RESEARCH ARTICLE

Varying Interpolation and Amalgamation in Polyadic MV-Algebras

Tarek Sayed Ahmed
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We prove several interpolation theorems for many valued infinitary logic with quantifiers, by studying expansions of MV-algebras in the spirit of polyadic and cylindric algebras. We prove for various reducts of polyadic MV-algebras of infinite dimensions that if $\mathfrak{A}$ is the free algebra in the given signature, $X_1, X_2 \subseteq \mathfrak{A}$, $a$ is in the subalgebra of $\mathfrak{A}$ generated by $X_1$, and $b$ is in the subalgebra of $\mathfrak{A}$ generated by $X_2$ and $a \leq b$, then there exists an interpolant $c$ in the subalgebra generated by $X_1 \cap X_2$ and $n \in \mathbb{N}$ such that $a^n \leq c \leq nb$. We call this a varying interpolation property because the integer $n$ depends on the inequality $a \leq b$. We also address cases where this interpolation property fails, but other weaker (also varying) ones hold. One such interpolation theorem says that though an interpolant $c$ may not be found as above, an interpolant can always be found if finitely many universal quantifiers are applied to $a^n$ making it smaller and the same number of existential quantifiers are applied to $nb$ making it bigger. This number of quantifiers also varies; it depends on the inequality $a \leq b$. Several amalgamation theorems for classes (mostly varieties) of polyadic MV-algebras are obtained. Completeness theorems, relative to Hilbert style axiomatizations, for the corresponding infinitary many valued predicate logics are derived using the methodology of algebraic logic.

Keywords: Many valued logic; MV-algebras; polyadic algebras; algebraic semantics; varying interpolation property; generalized super amalgamation.

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1. Introduction

MV-algebras were introduced by Chang in 1958 (Chang, 1958) to provide an algebraic reflection of the completeness theorem of the Lukasiewicz infinite valued propositional logic. In recent years the range of applications of MV-algebras has been enormously extended with profound interaction with other topics, ranging from lattice ordered abelian groups, $C^*$ algebras, to fuzzy logic.

In this paper we study MV-algebras in connection to many valued predicate logic. We prove several interpolation theorems for many valued logic using the machinery of algebraic logic.

Formalized many valued logic can be traced back to the work of Lukasiewicz in 1920 and the independent work of Post in 1921, when three valued logic was studied. Heyting, a few years later, introduced a three valued propositional calculus related to intuitionistic logic. Gödel proposed an infinite hierarchy of finitely-valued systems; his goal was to show that intuitionistic logic is not a many valued logic.

In the last few decades many valued logics have acquired tremendous interest; in 1965 Zadeh had published his landmark paper on fuzzy sets and the trend of
fuzzy logic started. Today the various approaches to many valued logics, expressed algebraically by $MV$-algebras, are aspiring to provide fuzzy logic the theoretical rigorous foundations that were lacking for a long time.

One of the earliest papers (if not the first) that deals with an application of polyadic algebras to $MV$-algebras is (Shwartz, 1980). Another application of polyadic algebras to $MV$-algebras and Pavelka Rational logic is (Dragulici & Georgescu, 2001). Here our proven interpolation theorems substantially generalize the representability results proved in the aforementioned papers.

An $MV$-algebra, has a dual behaviour: it can be viewed, in one of its facets, as a 'non-idempotent' generalization of a Boolean algebra possessing a strong lattice structure. The lack of idempotency enables $MV$-algebras to be compared to monoidal structures like monoids and abelian groups. Indeed, the category of $MV$-algebras has been shown to be equivalent to the category of $l$-groups. At the same time the lattice structure of Boolean algebras can be recovered inside $MV$-algebras, by an appropriate term definability of primitive connectives. In this respect, they have a strong lattice structure (distributive and bounded), which make the techniques of lattice theory readily applicable to their study.

Boolean algebras work as the equivalent semantics of classical propositional logic. To study classical first order logic, Tarski (Henkin, Monk, & Tarski, 1971; Henkin, Monk, & Tarski, 1985) introduced cylindric algebras, while Halmos (Halmos, 1962) introduced polyadic algebras. Both of those can be viewed as Boolean algebras with extra operations that reflect algebraically existential quantifiers. In this paper we show, in the context of many reducts of polyadic algebras, that when we replace the notion of a Boolean algebra with an $MV$-algebra, some results proved in the classical case, survive such a replacement. But in all cases, non-trivial modifications to accommodate non-idempotency, are required.

By studying reducts of polyadic algebras together with weakening the Boolean structure to be an $MV$-algebra, we are in the realm of many valued quantifier infinitary logics. Monadic $MV$–algebras where we have only one existential quantifier were studied in some depth in (Nola & Grigola, 2004; Belluce, Grigola, & Letteri, 2005). Many-valued logics are non-classical logics. The most two basic (semantical) assumptions of classical logic propositional as well as first order are the principles of bivalence and of compositionality. The principle of bivalence is the assumption that each sentence is either true or false, i.e has exactly one truth value. Many valued logics differ from classical ones by the fundamental fact that it does not restrict the number of truth values to only two: they allow for a larger set (possibly infinite) of truth degrees. However, like classical logic they accept the principle of compositionality (or truth-functionality), namely, that the truth of a compound sentence is determined by the truth values of its component sentences (and so remains unaffected when one of its component sentences is replaced by another sentence with the same truth value).

Algebraically the Craig interpolation property for cylindric algebras takes the form: If $\mathfrak{A}$ is a certain free algebra (corresponding to the Tarski–Lindenbaum quotient algebra of pure first order logic) $X_1, X_2 \subseteq \mathfrak{A}$, $a$ is in the algebra generated by $X_1$ $b$ is in algebra generated by $X_2$, and $a \leq b$, then there exists an interpolant, namely, an an element $c$ in the subalgebra generated by $X_1 \cap X_2$, such that $a \leq c \leq b$. But for polyadic $MV$–algebras due to the lack of idempotency, the interpolation theorems we prove have a different form; the conclusion will be always: there exists $c$ in the common subalgebra and $n \in \mathbb{N}$ such that $a^n \leq c \leq nb$. Here $a^n$ is $a \odot a \odot a \ldots n$ times, and $n$ is $b \oplus b \oplus b \ldots n$ times, where $\odot$ and $\oplus$ are the binary operations of strong conjunction and disjunction, respectively. $a^n$ is called $n$–fold multiplication while $na$ is called $n$–fold addition. Here the interpolation property
differs from the classical (Craig) interpolation property, in the sense, that it is not uniform.

The interpolant for an implication $\phi \rightarrow \psi$ (some formulas $\phi$ and $\psi$) is actually found in the classical sense to another implication, namely, $\phi^n \rightarrow n\psi$ for some $n \in \mathbb{N}$, where $n$ depends on the original implication $\phi \rightarrow \psi$. One can associate with $\phi \rightarrow \psi$ the least such $n$, and call it the the index of $\phi \rightarrow \psi$. Then the index is not unique (uniform) with respect to all implications. An interpolant for a different implication may render a different index. Accordingly, as we already did in the title, below we shall refer to this interpolation property as the varying interpolation property.

Of course in Boolean algebras $a^n = a = na$ and this new interpolation property reduces to the usual Craig interpolation property; for all implications regardless of the interpolent, the index $n$ collapses to the unique value $n = 1$. Results addressing the Boolean case (Sayed Ahmed, 2004, 2007, 2010a; Daigneault & Monk, 1963; Ferenczi, 2012; Halmos, 1962; Johnson, 1970) become special cases of our new results established below.

From our interpolation results we infer that Lukasiewicz calculus together with some of its infinitary extensions, have a varying (Craig) interpolation property as described above. For many valued predicate logics, the main types of logical calculi are Hilbert style calculi, Genzen type sequent calculi and Tableau calculi. We also show algebraically that the extensions we study are also complete, relative to a Hilbert style calculus.


We use fairly standard notation. Any possibly unfamiliar notation will be introduced at its first occurrence in the text. We denote algebras by Gothic letters, and when we write $A$ for an algebra, then we shall be tacitly assuming that the Roman $A$ denotes its universe. In what follows for an algebra $A$ and $X \subseteq A$, we write $\mathfrak{S}gA^X$, or simply $\mathfrak{S}gX$ if $A$ is clear from context, for the subalgebra of $A$ generated by $X$.

Throughout the paper, unless otherwise explicitly indicated, ordinals considered are infinite.

Layout

(1) In section 2, we review the basic algebraic notions and concepts of $M\!V$-algebras and polyadic $M\!V$-algebras needed in the sequel, and we fix the notation.

(2) In section 3, we show that the so–called dimension restricted free polyadic $M\!V$-algebras (to be defined below) have the stronger varying interpolation property as stated in the first part of the abstract, theorem 19. As a special case we obtain the varying interpolation theorem for $M\!V$-predicate logic. We prove (algebraically) an entirely analogous varying interpolation theorem for certain logics that can allow infinitary predicates of which Pavelka Predicate calculus is a special case, theorem 27. In such a calculus truth constants $r \in [0, 1]$ are incorporated in the syntax.

(3) In section 4, we show that the free quasi-polyadic $M\!V$-algebras without the italicized condition (of dimension restriction) in the previous item have the weaker form of the varying interpolation as in the second part of the
abstract, theorem 36, and that the stronger form in the abstract does not hold for these free algebras, corollary 41. Positive and negative theorems on representability and the amalgamation property for various classes of quasi-polyadic MV-algebras are proved, theorems 28, 30, 31, 34, 36, 40, 42, 45, 46, 48, 50, 53, 56, 58, and 60.

(4) In the last section, we show that the free polyadic MV-algebras, the free cylindric–polyadic MV-algebras, and free rich polyadic MV-algebras (these terms will be defined below) have the strong form of the varying interpolation property as stated in the abstract. From this we conclude that algebras in such classes, defined equationally, are representable, theorems 63, 67 and have the so-called generalized super amalgamation property, corollary 68.

In the classical case the super amalgamation property is a strong form of amalgamation; the generalized super amalgamation property, introduced in this paper, is its many valued version.

(5) In the last section we summarize our results in tabular form.

