cap X_2) \setminus [((\mathfrak{F}^\mathfrak{S}g^\alpha X_1 \setminus \mathfrak{S}g^\alpha (X_1 \cap X_2)) \cup \mathfrak{F}^\mathfrak{S}g^\alpha X_2 \setminus \mathfrak{S}g^\alpha (X_1 \cap X_2) \setminus \mathfrak{F}^\mathfrak{S}g^\alpha (X_1 \cap X_2) \setminus \mathfrak{F}^\mathfrak{S}g^\alpha (X_1 \cap X_2)].$$
Then $P$ is proper, so let $P'$ be a maximal proper filter in $\mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$ containing $P$. Then there are maximal filters $M$ of $\mathfrak{S}g^\mathfrak{A}X_1$ and $N$ of $\mathfrak{S}g^\mathfrak{A}X_2$ such that (*)

$$\mathfrak{A}^\mathfrak{A}(X_1 \cap X_2) \subseteq M$$

and $M \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2) = P' = N \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$. By assumption, we have $\mathfrak{A}^\mathfrak{A}(M \cup N)$ is proper, and so it is not the case that $x \leq z$, for if $x \leq z$, then $x \cdot (-z) = 0$ and so by (*) we get $0 \in \mathfrak{A}^\mathfrak{A}(M \cup N)$ and so $\mathfrak{A}^\mathfrak{A}(M \cup N) = \mathfrak{A}$. This is a contradiction and we are done. For the converse. Assume that $\mathfrak{A}$ has WIP. Let $M$ be a filter of $\mathfrak{S}g^\mathfrak{A}X_1$ and $N$ be a filter of $\mathfrak{S}g^\mathfrak{A}X_2$, both maximal, such that $M \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2) = N \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$. Assume for contradiction that $\mathfrak{A}^\mathfrak{A}(M \cup N) = \mathfrak{A}$. Then there exist $x \in M$, $z \in N$ such that $x \cdot z = 0$. By WIP there is an element $y \in \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$ and a finite $\Gamma \subseteq \alpha$ such that $q_{\Gamma}(x) \leq y \leq c_{\Gamma}(z)$, hence $y \in \mathfrak{A}^\mathfrak{A}(X_1 \cap X_2)$ and $-y \in \mathfrak{A}^\mathfrak{A}(X_2 \cap X_2)$ for if $y \in N \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2) = M \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2) \subseteq M$, and $-y \in M \cap \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2) \subseteq M$. Hence $0 = -y \cdot y \in M$ which is impossible. We conclude that $\mathfrak{A}^\mathfrak{A}(M \cup N)$ is proper and maximal, and it induces the required maximal congruence.

We leave (1) $\implies$ (4) to the reader. (Hint: Cf. the proof of the last item in Theorem 2.5).

**Definition 3.6.** $\mathfrak{A} \in \text{LCA}_\alpha$ is called a *substitution algebra of dimension* $\alpha$, if for all non-zero $x$ in $A$, for all finite $\Gamma \subseteq \alpha$, there exist distinct $i, j \in \alpha \sim \Gamma$, such that $s^{(i)}_j x \neq 0$. Let $\text{Sc}_\alpha$ denote the class of substitution algebras.