2. Algebraic Preliminaries

In this section we recall some basic algebraic notions. We start by the definition of MV-algebras and briefly discuss connections with residuated lattices.

2.1 MV-algebras

**Definition 1** An MV-algebra is an algebra

\[ A = (A, \oplus, \odot, \neg, 0, 1) \]

where \( \oplus, \odot \) are binary operations, \( \neg \) is a unary operation and \( 0, 1 \in A \), such that the following identities hold:

1. \( a \oplus b = b \oplus a, \ a \odot b = b \odot a \).
2. \( a \oplus (b \oplus c) = (a \oplus b) \oplus c, \ a \odot (b \odot c) = (a \odot b) \odot c \).
3. \( a \oplus 0 = a, \ a \odot 1 = a \).
4. \( a \oplus 1 = 1, \ a \odot 0 = a \).
5. \( a \oplus \neg a = 1, \ a \odot \neg a = 0 \).
6. \( \neg(a \oplus b) = \neg a \odot \neg b, \ \neg(a \odot b) = \neg a \oplus \neg b \).
7. \( a = \neg \neg a \), \( \neg 0 = 1 \).
8. \( \neg(a \oplus b) \odot b = \neg(b \oplus a) \oplus a \).

Many of the equations have obvious meaning, item (5) is the law of excluded middle, item (6) are the de Morgan laws for the operations \( \oplus \) and \( \odot \) and item (7) is the law of double negation. The last identity is not so obvious as the other ones, we will return to it in a while.

MV-algebras form a variety that is a subvariety of the variety of so-called BL algebras introduced by Hajek, in fact, MV-algebras coincide with those BL algebras satisfying double negation law, namely, that \( \neg \neg x = x \) and contains all Boolean algebras.

**Example 2** A simple and central, in fact canonical, numerical example is \( A = [0, 1] \) with operations \( x \oplus y = \min(x + y, 1) \), \( x \odot y = \max(x + y - 1, 0) \), and \( \neg x = 1 - x \). In mathematical fuzzy logic, this MV-algebra is called the standard MV-algebra, as it forms the standard real-valued semantics of Lukasiewicz logic.
MV-algebras are residuated lattices satisfying some extra conditions:

**Definition 3** A residuated lattice is an algebra

\[(L, \cup, \cap, *, \rightarrow, 0, 1)\]

with four binary operations and two constants such that

(i) \((L, \cup, \cap, 0, 1)\) is a lattice with largest element 1 and the least element 0 (with respect to the lattice ordering defined the usual way: \(a \leq b\) iff \(a \cap b = a\)).

(ii) \((L, *, 1)\) is a commutative semigroup with largest element 1, that is \(\ast\) is commutative, associative, \(1 \ast x = x\) for all \(x\).

(iii) Letting \(\leq\) denote the usual lattice ordering, we have \(\ast\) and \(\rightarrow\) form an adjoint pair, i.e for all \(x, y, z\)

\[z \leq (x \rightarrow y) \iff x \ast z \leq y.\]

It is known that an MV-algebra is a pre-linear commutative bounded integral residuated lattice satisfying the additional identity

\[x \cup y = (x \rightarrow y) \rightarrow y.\]

In case of an MV-algebra, \(\ast\) is the so-called strong conjunction which we denoted here following standard notation in the literature by \(\odot\). \(\cap\) is called weak conjunction.

The other operations are defined by

\[\neg a = a \rightarrow 0\]

\[a \oplus b = \neg (\neg a \odot \neg b)\]

The \(\cup\) is called weak disjunction, while \(\oplus\) is called strong disjunction. The presence of weak and strong conjunction is a common feature of substructural logics without the rule of contraction, to which Lukasiewicz logic belongs. The last identity in definition 1 of MV-algebras states that the supremum lattice operation is commutative, for in fact we have:

\[a \cup b = \neg (\neg a \oplus b) \oplus b = (a \odot \neg b) \oplus b,\]

\[a \cap b = \neg \neg a \cup \neg b = (a \odot \neg b) \odot b.\]

Note that if \(\mathfrak{A}\) is an MV-algebra, and \(a, b \in \mathfrak{A}\), then \(a \odot b \leq a \cap b\) and \(a \cup b \leq a \oplus b\).

Since we do not have idempotency in MV-algebras, then for an MV-algebra \(\mathfrak{A}\), \(n \in \mathbb{N}\) and \(a \in A\), \(a^n\) denotes \(a \odot a \odot \ldots\) \(n\) times, and \(na\) denotes \(a \oplus a \oplus \ldots\) \(n\) times. Plainly, \(a^n \leq a \leq na\).

We recall the definition of a filter in an MV-algebra. Filters, as in the case of Boolean algebras, correspond to congruences.

**Definition 4** Let \(\mathfrak{B}\) be an MV-algebra. A filter of \(\mathfrak{B}\) is a non-empty subset \(F \subseteq A\) such that for all \(a, b \in B\),

(i) \(a, b \in F\) implies \(a \odot b \in F\).

(ii) \(a \in F\) and \(a \leq b\) imply \(b \in F\).

For an MV-algebra \(\mathfrak{A}\) and \(X \subseteq \mathfrak{A}\), we write \(\text{fl}^{\mathfrak{A}}X\) for the filter generated by \(X\). It is folklore that MV-algebras has the congruence extension property. Formulated via filters we have:

**Lemma 5** Let \(\mathfrak{A} \subseteq \mathfrak{B}\) be MV-algebras. Let \(F\) be a filter in \(\mathfrak{A}\). Then there exists a filter \(G\) in \(\mathfrak{B}\) such that \(G \cap \mathfrak{A} = F\).
An MV-algebra is *simple* if it has no proper congruences, equivalently, it has no proper filters. The following theorem is also folklore.

**Theorem 6** An MV-algebra $\mathfrak{A}$ is simple $\iff$ it is isomorphic to a subalgebra of the standard MV-algebra.

### 2.2 Polyadic MV-Algebras

Now we introduce expansions of MV-algebras that allow quantification. We define a variety of polyadic MV-algebras and we give several (concrete) examples of such algebras defined abstractly by a finite schema of equations. Then we give the central definition in this paper which we call *varying interpolation property*, VIP for short. This property coincides with the (ordinary) interpolation property IP in the classical case, that is, when we have idempotency.

**Definition 7** Let $\mathfrak{A}$ be an MV-algebra. An existential quantifier on $\mathfrak{A}$ is a function $\exists : A \to A$ that satisfies the following conditions for all $a, b \in A$:

1. $\exists 0 = 0$.
2. $a \leq \exists a$.
3. $\exists(a \circ \exists b) = \exists a \circ \exists b$.
4. $\exists(a \circ \exists b) = \exists a \circ \exists b$.
5. $\exists(a \circ a) = \exists a \circ \exists a$.
6. $\exists(a \circ a) = \exists a \circ \exists a$.
7. $\exists(a \rightarrow b) \leq \exists a \rightarrow \exists b$, where $a \rightarrow b$ is $\neg a \oplus b$.

MV-algebras with an existential quantifier as in definition 7 are studied under the names of *Monadic MV-algebras* in (Nola & Grigolia, 2004; Belluce et al., 2005). Let $\mathfrak{A}$ be a Monadic MV algebra, that is, an MV-algebra, with existential quantifier $\exists$. For $a \in A$, set $\forall a = \neg \exists \neg a$. Then $\forall$ is a unary operation on $\mathfrak{A}$ called a *universal quantifier* and it satisfies all properties of the existential quantifier, except for (2) which takes the form: $\forall a \leq a$ and (7). This follows directly from the axioms.

Next we define the algebras to be addressed in this paper. Their signature depends on a fixed in advance semigroup and their dimension is infinite. We let $X \subseteq Y$ denote that $X$ is a finite subset of $Y$, and we write $\wp_\omega(Y)$ for $\{X : X \subseteq Y \}$.

**Definition 8** Let $\alpha$ be an ordinal. Let $G \subseteq \alpha$ be a semigroup under the operation of composition of maps. Let $T \subseteq \wp(\alpha)$. An $\alpha$-dimensional polyadic MV-algebra of type $(G, T)$, an MV$_{G,T}$ for short, is an algebra of the form

$$(A, \oplus, \odot, \neg, 0, 1, s_\tau, c(J))_{\tau \in G, J \in T}$$

where $(A, \oplus, \odot, \neg, 0, 1)$ is an MV-algebra, $s_\tau : \mathfrak{A} \to \mathfrak{A}$ is an endomorphism of MV-algebras, $c(J)$ is an existential quantifier, such that the following hold for all $p \in A$, $\sigma, \tau \in G$ and $J, J' \in T$:

1. $s_{Id}p = p$.
2. $s_{\sigma \tau}p = s_\sigma s_\tau p$.
3. $c_{(J \cup J')}p = c(J)c(J')p$.
4. If $\sigma \upharpoonright J = \tau \upharpoonright J$, then $s_\sigma c(J)p = s_\tau c(J)p$.
5. If $\sigma \upharpoonright \sigma^{-1}(J)$ is injective, then $c(J)s_\tau p = s_\sigma c_{\sigma^{-1}(J)}p$. 

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Some comments on the definition:

(1) We refer to the operations $c_i$ as cylindrifiers and the $s_j$ as substitutions. The former terminology comes from the geometric interpretation of such operations in concrete algebras; it is forming cylinders.

(2) A transformation $\tau : \alpha \rightarrow \alpha$ is finite if the set $\{ i \in \alpha : \tau(i) \neq i \}$ is finite. We write $FT_\alpha$ for the set of all finite transformations on $\alpha$. In all cases considered, we will have $FT_\alpha \subseteq G$ and $\psi_\alpha(\alpha) \subseteq T$. In particular, for all $i \in \alpha$, the cylindrifier $c_{\{i\}}$, which we denote simply by $c_i$, will be present in the signature. For $i \neq j \in \alpha$, $[i,j]$ denotes the transposition that interchanges $i$ and $j$ and leaves everything else fixed, while $[i]j$ denotes the replacement that sends $i$ to $j$ and is the identity map otherwise. It is known that $FT_\alpha$ is generated (as a semigroup) by the set of replacements and transpositions on $\alpha$. So in all cases considered, we have $[i,j]$ and $[i]j \in G$ for each $i \neq j \in \alpha$. According to a wide spread custom in the literature of cylindric and polyadic algebras, we denote the substitution operation $s_{[ij]}$ by $s_j$.

(3) Given $\mathfrak{A} \in MV_{G,T}$ for some $G \subseteq \alpha$ and $T \subseteq \varphi(\alpha)$, the dimension $\alpha$ will be clear from context. By $\mathfrak{R}_{MV}\mathfrak{A}$ we understand the reduct obtained from $\mathfrak{A}$ by discarding cylindrifiers and substitution operations, so that $\mathfrak{R}_{MV}\mathfrak{A}$ is an $MV$-algebra.

(4) If $\mathfrak{R}_{MV}\mathfrak{A}$ is a Boolean algebra (equivalently the operations $\oplus$ and $\odot$ are idempotent), we will say that we are in the classical case. In the classical case, when $G = FT_\alpha$ and $T = \psi_\alpha(\alpha)$, then $MV_{T,G}$ is the class of quasi–polyadic algebras of dimension $\alpha$, denoted by $QA_\alpha$ (Sain & Thompson, 1990). If $G = \alpha$ and $T = \varphi(\alpha)$, then $MV_{G,T}$ is the class of polyadic algebras of dimension $\alpha$, denoted by $PA_\alpha$ (Halmos, 1962; Daigneault & Monk, 1963).

Theorem 9 Let $\mathfrak{A} = (A, \oplus, \odot, \neg, 0, 1, s_\tau, c_{\{j\}})_{\tau \in G, J \in T}$ be an $MV_{G,T}$. For each $J \in T$ and $x \in A$, let $q_{\{j\}}x = \neg c_{\{j\}}x$. Then the following hold for each $p \in A$, $\sigma, \tau \in G$ and $J, J' \in T$.

1. $q_{\{j\}}$ is a universal quantifier,
2. $q_{\{j\} \cup J'}p = q_{\{j\}}c_{\{j\} \cap J'}p$,
3. $q_{\{j\}}q_{\{j\}}p = q_{\{j\}}p$, $q_{\{j\}}c_{\{j\}}p = c_{\{j\}}p$,
4. If $\sigma \upharpoonright \alpha \setminus J = \tau = \alpha \setminus J$, then $s_\sigma q_{\{j\}}p = s_\sigma q_{\{j\}}p$,
5. If $\sigma \upharpoonright \sigma^{-1}(J)$ is injective, then $q_{\{j\}}s_\sigma p = s_\sigma q_{\{j\}}p$.

Proof Routine.

The following definition is the algebraic counterpart of the interpolation property which we shall deal with. It is not the usual interpolation property dealt with in the case of Boolean algebras with operators because of lack of idempotency. The definition coincides with usual one when the operations $\oplus$ and $\odot$ are idempotent, in which case we shall be dealing with the known classical case.

Definition 10 An algebra $\mathfrak{A} \in MV_{G,T}$ has the varying interpolation property, VIP for short, if for all $X_1, X_2 \subseteq \mathfrak{A}$ if $a \in \mathfrak{S}_g X_1$ and $b \in \mathfrak{S}_g X_2$ are such that $a \leq b$, then there exists $n, m \in \mathbb{N}$, $c \in \mathfrak{S}_g \mathfrak{S}_g (X_1 \cap X_2)$ such that $a^n < c < b^n$.

In definition 10 one can take $n = m$ in the conclusion. To see why, if $n < m$, then one replaces $a^n$ by $a^m$. The inequality is still valid, because $a^m \leq a^n$. Conversely, if $m < n$ then one replaces $ma$ by $na$; the inequality will also remain valid because $ma \leq na$. If $n = m$, then there is nothing more to prove.

Unlike the usual (Craig) interpolation property, henceforth denoted by $IP$, for (classical) predicate logic this property is not ‘uniform’ in the sense that if $\phi \leq \psi$ then the interpolant is found for another inequality, namely, $\phi^n \rightarrow \psi^m$, for some
n ∈ N and there is no unique n for all such implications. Nevertheless, this VIP reduces to the usual IP for first order logic where the interpolant is found with respect to this same inequality. In the classical case, n is unique for because of idempotency n = 1 uniformly for every implication considered.

But this phenomena occurs too in more general contexts than first order logic where the usual (Craig) interpolation property ramifies into several different interpolation properties, witness (Madárasz & Sayed Ahmed, 2012; Sayed Ahmed, 2012b); the former reference for an overview. By this phenomena, we mean:

An interpolant of an implication, φ → ψ say, is in fact an interpolant in the ‘usual Craig interpolation’ sense of another implication, obtained from the first one by appying certain ‘operators’ on φ and ψ. Plainly IP ⇒ VIP but the converse implication might not be true. That is if VIP fails, then we do not have IP in the classical sense.

The notion of VIP will be formulated more rigorously in definition 20, witness too remark 1 for further elaboration and comparison with other cases in the literature of algebraic logic dealing with a VIP, too. Below we shall encounter a weak form of varying interpolation referred to as WVIP (weak varying interpolation property) which is weaker than VIP as the name suggests, cf. definition 36.

We will investigate VIP for certain dimension restricted free algebras, which is an algebraic notion more general than that of free algebras. We now give standard examples of polyadic MV-algebras of dimension α, that are the prime source of introducing such algebras.

Example 11 (1) We start by the concrete algebras, referred to as set algebras, when the operations ⊕ and ⊙ are idempotent. That is we start with the (special) classical case. Let U be a set and α be an ordinal. Then B(αU) is the Boolean set algebra with unit αU. We denote the Boolean (concrete) operations of intersection, union and complementation by ∩, ∪ and ∼, respectively. Let τ : α → α, i,j < α and X ⊆ αU. Then

sτX = {s ∈ αU : s ♦ τ ∈ X},

ciX = {t ∈ αU : ∃s ∈ X and t(j) = s(j) for all j ≠ i}.

A polyadic set algebra of dimension α is an algebra of the form

(B(αU), ci, sτ)τ:α→α,i<α.

The class of representable polyadic algebras of dimension α, RPAα, for short, is defined as the class of subdirect products of set algebras of dimension α. The class of representable quasi-polyadic algebras of dimension α, RQAα is defined analogously, taking set algebras in the appropriate signature, that is, taking G = FTα and T = σα(α). It is known that both RPAα and RQAα are varieties.

(2) In the previous item the domain of the representable algebra was ϕ(V) which can of course be identified with the set F(V, 2) of all maps from V to 2 = {0, 1}.

Now we define the many valued version of such set algebras. Instead of the two element Boolean algebra 2, take L to be any fixed linearly ordered complete MV-algebra. Let X be a set and α be an infinite ordinal. For x, y ∈ αX and J ⊆ α, write x ≡ J y if x(l) = y(l) for all l /∈ J. The universe of the new algebra we define, denoted by F(αX, L), consists of the set of all maps from αX to L. The
MV operations are defined pointwise. For \( p : \alpha X \to L \):

\[
c_{(J)} p(x) = \bigvee \{ p(y) : x \equiv_J y \},
\]

and

\[
s_\tau p(x) = p(x \circ \tau).
\]

Such special algebras, taken in a type \((G, T)\), where \( G \subseteq \alpha \) and \( T \subseteq \wp(\alpha) \), will be denoted by \( \mathfrak{A}(\alpha X, L) \). The signature of the algebra determined by \((G, T)\) will be clear from context. The typical example is when \( L = [0, 1] \).

(3) We give a special case of the previous example when the defined algebras arise from models of MV predicate calculus. Let \( \mathfrak{M} = (M, p_M)_{p \in P} \) be a structure for predicate MV logic having a sequence of variables of order type \( \omega \) based on an MV-algebra \( \mathfrak{A} \). For a formula \( \phi \), let \( \phi^\mathfrak{M} = \{ s : \omega M \to \mathfrak{A} : ||\phi||_{\mathfrak{M}, s} = 1 \} \). Here \( ||\phi||_{\mathfrak{M}, s} \) is the truth value of \( \phi \) at \( s \), evaluated in an MV-algebra \( \mathfrak{A} \), the usual Tarskian way. Let \( Fm \) denote the set of formulas. Then \( \{ \phi^\mathfrak{M} : \phi \in Fm \} \) is the universe of a polyadic MV-algebra which we denote by \( \mathfrak{C}^\mathfrak{M} \) to emphasize the role played by \( \mathfrak{M} \) and call it the set algebra based on \( \mathfrak{M} \). The operations are read off from the semantics of connectives. For example:

\[
\phi^\mathfrak{M} \oplus \psi^\mathfrak{M} = (\phi \oplus \psi)^\mathfrak{M},
\]

and

\[
c_{(J)} \phi^\mathfrak{M} = (\exists J \phi)^\mathfrak{M},
\]

where \( J \) is a finite subset of \( \omega \). Such algebras are subalgebras of algebras in the preceding item having the type \( G = FT_\alpha \) and \( T = \wp_\omega(\alpha) \), when \( L = [0, 1] \) and \( \alpha = \omega \).

(4) When in the previous item we take \( L = 2 \), then such algebras (of first order logic) are Boolean set algebras having domain \( \wp^\omega(\mathfrak{M}) \), enriched with cylindrifiers and substitution operations as defined in the first item. Such algebras arise from a first order structure \( \mathfrak{M} \) as defined in (Henkin et al., 1985, §4.3).

Like rings of matrices in ring theory, Boolean fields of sets and cylindric set algebras, the algebras in the last example will constitute the canonical concrete structures for polyadic MV-algebras, the representable algebras. As shown such algebras have a metalogical origin and their representation as set algebras adds a multi-dimensional geometric intuition to their construction.

Besides the interpolation property expressed in definition 10, a question that we answer for all algebras considered is the following: If \( \mathfrak{A} \models \Sigma \), where \( \Sigma \) is the set of equations in definition 8, is \( \mathfrak{A} \) representable? An affirmative answer is a completeness theorem for the corresponding multi valued predicate calculus. The converse; a soundness theorem is not hard to show. Indeed it is fairly straightforward to check that the equational axioms stipulated above hold in such concrete algebras.