**Theorem 3.7.** For $\alpha \geq \omega$, $\text{Sc}_\alpha \subseteq \text{RLCA}_\alpha$.

**Proof.** We show that for $\mathfrak{A} \in \text{Sc}_\alpha$, there exists $\mathfrak{B} \in \text{LCA}_{\alpha+\omega}$ such that $\mathfrak{A} \in \mathfrak{N}_{\alpha+\omega} \mathfrak{B}$. Then we can assume $\mathfrak{A} \in \text{LD}_{\alpha+\omega}$, and then use theorem 2.5 to show representability of $\mathfrak{A}$, baring in mind that a neat reduct of a representable algebra is plainly representable. So let $\mathfrak{A} \in \text{Sc}_{\alpha}$. To construct the required $\omega$-dilation $\mathfrak{B}$ we proceed as follows using ultraproducts. Let $R$ be the set of all quadruples $(\rho, n, k, l)$ such that: $\rho \in \mathfrak{m}_\alpha$ is injective for some $m \in \omega$, $n \in \omega$, $k, l$ are injective (finite) sequences with $k, l \in n(\alpha \sim \text{rng} \rho)$ and $\text{rng} k \cap \text{rng} l = \emptyset$. For $\rho \in \mathfrak{m}_n$ $(n \in \omega)$ injective, put $X_{\rho, n} = \{(\sigma, m, k, l) \in R : \rho \subseteq \sigma$ and $n \leq m\}$. It is straightforward to check that the set consisting of all the $X_{\rho, n}$’s is closed under finite intersections. Accordingly, let $M$ be the proper filter of $\varphi(R)$ generated by the $X_{\rho, n}$’s so that $M = \{Y \subseteq R : X_{\rho, n} \subseteq Y$ for some $\rho$ and $n \in \omega\}$. For each $(\rho, n, k, l) \in R$, choose a bijection $t[(\rho, n, k, l)]$ from $\alpha + \omega$ onto $\alpha$ such that $t[(\rho, n, k, l)] \mid \text{rng} \rho \subseteq Id$, and $t[(\rho, n, k, l)](\alpha + j) = k_j$, for each $j < n$.

Let $\mathfrak{B} = \prod_{\phi \in R} \mathfrak{R}^{(\phi)} \mathfrak{A}/M$. Here $\mathfrak{R}^{(\phi)} \mathfrak{A}$ - the $t(\phi)$ reduct of $\mathfrak{A}$ - is an LCA_{\alpha+\omega}, and so $\mathfrak{B}$ - an ultraproduct of these - is also a LCA_{\alpha+\omega}. Note too, that for each $\phi \in R$, the algebra $\mathfrak{R}^{(\phi)} \mathfrak{A}$ has universe $\mathfrak{A}$. Let $f$ be the function from $\mathfrak{A}$ into $\mathfrak{B}$ defined by $f(x) = (s^{k_0}_{l_0} \circ \ldots \circ s^{k_{n-1}}_{l_{n-1}} x : (\rho, n, k, l) \in R)/M$.

Consider any $i < \alpha$. Then for each $(\rho, m, k, l) \in R$ such that $i \in \text{rng} \rho$ we have $s^{\rho_i}_{l_i} \ldots s^{\rho_{n-1}}_{l_{n-1}} c_i x = c_is^{\rho_i}_{l_i} \ldots s^{\rho_{m-1}}_{l_{m-1}} x$.
for every $x \in A$. Since $\{(\rho, m, k, l) \in R : i \in \text{rng} \rho\} \subseteq M$, we have $f(c_i x) = c_i f(x)$ for all $x \in A$ and $i < \alpha$. Preservation of boxes and diagonal elements is analogous. Now consider any $i \in (\alpha + \omega) \sim \alpha$. Then for each $(\rho, m, k, l) \in R$ such that $i < \alpha + m$ we have

$$c_{t(\rho, m, k, l)(i)}s_{i_0}^{k_0} \cdots s_{i_{m-1}}^{k_{m-1}} x = s_{i_0}^{k_0} \cdots s_{i_{m-1}}^{k_{m-1}} x,$$

since $t(\rho, m, k, l)(i) \in \text{rng} k \sim \text{rng} l$. Hence $f(\mathcal{A}) \subseteq \mathcal{A} \mathcal{R}_\alpha \mathcal{B}$. Consider now any $x \in A$ such that $x \neq 0$. Then for any given finite $\Gamma \subseteq \alpha$, and any $m < \omega$, there exist by assumption, using a simple induction, injective sequences $k, l \in m(\alpha \sim \Gamma)$ such that $\text{rng}(k) \cap \text{rng}(l) = \emptyset$ and $s_{i_0}^{k_0} \cdots s_{i_{m-1}}^{k_{m-1}} x \neq 0$, thus $f$ is injective and we are done. 

\[\square\]

**Theorem 3.8.** If $\mathcal{A}$ is semi-simple or $\mathcal{A} \in \text{LD}_{\mathcal{C}_\alpha}$, then $\mathcal{A} \in \mathcal{S}_{\mathcal{C}_\alpha}$.