### 2.3 Representable and Free Algebras

In this subsection we give more concrete examples of polyadic (set) MV-algebras. To define the representable algebras we need:
Definition 12 Let \( \alpha \) be an ordinal. A set of the form \( ^\alpha U \) is called a cartesian space of dimension \( \alpha \). Let \( p \in ^\alpha U \). A set of the form \( \{ s \in ^\alpha U : |\{ i \in \alpha : s_i \neq p_i \}| < \omega \} \), denoted by \( ^\alpha U(p) \), is called a weak space of dimension \( \alpha \).

When \( G = FT_\alpha \) (the set of finite transformations on \( \alpha \)) and \( T = \varphi_\omega(\alpha) \) we denote the class \( \text{MV}_{G,T} \) by \( \text{MA}_\alpha \). When \( G = \alpha \alpha \) and \( T = \varphi(\alpha) \), we denote the class \( \text{MV}_{G,T} \) by \( \text{PMA}_\alpha \).

Definition 13 (1) An algebra \( \mathfrak{A} \in \text{MA}_\alpha \) is representable if for all \( a \neq 0 \), there exist a set \( V \) that is weak space of dimension \( \alpha \), and a homomorphism from \( \mathfrak{A} \) to \( \mathfrak{F}(V, [0,1]) \) such that \( f(a) \neq 0 \).

(2) An algebra \( \mathfrak{A} \in \text{PMA}_\alpha \) is representable if for all \( a \neq 0 \), there exist a set \( U \), and a homomorphism from \( \mathfrak{A} \) to \( \mathfrak{F}(^\alpha U, [0,1]) \) such that \( f(a) \neq 0 \).

In the first item the algebra \( \mathfrak{F}(V, [0,1]) \) is taken in the type \( (\text{FT}_\alpha, \varphi_\omega(\alpha)) \) and in the second item the algebra \( \mathfrak{F}(^\alpha U, [0,1]) \) is taken in the type \( (^\alpha \alpha, \varphi(\alpha)) \). The class of representable algebras \( \text{MA}_\alpha \) will be denoted by \( \text{RMA}_\alpha \) and the class of representable \( \text{PMA}_\alpha \) will be denoted by \( \text{RPMA}_\alpha \). The soundness condition is now succintly expressed via \( \text{RMA}_\alpha(\text{RPMA}_\alpha) \subseteq \text{MA}_\alpha(\text{PMA}_\alpha) \).

We introduce next the notion of neat reducts for \( \text{MA}_\alpha \), a notion borrowed from cylindric algebras. This notion can (and will) be defined for \( \text{PMA}_\alpha \), too, but we postpone the definition (which is slightly more involved when we have infinitary substitutions) until needed, definition 61. This notion plays a pivotal role in the representation theory of both cylindric and polyadic algebras, and as we will see as the paper unfolds, this is the case too with \( \text{MA}_\alpha \).

Definition 14 (1) Let \( \alpha < \beta \) be ordinals. For \( \tau \) in \( \text{FT}_\alpha \), we write \( \bar{\tau} \) for \( \tau \cup \text{Id}_{\beta \setminus \alpha} \).

Plainly \( \bar{\tau} \in \text{FT}_\beta \). Let \( \mathfrak{A} = (A, \oplus, \odot, \neg, 0, 1, c_{(J)}, s_\tau)_{J \in T, \tau \in FT_\beta} \) be an algebra in \( \text{MA}_\beta \). Then \( \mathfrak{R}_\alpha \mathfrak{A} \) is the \( \text{MA}_\alpha \) obtained by discarding operations indexed by elements in \( \beta \setminus \alpha \). That is

\[
\mathfrak{R}_\alpha \mathfrak{A} = (A, \oplus, \odot, \neg, 0, 1, c_{(J)}, s_\tau)_{J \subseteq \alpha, \tau \in \text{FT}_\alpha}.
\]

(2) For \( \mathfrak{A} \in \text{MA}_\beta \) and \( x \in A \), then \( \Delta x \), the dimension set of \( x \), is defined by \( \Delta x = \{ i \in \beta : c_ix \neq x \} \). Let \( B = \{ x \in A : \Delta x \subseteq \alpha \} \). Then \( B \) is a subuniverse of \( \mathfrak{R}_\alpha \mathfrak{A} \), the \( \text{MA}_\alpha \) having universe \( B \) is called the \( \alpha \)-neat reduct of \( \mathfrak{A} \) and it is denoted by \( \mathfrak{R}_\alpha \mathfrak{A} \). When \( \mathfrak{D} \subseteq \mathfrak{R}_\alpha \mathfrak{A} \), we say that \( \mathfrak{A} \) is a \( \beta \)-dilation of \( \mathfrak{D} \) and that \( \mathfrak{D} \) neatly embeds into \( \mathfrak{A} \).

For a pair of ordinals \( \alpha < \beta \), and \( K \subseteq \text{MA}_\beta \), \( \text{NR}_\alpha K \) denotes the class \( \{ \mathfrak{R}_\alpha \mathfrak{B} : \mathfrak{B} \in K \} \subseteq \text{MA}_\alpha \). It is known (in the classical case) that \( \text{RQA}_\alpha = \text{SNR}_\alpha \text{QA}_{\alpha+\omega} \), where \( \text{SN} \) is the operation of forming subalgebra, a result that we extend to \( \text{MA}_\alpha \), cf. theorem 28.

For \( \alpha \geq \omega \), \( \text{DC}_\alpha \) denotes the class \( \{ \mathfrak{A} \in \text{MA}_\alpha : |\Delta x| \geq \omega, \forall x \in \mathfrak{A} \} \). An algebra \( \mathfrak{A} \in \text{DC}_\alpha \) is called dimension complemented.

Theorem 15 Let \( \omega \leq \alpha < \beta \) be ordinals. Let \( \mathfrak{A} \in \text{DC}_\alpha \). Then there exists a \( \beta \)-dilation \( \mathfrak{B} \in \text{MA}_\beta \) of \( \mathfrak{A} \), such that \( \mathfrak{A} \subseteq \mathfrak{R}_\alpha \mathfrak{B} \) and for all \( X \subseteq A \), one has \( \text{SG}^\beta X = \mathfrak{R}_\alpha \text{SG}^\beta X \). In particular, \( \mathfrak{A} = \mathfrak{R}_\alpha \mathfrak{B} \).

Proof (Henkin et al., 1971, Theorems 2.6.49, 2.6.66).

Next we define a notion more general than free algebras: that of dimension restricted free algebras. The definition, originally formulated for cylindric algebras, is due to Henkin et al. (Henkin et al., 1971, Definition 2.5.31) to capture the notion
algebras of formulas in pure first order logic which are not free, but satisfy a certain universal property.

For a class $K$, (recall that) $S$ stands for the operation of forming subalgebras of $K$, and let $PK$ stand for the operation of forming direct products of algebras in $K$. The algebras we define next (like ordinary free algebras) are constructed from the absolutely free algebra in the signature of $MA_\alpha$.

**Definition 16** Let $\beta$ be a non–zero cardinal and $\alpha$ be an (infinite) ordinal. Let $\alpha \frak{tr}_\beta$ be the absolutely free algebra on $\beta$ generators and of the signature of $MA_\alpha$. For an algebra $\frak{A}$, we write $R \in Co\frak{A}$ if $R$ is a congruence relation on $\frak{A}$. Let $\rho \in \beta \chi(\alpha)$. Let $L$ be a class having the same signature as $MA_\alpha$. Let

$$Cr_\beta^{(\rho)}L = \bigcap \{R : R \in Co_{\alpha \frak{tr}_\beta}, \alpha \frak{tr}_\beta/R \in SPL, c_k^{\frak{tr}}\frak{r} \otimes \eta/R = \eta/R \text{ for each } \eta < \beta \text{ and each } k \in \alpha \setminus \rho(\eta)\}$$

and

$$\frak{tr}_\beta^\rho L = \alpha \frak{tr}_\beta/Cr_\beta^{(\rho)}L.$$

We say that the sequence $x = (\eta/Cr_\beta^{\rho}L : \eta < \beta) L$ - freely generates $\frak{tr}_\beta^\rho L$ under $\rho$. When $L = MA_\alpha$, the hitherto constructed algebra $\frak{tr}_\beta^\rho MA_\alpha$ is called the free algebra on $\beta$ generators with dimension restricting function $\rho$, or simply a dimension restricted free algebra on $\beta$ generators, when $\rho$ is clear from context.

The following theorem characterizes the dimension restricted free algebras up to isomorphism:

**Theorem 17** Let $\frak{A} = \frak{tr}_\beta^\rho MA_\alpha$ and let $x = (\eta/Cr_\beta^{\rho}MA_\alpha : \eta < \beta)$ be as in definition 16, that is, $x MA_\alpha$-freely generates $\frak{A}$ under the dimension restricting function $\rho$. Then $\frak{A}$ is characterized up to isomorphism by the following universal property. For any algebra $\frak{B} \in MA_\alpha$, for any $y = \{y_i : i < \beta\} \in ^\beta \frak{B}$ whenever $\Delta y_i \subseteq \rho(i)$ for all $i < \beta$, then there is a unique homomorphism $h : \frak{A} \rightarrow \frak{B}$ such that $h \circ x = y$.

**Proof** (Henkin et al., 1971, Theorems 2.5.36–2.5.37). $\square$

**Example 18** The function $\rho$ in definition 16 can be referred to to as the the rank function. Viewed as an extended algebra of formulas $\rho$ determines the rank of atomic formulas (the generators) in the dimension restricted free algebra $\frak{tr}_\beta^\rho MA_\alpha$. For $i \in \beta$, the rank of (the atomic formula) $i$ is $\rho(i)$.

1. When in definition 16 $\rho(i) = \alpha$ for all $i \in \beta$, this coincides with usual notion of the free algebra on $\beta$ generators. That is $\frak{tr}_\beta^\rho MA_\alpha = \frak{tr}_\beta MA_\alpha$ when $\rho = \beta \times \{\alpha\}$. (Here $\rho$ is not restricting the dimension). The corresponding logic in the classical cases (when $L = QA_\alpha$) is referred to in (Henkin et al., 1985, Section 4.3) as a typeless logic of infinitary relations. Here the arity of atomic relational formulas is $\alpha$.