**Proof.** Let $\mathcal{A} \in \text{LD}_{\mathcal{C}_\alpha}$, $a \in \mathcal{A}$ be non zero, and $\Gamma \subseteq \alpha$. Choose $i, j \in \alpha \sim \Delta x$. Then $s_i^j x = x \neq 0$. Assume that $\mathcal{A}$ is semisimple. We show that $\mathcal{A} \in \mathcal{S}_{\mathcal{C}_\alpha}$. Let $\Gamma$ be a finite subset of $\alpha$ and $x \in \mathcal{A} \sim \{0\}$. Using Zorn’s lemma one can find a maximal filter $F$ of $\mathcal{A}$ such that $x \notin F$. Since $F$ is maximal then $\mathcal{A}/F$ is simple. But $x \notin F$, hence there exists a finite $\Delta \subseteq \alpha$, such that $c_{(\Delta)}(x/F) = c_{(\Delta)}x/F = 1$. Let $i, j \in \alpha \sim (\Gamma \cup \Delta)$, then we claim that $s_i^j x \neq 0$. If not, then

$$0 = (c_{(\Delta)}s_i^j x)/F = (s_i^j c_{(\Delta)} x)/F = c_{(\Delta)} x/F = 1,$$

which is impossible. 

\[\square\]

**Theorem 3.9.**

1. Let $\mathcal{A}_0$ and $\mathcal{A}_1 \in \mathcal{S}_{\mathcal{C}_\alpha}$ Then there exist $\mathcal{B}_0$, $\mathcal{B}_1 \in \mathcal{K} \subseteq \text{LCA}_{\alpha+\omega}$ such that $\mathcal{B}_0 \to \mathcal{A}_0 \mathcal{R}_\alpha \mathcal{B}_0$ and $\mathcal{B}_1 \to \mathcal{A}_1 \mathcal{R}_\alpha \mathcal{B}_1$ such that for every injective homomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ there exists an injective homomorphism $g : \mathcal{B}_0 \to \mathcal{B}_1$ such that $g \circ j_0 = j_1 \circ f$.

2. $\mathcal{S}_{\mathcal{C}_\alpha}$ has the amalgamation property.

3. Simple algebras has the amalgamation property. Thus using Lemma 3.5, (we reprove that) $\mathcal{A}_0 \mathcal{R}_\beta \text{LCA}_{\alpha}$ has WIP.

**Proof.** Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be as in the hypothesis. Fix $i \in \{0, 1\}$. Let $\mathcal{B}_i'$ be the dilations constructed in theorem 3.7, of $\mathcal{A}_i$, so that, using the notation in op.cit, $\mathcal{B}_i' = \Pi_{\phi \in R} \mathcal{A}^{\phi(\phi)} \mathcal{A}_i/M$. Let $j_i$ be the neat embedding functions from $\mathcal{A}_i$ into $\mathcal{B}_i'$ defined, as in op.cit by

$$f_i x = (s_{i_0}^{k_0} \circ \ldots \circ s_{i_{n-1}}^{k_{n-1}} x : (\rho, n, k, l) \in R)/M.$$

($f_i$ is an embedding form $\mathcal{A}_i$ to $\mathcal{A}_0 \mathcal{R}_\phi \mathcal{B}_i'$). Now let $h$ be the function from $\mathcal{B}_0'$ into $\mathcal{B}_1'$ defined by:

$$h((x_\phi : \phi \in R)/M) = (f x_\phi : \phi \in R)/M.$$
Then it is easy to check that \( h \) is a well defined homomorphism such that \( h \circ j_0 = j_1 \). Take \( \mathcal{B}_i = \mathcal{G}^\mathfrak{B}_i \{ j_i(\mathfrak{A}_i) \} \) for \( i \in \{0, 1\} \) and define \( g = h \upharpoonright \mathcal{B}_0 \). Then it is not hard to check that \( g \) is the desired “lifting” function.

Assume that \( \mathfrak{A}_0 \subseteq \mathfrak{A}_1, \mathfrak{A}_2 \) all in \( \mathcal{S}_{\mathcal{C}_\alpha} \). For \( i \in \{0, 1, 2\} \), By Theorem 3.7, let \( \mathfrak{A}_i^+ \in \mathcal{C}_{\mathcal{A}_\alpha} \) be such that \( \mathfrak{A}_i \subseteq \mathfrak{N}_\alpha \mathfrak{A}_i^+ \). We can assume that \( \mathfrak{A}_0^+, \mathfrak{A}_1^+, \mathfrak{A}_2^+ \in \mathcal{K} \) (by assuming that \( \mathfrak{A}_i^+ \) is generated by \( \mathfrak{A}_i \)) and that \( \mathfrak{A}_0^+ \) embeds into both \( \mathfrak{A}_1^+ \) and \( \mathfrak{A}_2^+ \). By Theorem 3.3, there is an amalgam \( \mathcal{E} \) for the last three algebras. Then \( \mathcal{D} = \mathfrak{N}_\alpha \mathcal{E} \) will be the required amalgam for the original algebras. So far, we obtained a representable amalgam. But \( \mathcal{S}_{\mathcal{C}_\alpha} \subseteq \mathcal{R}_{\mathcal{C}_{\mathcal{A}_\alpha}} \), so we want to obtain an amalgam in (the smaller) class \( \mathcal{S}_{\mathcal{C}_\alpha} \). We this amalgam as a homomorphic image of \( \mathcal{D} \) as follows: Define a transfinite sequence of filters \( M_i \) where \( i \) ranges over arbitrary ordinals. \( M_0 = \{0\} \) and for each \( i > 0 \) \( M_i \) is the set of all elements \( x \in D \) such that for some finite \( \Gamma \subseteq \alpha \) and all distinct \( k, l < \alpha \sim \Gamma \), \( s_k x \) is not in \( \bigcup_{j<p} M_j \). Then \( M_i \) is a filter in \( \mathcal{D} \). Since \( M_i \subseteq M_j \) whenever \( i < j \) it follows that \( M_i = M_{i+1} \) for sufficiently large \( i \), if \( \beta \) is the least such ordinal with this property, then \( \mathcal{D}/M_\beta \) is in \( \mathcal{S}_{\mathcal{C}_\alpha} \) and is the required amalgam. Given simple algebras \( \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \) and \( \mathfrak{A}_0 \subseteq \mathfrak{A}_2 \), we have by theorem 3.9, since the class of simple algebras is contained in \( \mathcal{S}_{\mathcal{C}_\alpha} \) an amalgam \( \mathcal{E} \in \mathcal{S}_{\mathcal{C}_\alpha} \) of \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) over \( \mathfrak{A}_0 \). The following reasoning ensures that \( \mathcal{E} \) can be chosen to be simple. Let \( \mathfrak{A} \subseteq \mathcal{E} \). Suppose that \( \mathfrak{A} \) is simple, and let \( I \) be a filter of \( \mathcal{E} \). Then \( \mathfrak{A} \) can be embedded in \( \mathcal{E}/I \). Indeed, \( I \cap A \) is a filter of \( \mathfrak{A} \), and is proper since \( 0 \notin I \). Hence \( I \cap A = \{1\} \) because \( \mathfrak{A} \) is simple. But then \( \mathfrak{A} \cong \mathfrak{A}/(I \cap A) \) and the latter embeds in \( \mathcal{E}/I \). By taking a maximal filter \( I \) we are done. \( \square \)

**Theorem 3.10.** Let \( \alpha > 0 \). Then the following conditions are equivalent for any variety \( V \subseteq \mathcal{L}_{\mathcal{C}_{\mathcal{A}_\alpha}} \).

1. Free algebras in \( V \) have UIP.
2. \( V \) has AP.
3. \( V \) has the CP.

If \( \alpha < \omega \), then they are equivalent to the conditions in theorem 3.5.

**Proof.** (1) \( \implies \) (2) is like the proof of the first part of the third item of theorem 2.5. (2) \( \implies \) (3) is like the proof of (1) \( \implies \) (4) in theorem 3.5 (dropping maximality of congruences). Now we show that (3) \( \implies \) (1). Assume that \( V \) has CP and let \( \mathfrak{A} \) be the free algebra on \( \beta \) generators, for some non-zero cardinal \( \beta \). Let \( x \in \mathcal{G}^\mathfrak{A} X_1, z \in \mathcal{G}^\mathfrak{A} X_2 \) and assume that \( x \leq z \). Then \( z \in \mathfrak{G}^\mathfrak{A} \{x\} \cap \mathcal{G}^\mathfrak{A} X_1 \). Let \( M = \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} X_1} \{x\} \) and \( N = \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} (X_1 \cap X_2)} \). Then

\[
M \cap \mathcal{G}^\mathfrak{A} (X_1 \cap X_2) = N \cap \mathcal{G}^\mathfrak{A} (X_1 \cap X_2).
\]