2. When in definition 16, we have $|\rho(i)| < \omega$ for all $i < \beta$, then $\frak{tr}_\beta^\rho MA_\alpha$ is isomorphic to the Tarski–Lindenbaum quotient algebra of predicate MV logic (having a sequence of variables of order type $\alpha$) taken in a signature having $\beta$ — many relation symbols $\{R_i : i < \beta\}$, say, where the arity of $R_i$ is $\rho(i)$. Here the quotient is taken relative to validity $\models$ defined the usual way.
The following results in the classical case are known. Let $\alpha$ be an infinite ordinal and let $\beta$ be a non–zero cardinal. Then $\mathfrak{Fr}_\beta \mathbb{PA}_\alpha$ has IP, while $\mathfrak{Fr}_\beta \mathbb{QA}_\alpha$ does not have IP when $\beta \geq 4$ (Sayed Ahmed, 2011a) but it has the weaker form of interpolation as in the second part of the abstract with $n = 1$. Furthermore, if $\rho : \beta \to \wp(\alpha)$ is a dimension restricting function such that $\mathfrak{A} = \mathfrak{Fr}_\beta \mathbb{QA}_\alpha$ is dimension complemented, then $\mathfrak{A}$ has IP, cf. (Sayed Ahmed, 2010a, 2011a, 2007). In what follows all four results will be generalized to the ‘many valued’ case. We start by the last result.

3. Varying Interpolation Property for Dimension Restricted Free $\mathbb{MA}_\alpha$s

In this subsection we prove that certain (dimension restricted) free algebras have the $VIP$. The next theorem shows that when such algebras are dimension complemented, then they have VIP. The proof of our next theorem is the many valued version of the proof in (Sayed Ahmed, 2007) which has a lot of affinity to the proof in (Sayed Ahmed, 2004). Both are Henkin constructions. However, in the last two references interpolation was proved for countable languages. Here algebras considered may be uncountable.

This difficulty is conquered by dilating to regular cardinals, creating enough space for ‘witnesses of existential formulas.’

According to a wide spread custom (in algebraic logic), from now on we write $s^i_j$ for the substitution corresponding to the replacement $[i|j]$, namely, the operation $s_{[i|j]}$.

We first give the general idea which in fact covers all positive results on varying interpolation for free algebras hitherto obtained: For a start, we need the following algebraic counterpart of the notion of satisfiability, that makes the use of model theoretic jargon (whenever needed) feasible and helpful. Recall that $\mathfrak{A} \in \mathbb{MA}_\alpha$ is representable if for all $a \neq 0$, there exist a set $V$ that is a weak space of dimension $\alpha$, and a homomorphism from $\mathfrak{A}$ to $\mathfrak{B} = \mathfrak{Fr}(V, [0,1])$ such that $f(a) \neq 0$. We say that $(\mathfrak{B}, f)$ or simply $\mathfrak{B}$ is a many valued model of $(\mathfrak{A}, a)$. For further brevity, we might refer to $\mathfrak{B}$ simply as a model of $(\mathfrak{A}, a)$.

Idea of proof: Let $\beta$ be a non-zero cardinal. For the sake of brevity we denote $\mathfrak{Fr}_\beta \mathbb{MA}_\alpha$ by $\mathfrak{A}$. Let $X_1, X_2 \subseteq \beta$. We assume without loss that $\beta = X_1 \cup X_2$. Let $a \in \mathfrak{Fr}^\alpha X_1$ and $b \in \mathfrak{Fr}^\alpha X_2$ be such that $\mathfrak{A} \models a \leq b$. We must show that there exists an interpolant of $a$ and $b$, i.e. some $c \in \mathfrak{Fr}^\alpha(X_1 \cap X_2)$ and some $n \in \mathbb{N}$ such that $\mathfrak{A} \models a^n \leq c \leq nb$. Like the standard proof of the interpolation theorem for first order logic, the proof is by contradiction. We assume no such $c$ exists. Then we construct a model of $(\mathfrak{A}, a - b)$. This contradicts $\mathfrak{A} \models a \leq b$.

We divide the proof into three parts (1), (2) and (3). Using Hodges’ terminology (Hodges, 1997), in parts (1) and (2) we force with $\mathfrak{A}$, preparing for constructing the model. This is done by adding enough supply of witnesses, in preparation of defining a notion of forcing. Algebraically we neatly embed $\mathfrak{A}$ into “enough” spare dimensions, or expressed differently, we take a suitable (minimal) $\kappa$–dilation of $\mathfrak{A}$, where $\kappa$ is a regular cardinal $> \max\{|\alpha|, |\mathfrak{A}|\}$.

The neat embedding part is done in part (1). In part (2) we define a notion of forcing and construct two appropriate maximal filters, that are in some sense potential representations. In part (3) we construct the desired model from these maximal filters using a third maximal filter that agree with both maximal filters on the algebra $\mathfrak{Fr}^\alpha(X_1 \cap X_2)$.

In more detail:

1. Adding witnesses: We choose a cardinal $\kappa \geq \max\{|\alpha|, |\mathfrak{A}|\}$ as indicated. This cardinal $\kappa$ determines the supply of added individual constants (witnesses to
existential formulas). By theorem 15, there exists $\mathcal{B} \in \text{MV}_\kappa$ such that $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathcal{B}$, $\mathfrak{A}$ generates $\mathcal{B}$, and the following holds (*):

$$(\forall X \subseteq \mathfrak{A})(\mathcal{G}^\mathfrak{A} X = \text{Nr}_\alpha \mathcal{G}^\mathfrak{B} X).$$

We assume that there exists no interpolant in $\mathcal{G}^\mathfrak{A} (X_1 \cap X_2)$. Then by (*) we will be able to show that there exists no interpolant even in the bigger algebra $\mathcal{G}^\mathfrak{B} (X_1 \cap X_2)$.

(2) **Defining a notion of forcing:** Using the $\kappa$–many spare dimensions (witnesses), we extend $a$ and $-b$ to maximal filters $F_1$ and $F_2$, respectively of the bigger algebra $\mathcal{B}$. In (model–theoretic) forcing terminology, these two maximal filters are called generic. The maximal filters $F_1$ and $F_2$ are constructed in a step-by-step fashion to agree on $\mathcal{G}^\mathfrak{B} (X_1 \cap X_2)$ with (*) providing the base of this construction. That is

$$(\forall x \in \mathcal{G}^\mathfrak{B} (X_1 \cap X_2))(x \in F_1 \iff x \in F_2).$$

We call $F_1$ and $F_2$ Henkin filters emphasizing the analogy with Henkin constructions. The role of $F_1$ and $F_2$ is to ‘eliminate quantifiers’ using the spare dimensions added, in a sense to be made precise in the more detailed proof. Though conceptually easy to grasp, this part of the proof is the most involved part computationally.

(3) **Forming the desired model of $\langle \mathfrak{A}, a - b \rangle$:**

In this last part of the proof we build the desired model of $\langle \mathfrak{A}, a - b \rangle$. Let $F$ be a maximal filter of $\mathfrak{A}$. We say that $\langle \mathcal{B}, f \rangle$ is a model of $\langle \mathfrak{A}, d \rangle$ corresponding to $F$, if $\mathcal{B}$ is a model of $\langle \mathfrak{A}, d \rangle$, $\mathcal{B} = \mathcal{G}(V, \mathfrak{A}/F)$ (note that $\mathfrak{A}/F \cong [0,1]$), where $V$ is a weak space, and $f : \mathfrak{A} \rightarrow \mathcal{G}(V, \mathfrak{A}/F)$ is defined via $f(b)(x) = s^\mathfrak{B} b/F$ with $x \in V$, and $s_x : \mathfrak{A} \rightarrow \mathfrak{A}$ the substitution operation defined on $\mathfrak{A}$ corresponding the transformation $x \in V$.

We assume without loss of generality that $X_1 \cup X_2 = \beta$. We form a model of $(\mathcal{G}^\mathfrak{A} X_1, a)$ corresponding to $F_1$ and another of $(\mathcal{G}^\mathfrak{A} X_2, -b)$ corresponding to $F_2$. Here quantifier elimination plays a key role to show that such models can be built using the extra $\kappa$ extra dimensions. This consists of forming the so–called canonical models of the ‘rich complete theories’ corresponding to $F_1$ and $F_2$.

We construct a third maximal filter $H^*$ of $\mathcal{G}^\mathfrak{B} (X_1 \cap X_2)$ such that

$$F_1 \cap \mathcal{G}^\mathfrak{B} (X_1 \cap X_2) \cap H^* = F_2 \cap \mathcal{G}^\mathfrak{B} (X_1 \cap X_2).$$

Using the freeness of $\mathfrak{A}$, we will be able to ‘paste’ the aforementioned two models, constructing, using the hitherto obtained maximal filter $H^*$, the required model of $(\mathcal{G}^\mathfrak{A} (X_1 \cup X_2), a - b) = (\mathfrak{A}, a - b)$. This last model is the model corresponding to $H^*$ (in the above sense). But this contradicts the hypothesis that $a \leq b$ and we are done.