By identifying ideals with congruences, and using the congruence extension property, there is a filter \( P \) of \( \mathfrak{A} \) such that \( P \cap \mathcal{G}^\mathfrak{A} X_1 = N \) and \( P \cap \mathcal{G}^\mathfrak{A} X_2 = M \). It follows that \( \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} (\bigcup M)} \cap \mathcal{G}^\mathfrak{A} X_1 \subseteq P \cap \mathcal{G}^\mathfrak{A} X_1 = N \). Hence \( \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} (\bigcup M)} \cap \mathcal{G}^\mathfrak{A} X_1 \subseteq N \), and we have \( z \in \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} X_1} [\mathfrak{G}^{\mathfrak{G}^\mathfrak{A} X_2} \{x\} \cap \mathcal{G}^\mathfrak{A} (X_1 \cap X_2)]. \) This implies that there is an element \( y \) such that \( z \geq y \in \mathcal{G}^\mathfrak{A} (X_1 \cap X_2) \), and \( y \in \mathfrak{G}^{\mathfrak{G}^\mathfrak{A} X_2} \{x\} \). Hence, there exists a finite \( \Gamma \subseteq \alpha \) such that \( y \geq q(\Gamma) x \), so we get \( q(\Gamma) x \leq y \leq z \). If \( \alpha < \omega \), then \( \mathcal{L}_{\mathcal{C}_{\mathcal{A}_\alpha}} \) is a discriminator variety, with discriminator term \( c_{(\alpha)} \). In particular, every
subdirectly indecomposable algebra is simple and hence every algebra is semisimple.

\[\square\]

4. A summary of results and conclusion.

4.1. A summary of results hitherto obtained in tabular form. In the next two tables, \(n\) is finite > 1 and \(\alpha\) is an infinite ordinal.

In Table 1, \(IP\) is short for interpolation property. The top row addresses all interpolation properties introduced and investigated throughout this paper. \(IP\) is the interpolation property, \(UIP\) is universal \(IP\), \(WIP\) is weak \(IP\), \(RIP\) is restricted \(IP\). By restricted \(IP\), we understand \(IP\) when the two sets \(X_1\) and \(X_2\) involved in the definition of \(IP\) are required to be disjoint. That is to say, \(\mathfrak{A}\) has the restricted interpolation property, \(RIP\) for short, if for all \(X_1, X_2\) such that \(X_1 \cap X_2 = \emptyset\) for all \(x \in \mathcal{S}g^A X_1, z \in \mathcal{S}g^A X_2\) if \(x \leq z\), then there exists \(y \in \mathcal{S}g^A (X_1 \cap X_2)\) such that \(x \leq y \leq z\). \(RUIP\) and \(RWIP\) is obtained from \(UIP\) and \(WIP\) in the same way. The results on different interpolation properties is proved in Theorem 2.5. The case of restricted \(IP\) for \(RCA_\alpha\) follows from the fact that any three representable algebras \(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\), such that \(\mathfrak{C} \subseteq \mathfrak{A} \cup \mathfrak{B}\), can be amalgamated in \(LRCA_\alpha\) if \(\mathfrak{C}\) is a minimal algebra, i.e generated by the diagonal elements. Denoting the class of minimal algebras by \(LMn_\alpha\), this can be proved exactly like the \(CA\) case [17], using that \(LMn_\alpha \subseteq LDC_\alpha \subseteq SUPAPbase(\text{RCA}_\alpha)\). That \(CA_\alpha\) does not have \(RIP\) follows from discrete modalizing a result of Simon proved for \(CA_\alpha\) [30, Chapter 5], namely, there is a \(\mathfrak{C} \in Mn_\alpha, \mathfrak{A}, \mathfrak{B} \in CA_\alpha\) such that \(\mathfrak{C} \subseteq \mathfrak{A} \cup \mathfrak{B}\), but \(\mathfrak{A}\) and \(\mathfrak{B}\) have no amalgam in \(CA_\alpha\) over \(\mathfrak{C}\).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Dimension restricted free algebra & \(IP\) & \(UIP\) & \(WIP\) & \(RIP\) & \(RUIP\) & \(RWIP\) \\
\hline
\(\mathfrak{A}, K, RLCA_\alpha \subseteq K \subseteq LCA_\alpha\) & no & no & no & no & no & no \\
\hline
\(\mathfrak{A}, LCA_\alpha\) & yes & yes & yes & yes & yes & yes \\
in \(LLf_\alpha\) & & & & & & \\
\hline
\(\mathfrak{A}, LCA_\alpha\) & yes & yes & yes & yes & yes & yes \\
in \(LDC_\alpha\) & & & & & & \\
\hline
\(\mathfrak{A}, RLCA_\alpha\) & no & no & yes & yes & yes & yes \\
\hline
\(\mathfrak{A}, LCA_\alpha\) & no & no & yes & yes & no & no \\
\hline
\end{tabular}
\caption{Table 1}
\end{table}