**Henceforth, all positive interpolation theorems proved for (various notions of) free algebras follow exactly the same pattern in the above outline which is basically a Henkin construction extrapolated to the many valued context, cf. theorems 63 and 67.**

Now we give the details. The naming of the three items to which the next proof is divided reflect the algebraic counterpart of the three items in the above outline that were formulated instead using metalogical jargon:

**Theorem 19** Let $\alpha$ be an infinite ordinal. Let $\beta$ be a non–zero cardinal. Let $\rho : \beta \rightarrow \varphi(\alpha)$ such that $\mathfrak{A}_{\beta}^\alpha \text{MA}_\alpha \in \text{DC}_\alpha$. Then $\mathfrak{A}_{\beta}^\alpha \text{MA}_\alpha$ has the VIP. In particular, if
Then there exists a finite \( \Gamma \subseteq C \kappa_1 \) such that for all \( \rho \in C \), \( v \in H \), we have

\[
|\rho(i)| < \omega \quad \text{for all } i \in \beta, \text{ then } \mathfrak{t}_\beta^\kappa \text{MA}_\alpha \text{ has the VIP.}
\]

**Proof** (1) **Forming a (suitable) dilation of the free algebra:** Let \( \mathcal{B} = \mathfrak{t}_\beta^\kappa \text{MA}_\alpha \). Let \( a \in \mathcal{G}^\mathcal{B} X_1 \) and \( b \in \mathcal{G}^\mathcal{B} X_2 \) be such that \( a \leq b \). Notice that this is the lattice order, and that \( a \leq b \iff a \cap -b = 0 \). We want to find an interpolant in \( \mathcal{G}^\mathcal{B} (X_1 \cap X_2) \). Assume that \( \kappa \) is a regular cardinal \( \geq \max(|\mathcal{B}|, |\alpha|) \). By theorem 15 there exists a \( \kappa \)-dilation \( \mathcal{C} \in \text{MA}_\kappa \) of \( \mathcal{B} \) such that \( \mathcal{B} = \mathfrak{t}_{\kappa_1} \mathcal{C} \), and \( \mathcal{B} \) generates \( \mathcal{C} \). If an interpolant exists in the big algebra \( \mathcal{C} \), then an interpolant exists in the smaller one \( \mathcal{B} \). For assume there exists \( c \in \mathcal{G}^\mathcal{C} (X_1 \cap X_2) \) such that that \( a \leq c \leq b \). Then there exists a finite \( \Gamma \subseteq \kappa \setminus \alpha \) such that \( a \leq c_\Gamma c \leq b \). Now by theorem 15 we have

\[
c_\Gamma c \in \mathfrak{t}_{\kappa_1} \mathcal{G}^\mathcal{C} (X_1 \cap X_2) = \mathcal{G}^{\mathfrak{t}_{\kappa_1} \mathcal{C}} (X_1 \cap X_2) = \mathcal{G}^\mathcal{B} (X_1 \cap X_2).
\]

So assume that no interpolant exists in \( \mathcal{B} \), then no interpolant exists in \( \mathcal{C} \). We will reach a contradiction. Recall that \( \kappa \geq \max(|\mathcal{B}|, |\alpha|) \).

(2) **Constructing the two Henkin filters that eliminate quantifiers:** Arrange \( \kappa \times \mathcal{G}^\mathcal{C} (X_1) \) and \( \kappa \times \mathcal{G}^\mathcal{C} (X_2) \) into \( \kappa \)-termed sequences

\[
\langle(k_i, x_i) : i \in \kappa\rangle \quad \text{and} \quad \langle(l_i, y_i) : i \in \kappa\rangle \text{ respectively.}
\]

Since \( \kappa \) is regular, we can define by recursion \( \kappa \)-termed sequences

\[
\langle u_i : i \in \kappa\rangle \quad \text{and} \quad \langle v_i : i \in \kappa\rangle,
\]

such that for all \( i \in \kappa \) we have:

\[
u_i \in \kappa \setminus (\Delta a \cup \Delta b) \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}
\]

and

\[
u_i \in \kappa \setminus \Delta a \cup \Delta b \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}.
\]

For an MV algebra \( \mathcal{D} \) and \( Y \subseteq \mathcal{D} \), recall that \( \mathfrak{f}^\mathcal{D} Y \) denotes the MV filter generated by \( Y \) in \( \mathcal{C} \). Recall that \( \mathfrak{t}^\mathcal{D}_{\text{MV}} \mathcal{D} \) denotes the MV reduct of \( \mathcal{D} \) obtained by discarding the operations of cylindrifiers and substitutions. Now let

\[
Y_1 = \{a\} \cup \{-c_k, x_i + s_k^i x_i : i \in \kappa\},
Y_2 = \{-b\} \cup \{-c_l, y_i + s_l^i y_i : i \in \kappa\},
H_1 = \mathfrak{f}^\mathfrak{t}^\mathfrak{D}_{\text{MV}} \mathcal{G}^\mathcal{C} (X_1) Y_1, \quad H_2 = \mathfrak{f}^\mathfrak{t}^\mathfrak{D}_{\text{MV}} \mathcal{G}^\mathcal{C} (X_2) Y_2,
H = \mathfrak{f}^\mathfrak{t}^\mathfrak{D}_{\text{MV}} \mathcal{G}^\mathcal{C} (X_1 \cap X_2) [H_1 \cap \mathcal{G}^\mathcal{C} (X_1 \cap X_2) \cup (H_2 \cap \mathcal{G}^\mathcal{C} (X_1 \cap X_2)].
\]

We claim that \( H \) is a proper filter of \( \mathcal{G}^\mathcal{C} (X_1 \cap X_2) \). To prove this it is sufficient to consider any pair of finite, strictly increasing sequences of natural numbers

\[
\eta(0) < \eta(1) \cdots < \eta(n-1) < \omega \quad \text{and} \quad \xi(0) < \xi(1) < \cdots < \xi(m-1) < \omega,
\]

and to prove that the following condition holds:
(1) For any \( b_0, b_1 \in \mathcal{S}_g^\mathcal{E}(X_1 \cap X_2) \) such that
\[
a^l \odot \prod_{i<n} \left( -c_{k_{\eta(i)}} x_{\eta(i)} \oplus s_{u_{\eta(i)}} x_{\eta(i)} \right)^{l_i} \leq b_0
\]
and
\[
(-b)^k \odot \prod_{i<m} \left( -c_{l_{\xi(i)}} y_{\xi(i)} \oplus s_{u_{\xi(i)}} y_{\xi(i)} \right)^{k_i} \leq b_1,
\]
where for \( i < n \) and \( j < m, l_i, k_j \) as well as \( l \) and \( k \) are finite ordinals > 0, we have
\[
b_0 \odot b_1 \neq 0.
\]

In (1) we have four parameters \( k, l, m \) and \( n \). We fix \( k, l \in \mathbb{N} \) and proceed by induction on \( n + m \). Assume that \( n + m = 0 \). Then (1) simply expresses the fact that no interpolant of \( a \) and \( b \) exists in \( \mathcal{S}_g^\mathcal{E}(X_1 \cap X_2) \). In more detail: if \( n + m = 0 \), then \( a^l \leq b_0 \) and \( [-b]^k \leq b_1 \). So if \( b_0 \odot b_1 = 0 \), we get \( a^l \leq b_0 \leq a^l \leq -b \leq -b^k = kb \) and \( b_0 \) would be the desired interpolant.

Now assume that \( n + m > 0 \) and for the time being suppose that \( \eta(n-1) > \xi(m-1) \). Apply \( c_{u_{\eta(n-1)}} \) to both sides of the first inclusion of (1). By \( u_{\eta(n-1)} \notin \Delta a \)
\( \iff \ c_{u_{\eta(n-1)}} a = a \), and hence \( c_{u_{\eta(n-1)}} (a^l) = \left[ c_{u_{\eta(n-1)}} a \right]^l = a^l \), upon noting that \( c_i(x \circ y) = c_i x \circ c_i y \), we get (2)
\[
a^l \odot c_{u_{\eta(n-1)}} \prod_{i<n} \left( -c_{k_{\eta(i)}} x_{\eta(i)} \oplus s_{u_{\eta(i)}} x_{\eta(i)} \right)^{l_i} \leq c_{u_{\eta(n-1)}} b_0,
\]

hence,
\[
a^l \odot \prod_{i<n} c_{u_{\eta(n-1)}} \left( -c_{k_{\eta(i)}} x_{\eta(i)} \oplus s_{u_{\eta(i)}} x_{\eta(i)} \right)^{l_i} \leq c_{u_{\eta(n-1)}} b_0.
\]

Now apply \( q_{u_{\eta(n-1)}} \) to the second inclusion of (1). By observing that \( u_{\eta(n-1)} \notin \Delta b = \Delta (-b) \), and \( q_{u_{\eta(n-1)}} \left( [-b]^k \right) = \left[ q_{u_{\eta(n-1)}} (-b) \right]^k = (-b)^k \), we get, by the same token, (3)
\[
(-b)^k \odot \prod_{j<m} q_{u_{\eta(n-1)}} \left( -c_{l_{\xi(j)}} y_{\xi(j)} \oplus s_{u_{\xi(j)}} y_{\xi(j)} \right)^{k_i} \leq q_{u_{\eta(n-1)}} b_1.
\]

Before going on, we formulate (and prove) a claim that will enable us to eliminate the quantifier \( c_{u_{\eta(n-1)}} \) (and its dual) from (2) (and (3)) above.

**Claim 1.** For \( j < m \), let \( t_j = -c_{l_{\xi(j)}} y_{\xi(j)} \oplus s_{u_{\xi(j)}} y_{\xi(j)} \). Then
\[
q_{u_{\eta(n-1)}} t_j = t_j \text{ for all } j < m.
\]

**Proof of Claim 1.** Let \( j < m \). Then we have
\[
q_{u_{\eta(n-1)}} \left( -c_{l_{\xi(j)}} y_{\xi(j)} \right) = -c_{l_{\xi(j)}} y_{\xi(j)}
\]
\[
c_{u_{\eta(n-1)}} \left( s_{u_{\xi(j)}} y_{\xi(j)} \right) = s_{u_{\xi(j)}} y_{\xi(j)}.
\]
Indeed, computing we get
\[ q_{u_{n(n-1)}}(-c_{\xi(j)}y_{\xi(j)}) = -c_{u_{2n-1}} - (-c_{\xi(j)}y_{\xi(j)}) \]
\[ = -c_{u_{n(n-1)}}c_{\xi(j)}y_{\xi(j)} \]
\[ = -c_{\xi(j)}y_{\xi(j)}. \]

Similarly, we have
\[ q_{u_{n(n-1)}}(s_{\xi(j)}y_{\xi(j)}) = -c_{u_{n(n-1)}} - (s_{\xi(j)}y_{\xi(j)}) \]
\[ = -c_{u_{n(n-1)}}(s_{\xi(j)} - y_{\xi(j)}) \]
\[ = -s_{\xi(j)} - y_{\xi(j)} \]
\[ = s_{\xi(j)}y_{\xi(j)}. \]