In Tables 2 and 3, we summarize results obtained on various amalgamation properties for several classes of algebras. \(EP\) is the \(AP\) when the common subalgebra is minimal. \(SEP\) is \(SAP\) when the common subalgebra is minimal. \(ES\) is short for epimorphisms are surjective. \(SS_\alpha\) denotes the class of semisimple algebras. ‘\((S)AP\) w.r.t rep’, means (strong) amalgamation with respect to the class of representable algebras. ‘\(AP\) semisimple’ abbreviates that the class of semisimple algebras has \(AP\) which is true by Theorem 3.9. For finite dimensions, \(2 < n < \omega\), in both tables all results follow by \textit{discrete modalizing} the constructions used in the
CA case [17, 16] and the references therein. All other results are either in the text, cf. Theorems 2.5, 3.9, or can be obtained by discrete modalizing results proved in [18, 17, 16].

4.2. Conclusion; comparison to current research and proposing future research. An important instance of modal cylindric algebras is when $L = S4$, namely, Topological Logic. Motivated by questions like: which spatial structures may be characterized by means of modal logic, what is the logic of space, how to encode in modal logic different geometric relations, topological logic provides a framework for studying the confluence of the topological semantics for $S4$ modalities, based on topological spaces rather than Kripke frames, with the $S4$ modality induced by the interior operator of so-called Alexandrov topologies. In an Alexandrov topology the intersection of any number of closed sets is closed; so Alexandrov topologies are only significantly distinct from ordinary topologies when the underlying sets are infinite. Topological logic was introduced by Makowsky, Ziegler and Sgro [29, 28]. Such logics have a classical semantics with a topological flavour, addressing spatial logics and their study was approached using algebraic logic by Georgescu [10], the task that we further pursued in this paper.

The project of relating topology to modal logic begins with work of Alfred Tarski and J.C.C. McKinsey [31]. Strictly speaking Tarski and McKinsey did not work with modal logic, but rather with its algebraic counterpart, namely, Boolean algebras with operators which is the approach we adopt here; the operators they studied where the closure operator induced on what they called the algebra of topol-
ogy, certainly a very ambitious title, giving the impression that the paper aspired to completely algebraise topology. Other topological interpretations of propositional topological logic were recently extended in a natural way to arbitrary theories of full first order logic by Awodey and Kishida using so-called topological sheaves to interpret domains of quantification [3]. They prove that $S4_v$ (predicate $S4$ logic) is complete with respect to such extended topological semantics, using techniques related to recent work in topos theory.

In this paper we also studied algebraically a predicate version of the modal topological logic (and other modal predicate logics) as described above. One way of doing this is to deal with the same syntax in [3], but alter the semantics, dealing with usual Kripke semantics, which is what we did in this paper. Let $n < \omega$. In a sequel to this paper (in preparation), we intend to investigate finite dimensional modal cylindric algebras, in analogy to the well developed theory of $\text{CA}_n$. We shall investigate, among other things, finite axiomatizability, Sahlqvist axiomatizability, canonicity, atom-canonicity (and other notions applicable to $\text{CA}_n$s and $n$-dimensional multimodal logic) for various subvarieties of $\text{LCA}_n$, such as $\text{RLCA}_n$ and $\text{SNr}_k\text{LCA}_n$ for $2 < k < n$.

References


Received 26 February, 2019 and in revised form 17 March, 2019.