By \( q_i(q_x \oplus y) = q_i x \oplus q_i y \) we get from the above that
\[ q_{u_{n(n-1)}}(t_j) = q_{u_{n(n-1)}}(-c_{\xi(j)}y_{\xi(j)} \oplus s_{\xi(j)}y_{\xi(j)}) \]
\[ = q_{u_{n(n-1)}} - c_{\xi(j)}y_{\xi(j)} \oplus q_{u_{n(n-1)}}s_{\xi(j)}y_{\xi(j)} \]
\[ = -c_{\xi(j)}y_{\xi(j)} \oplus s_{\xi(j)}y_{\xi(j)} = t_j. \]

\[ \Box \]

**Claim 2.** For each \( i < n \), let
\[ z_i = -c_{k_{\eta(i)}}x_{\eta(i)} \oplus s_{u_{n(i)}}x_{\eta(i)}. \]

Then
\[ c_{u_{n(n-1)}}z_i = z_i \text{ for } i < n - 1 \text{ and } c_{u_{n(n-1)}}z_{n-1} = 1. \]

**Proof of Claim 2.** Let \( i < n - 1 \). Then by the choice of witnesses we have
\[ u_{n(n-1)} \neq u_{\eta(i)}. \]

Also it is easy to see that for all \( i, j \in \alpha \) we have
\[ \Delta c_{\eta(i)}x \subseteq \Delta x \text{ and that } \Delta s_{\eta(i)}x \subseteq \Delta x \setminus \{i\} \cup \{j\}. \]

In particular,
\[ u_{\eta(n-1)} \notin \Delta c_{k_{\eta(i)}}x_{\eta(i)} \text{ and } u_{\eta(n-1)} \notin \Delta s_{u_{n(i)}}x_{\eta(i)}. \]

It thus follows that
\[ c_{u_{n(n-1)}}(-c_{k_{\eta(i)}}x_{\eta(i)}) = -c_{k_{\eta(i)}}x_{\eta(i)} \text{ and } c_{u_{n(n-1)}}(s_{u_{n(i)}}x_{\eta(i)}) = s_{u_{n(i)}}x_{\eta(i)}. \]
Finally, by properties of $c_{u_{i}(n-1)}$, we get

$$c_{u_{i}(n-1)} z_i = z_i \text{ for } i < n - 1.$$ 

**Proof of $c_{u_{i}(n-1)} z_{n-1} = 1$.**

Computing we get, by $u_{i}(n-1) \not\in \Delta x_{(n-1)}$, and by familiar axioms of substitutions, the following:

$$c_{u_{i}(n-1)} \left( -c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus s_{u_{i}(n-1)} x_{\eta_{i}(n-1)} \right)$$

$$= c_{u_{i}(n-1)} - c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus c_{u_{i}(n-1)} s_{u_{i}(n-1)} x_{\eta_{i}(n-1)}$$

$$= -c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus c_{u_{i}(n-1)} s_{u_{i}(n-1)} x_{\eta_{i}(n-1)}$$

$$= -c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus c_{u_{i}(n-1)} s_{\eta_{i}(n-1) n} x_{\eta_{i}(n-1)}$$

$$= -c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus c_{u_{i}(n-1)} x_{\eta_{i}(n-1)}$$

$$= -c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} \oplus c_{k_{i}(n-1)} x_{\eta_{i}(n-1)} = 1.$$ 

By the above proven claims, and using the notation introduced in claim 2, we have

$$c_{u_{i}(n-1)} \left( \prod_{i < n} z_i^l \right)$$

$$= c_{u_{i}(n-1)} \prod_{i < n} \left( z_i^l \circ z_{l_{n-1}^n} \right)$$

$$= c_{u_{i}(n-1)} \prod_{i < n} z_i^l \circ (c_{u_{i}(n-1)} z_{l_{n-1}^n})$$

$$= c_{u_{i}(n-1)} \prod_{i < n} z_i^l \circ (c_{u_{i}(n-1)} z_{l_{n-1}^n})$$

$$= \prod_{i < n} c_{u_{i}(n-1)} z_i^l \circ (c_{u_{i}(n-1)} z_{l_{n-1}^n})$$

$$= \prod_{i < n} z_i^l.$$ 

Combined with (2) we obtain

$$a^l \circ \prod_{i < n-1} \left( -c_{k_{i}(n)} x_{\eta_{i}(i)} \oplus s_{u_{i}(n)} x_{\eta_{i}(i)} \right) \leq c_{u_{i}(n-1)} b_{i}.$$ 

On the other hand, from our proven claims and (3), it follows that

$$(-b)^k \circ \prod_{j < m} \left( -c_{l_{j}(i)} y_{\xi_{j}(j)} \oplus s_{u_{i}(n)} y_{\xi_{j}(j)} \right) \leq q_{u_{i}(n-1)} b_{1}.$$
Now making use of the induction hypothesis, we get
\[ c_{u_{\eta(n-1)}} b_0 \odot q_{u_{\eta(n-1)}} b_1 \neq 0; \]
and hence that
\[ b_0 \odot q_{u_{\eta(n-1)}} b_1 \neq 0. \]
From
\[ b_0 \odot q_{u_{\eta(n-1)}} b_1 \leq b_0 \odot b_1, \]
we reach the desired conclusion, i.e. that
\[ b_0 \odot b_1 \neq 0. \]
The other case, when \( \eta(n-1) \leq \xi(m-1) \) can be treated analogously and is therefore left to the reader. We have proved that \( H \) is a proper filter.

Proving that \( H \) is a proper filter of \( SgC(X_1 \cap X_2) \), let \( H^* \) be a maximal filter of \( SgC(X_1 \cap X_2) \) containing \( H \). We obtain by the congruence extension property, lemma 5, filters \( F_1 \) and \( F_2 \) of \( SgC X_1 \) and \( SgC X_2 \), respectively, such that
\[ H^* \subseteq F_1, \quad H^* \subseteq F_2 \]
and (**)
\[ F_1 \cap SgC(X_1 \cap X_2) = H^* = F_2 \cap SgC(X_1 \cap X_2). \]
We can assume that each of \( F_1 \) and \( F_2 \) are maximal. If \( F_1 \) say is not maximal, then invoking Zorn’s lemma we can extend it to \( G \) which is maximal and we would still have \( G \cap SgA(X_1 \cap X_2) = H^* \). For if not, then \( G \cap SgA(X_1 \cap X_2) = SgA(X_1 \cap X_2) \) by maximality of \( H^* \) and so \( 0 \in G \) which is impossible because \( G \) is proper.

Now for all \( x \in SgC(X_1 \cap X_2) \) we have
\[ x \in F_1 \iff x \in F_2. \]
Also from how we defined our maximal filters, \( F_i \) for \( i \in \{1, 2\} \) satisfy the following condition which is precisely the elimination of quantifiers condition:

(*) For all \( k < \kappa \), for all \( x \in SgC X \) if \( c_k x \in F_i \) then \( s_k^l x \) is in \( F_i \) for some \( l \notin \Delta x \).

(3) Constructing and amalgamating two representations getting the required representation (model of \( a - b \)) finishing the proof: For the sake of brevity we let \( V \) denote the set of all finite transformations on \( \kappa \), i.e
\[ V = FT_\kappa = \{ \sigma \in \kappa^\kappa : |\{ \sigma(i) \neq i \}| < \omega \}. \]
Every \( \tau \in V \) defines a unary operation \( s_\tau \) on \( C \) satisfying the equational axiomatization of \( MA_\kappa \). Let \( D_i = SgC X_i \), \( i = 1, 2 \). Then let \( \psi_i \) be the map defined as follows:
\[ \psi_i : D_i \to \mathfrak{F}(V, D_i / F_i) \]
\[ \psi_1(a)(x) = s_x^i a / F_i. \]

For brevity, we omit the superscript \( i \), so that
\[ \psi : \mathcal{D} \rightarrow \mathcal{G}(V, \mathcal{D} / F). \]

We have \( \mathcal{D} / F \) is a simple algebra, hence it is isomorphic to a subalgebra of \([0, 1]\). So we consider \( \psi \) as a mapping from \( \mathcal{D} \) into \( \mathcal{G}(V, [0, 1]) \). We check that it is a homomorphism. We only check cylindrifiers (the other operations are straightforward to check.) Now, we have
\[ \psi(c_k a)(x) = s_x c_k a / F. \]

Let
\[ l \in \{ \mu \in \kappa : x^{-1}\{\mu\} = \{\mu\} \} \setminus \Delta a. \]

Such an \( l \) clearly exists. Let
\[ \tau = x \circ [k, l]. \]

Then by familiar substitution rules we have
\[ c_l s_\tau a = s_\tau c_k a = s_x c_k a, \]
and by the choice of \( F \), we have
\[ c_l s_\tau a \in F \iff s_x^l s_\tau a \in F. \]

We use the following helpful notation. For a function \( f \), the function \( g = f(a \rightarrow u) \) is defined by \( g(x) = f(x) \) for \( x \neq a \) and \( g(a) = u \). Now we have
\[ \psi(c_k a)(x) = s_x c_k a / F \]
\[ = c_l s_\tau a / F \]
\[ = s_x^l s_\tau a / F \]
\[ = s_x(k \rightarrow u) a / F \]
\[ = \psi a(x(k \rightarrow u))/F \]
\[ \leq c_k \psi(a)(x) \]

Conversely if \( y \equiv_k x \), then
\[ \psi(a)(y) = s_y a / F \leq s_x c_k a / F = \psi(c_k a)(x). \]

We have proved that \( \psi \) is a homomorphism. Then putting back superscripts, we show that \( \psi_1 \) and \( \psi_2 \) agree on their common part \( \mathcal{G}g^\mathcal{C}(X_1 \cap X_2) \).
Let $a \in \mathcal{S} \mathcal{g} \mathcal{C}(X_1 \cap X_2)$. Then

$$
\psi_1(a)(x) = s_xa/F_1
= s_xa/F_1 \cap \mathcal{S} \mathcal{g} \mathcal{C}(X_1 \cap X_2)
= s_xa/H^*
= s_xa/F_2 \cap \mathcal{S} \mathcal{g} \mathcal{C}(X_1 \cap X_2)
= s_xa/F_2
= \psi_2(a)(x).
$$

Assuming that $X_1 \cup X_2$ generates $\mathcal{C}$, we have $\psi_1 \cup \psi_2$ defines a function on $\mathcal{C}$ into $\mathfrak{F}(V,[0,1])$, since they agree on the common part; by freeness they can be pasted to give $\psi : \mathcal{C} \rightarrow \mathfrak{F}(V,[0,1])$ such that $\psi(a \odot -b) \neq 0$, because the identity substitution is in $\psi(a \odot -b)$ by definition, but this contradicts that $a \leq b$ and the proof is complete.

□

Remark 1 As indicated above, in the many valued (MV) case the interpolant depends on the implication $\phi \rightarrow \psi$, say. There exist $n \in \mathbb{N}$ and $\beta$ in the common language such that $\phi^n \rightarrow \beta$ and $\beta \rightarrow n \psi$. Call the least such $n$, the index of $\phi \rightarrow \psi$. So here the index varies with every implication; hence the usage of the term varying interpolation property. Another inequality may have a different index. It is also likely (and indeed plausible), that we cannot control the index, in the sense that for any $m > 1$, there exists an inequality as above with index $m$. (The latter statement is not proved in the paper.)

Worthy of note is that a similar interpolation property in the classical case for certain infinitary logics was proved in (Sayed Ahmed, 2012b) which has an MV analogue stated in the second part of the abstract of this paper and proved in theorem 35 encountered below. In the classical (one valued) case for the implication $\phi \rightarrow \psi$ in this possibly infinitary logic, the interpolant is found with respect to a different implication obtained by quantifying each formula in the original implication; where one applies to $\phi$ finitely many universal quantifiers, and to $\psi$ one applies the same number of existential quantifiers (here quantification is allowed only on finitely many variables). In more detail, if $\phi \rightarrow \psi$, then there exists an $n \in \mathbb{N}$ and an interpolant $\beta$ in the common language, such that $\forall x_1, \ldots, x_n \phi \Rightarrow \beta$ and $\beta \Rightarrow \exists x_1, \ldots, x_n \psi$. This interpolation theorem is mentioned explicitly in the abstract of the aforementioned paper (Sayed Ahmed, 2012b). By the same token, this too is a varying interpolation theorem, but this term was not used in op.cit. In particular, the term ‘interpolation’ in (Sayed Ahmed, 2012b) is not used in the standard way, too; though referred to simply as ‘interpolation’. Furthermore, in this context it can be proved that there is no upper bound on the finite number of quantified variables: For all $m > 0$, there is a $k > m$ and an implication $\phi \rightarrow \psi$, such that $\forall x_1, \ldots, x_k \phi \Rightarrow \beta$ and $\beta \Rightarrow \exists x_1, \ldots, x_k \psi$ and any number of quantified variables $< k$ does not work here.

This equally applies to the VIP proved in theorem 35. In this case things are even more complex. This situation is a synthesis of the two aforementioned varying interpolation theorems because both the number of used quantifiers and the index are not uniform.

This prompts:
Definition 20 A logic \( \mathbf{L} \) for which the connective \( \rightarrow \) is definable, has the Varying Craig interpolation property if for all \( \theta = \phi \rightarrow \psi \), with \( \phi_1 \in L_1 \) and \( \phi_2 \in L_2 \), where \( L_1 \) and \( L_2 \) are signatures in \( \mathbf{L} \) there exist unary functions \( f_\theta, g_\theta \) defined on the set of formulas such that \( f_\theta(\phi) \rightarrow \phi \) and \( \psi \rightarrow g_\theta(\psi) \), and there exists \( \beta \) in the common language \( L_1 \cap L_2 \), such that \( f_\theta(\phi) \rightarrow \beta \) and \( \beta \rightarrow g_\theta(\psi) \).

It is legitimate to call this property an interpolation property (though varying). From the algebraic point of view, the usual interpolation property corresponds to amalgamation properties in corresponding classes of algebras (Madárasz & Sayed Ahmed, 2012; Andréka, Németi, & Sain, 2000). In the many \( \mathcal{M} \) context amalgamation properties can be defined; in fact it can be defined for any class of algebras. As will be shown below it is precisely the local varying interpolation property on free algebras that corresponds, using so–called bridge theorems in algebraic logic, to the global property of amalgamation in classes of \( \mathcal{M} \) algebras (Blok & Pigozzi, 1989).

Corollary 21 Every algebra \( \mathfrak{A} \in \text{Dc}_\alpha \) is representable.

Proof Let \( \mathfrak{A} \) be given and \( \alpha \neq 0 \) be in \( A \). Let \( \kappa \) be a regular cardinal \( \geq \max\{\{\alpha\}, |A|\} \). Let \( \mathfrak{B} \in \text{MA}_\kappa \) be such that \( \mathfrak{A} = \text{Fr}_\alpha \mathfrak{B} \). Let \( \langle (k_i, x_i) : i \in \kappa \rangle \) be an enumeration of \( \kappa \times B \). Since \( \kappa \) is regular, we can define by recursion a \( \kappa \)–termed sequence \( \langle u_i : i \in \kappa \rangle \) such that for all \( i \in \kappa \) we have: \( u_i \in \kappa \backslash (\Delta a \cup \bigcup_{j < i} \Delta x_j \cup \{ u_j : j < i \}) \). Let \( Y = \{ a \} \cup \{-c_{k_i} x_i \oplus \delta_{u_i} x_i : i \in \kappa \} \). Let \( H \) be the filter generated by \( Y \); then \( H \) is proper, take the maximal filter containing \( H \) and \( a \), and define \( \psi(b)x = s_x b/F \) where \( b \in B \) and \( x \in V \), and \( V \) is as defined in the previous proof. Then \( \psi(a) \neq 0 \), and \( \psi \) establishes the representability of \( \mathfrak{B} \), hence of \( \mathfrak{A} \). \( \square \)

It can be shown that for \( \alpha < \beta \), if \( \mathfrak{A} \in \text{Dc}_\alpha \), \( \mathfrak{A} \subseteq \text{Fr}_\alpha \mathfrak{B} \) with \( \mathfrak{B} \in \text{MA}_\beta \) and \( A \) (as a set) generates the algebra \( \mathfrak{B} \), then \( \mathfrak{B} \in \text{Dc}_\beta \) and \( \mathfrak{B} \) is unique up to isomorphisms that fix \( A \) pointwise. That is, if \( \mathfrak{B}' \in \text{MA}_\beta \), \( \mathfrak{A} \subseteq \text{Fr}_\alpha \mathfrak{B}' \), and \( A \) generates \( \mathfrak{B}' \), then there exists an isomorphism \( f : \mathfrak{B} \to \mathfrak{B}' \) such that \( f(a) = a \) for all \( a \in A \). In particular, \( \mathfrak{B}' \in \text{Dc}_\beta \). This property is referred to as the UNEP in (Sayed Ahmed, 2012a, Definition 5.2.1, Theorem 5.2.4) for cylindric algebras, short for Unique Neat Embedding Property, witness definition 43. Using this observation, one can show that in our previous proof the \( \kappa \)–dilation \( \mathfrak{C} \in \text{Dc}_\alpha \) for which \( \text{Fr}_\alpha \mathfrak{C} = \text{Fr}_\alpha \mathfrak{C} \) that was used to construct the maximal MV filters \( F_1 \) and \( F_2 \), is actually isomorphic to \( \text{Fr}_\beta \mathfrak{A}_\eta \). In particular, the sequence \( x = (\eta/\text{Fr}_\eta \mathfrak{A}_\eta : \eta < \beta) \) \( \mathfrak{A}_\eta \)-freely generates \( \text{Fr}_\alpha \text{Fr}_\beta \mathfrak{A}_\eta \) under the dimension restricting function \( \rho \). We will
encounter the UNEP in the proof of theorem 45 below and in theorem 60 it will be given a categorical formulation.

3.1 Interpolation for Pavelka Predicate Calculus

The Pavelka propositional calculus was introduced to incorporate in the syntax truth constants $\vec{r}$ for any $r \in [0, 1]$. The language then becomes uncountable, but Hajek simplified this by eliminating from the syntax the irrational truth values. The predicate calculus for Pavelka logic was studied by Novak, and algebraically by Dragulici and Georgescu. We pursue the latter approach, but in a more general setting. Let us denote the $MV$ algebra $[0, 1] \cap \mathbb{Q}$ by $L$. The following two definitions and lemma are taken from (Dragulici & Georgescu, 2001).

Definition 22 A Pavelka algebra is a structure $(\mathfrak{A}, \{\vec{r} : r \in L\})$ where $\mathfrak{A}$ is an $MV$ algebra and $\{\vec{r} : r \in L\}$ is a subset of $A$ such that $0 = 0$, $\vec{r} \oplus \vec{s} = \vec{r} \oplus \vec{s}$ and $\neg \vec{r} = \neg \vec{r}$, for all $r, s \in L$.

Lemma 23 Let $\mathfrak{B} = (\mathfrak{A}, \{\vec{r} : r \in L\})$ be a Pavelka algebra. Assume that $P$ is a proper filter of $\mathfrak{B}$ and that $r, s \in L$. Then $\vec{r} \in P \iff r = 1$ and $\vec{r}/P \leq \vec{s}/P \iff r \leq s$.

Definition 24 An existential quantifier on a Pavelka algebra $(\mathfrak{A}, \{\vec{r} : r \in L\})$ is an existential quantifier on $\mathfrak{A}$ such that $\exists r = r$ for every $r \in L$.

In our signature, we consider only substitutions indexed by replacements. This is not a restriction, when algebras considered are dimension complemented, because all finite substitutions are definable from those as will be shown in a moment.

Definition 25 Let $\alpha$ be an ordinal. An $MV$ substitution Pavelka algebra of dimension $\alpha$, an $\mathfrak{SA}_\alpha$, for short, is an algebra of the form

$$\mathfrak{B} = (\mathfrak{A}, \{\vec{r} : r \in L\}, c_i, s^j_i)_{i, j < \alpha}$$

where $(\mathfrak{A}, \{\vec{r} : r \in L\})$ is a Pavelka algebra and $c_i, s^j_i$ are unary operations on $\mathfrak{A}$ $(i, j < \alpha)$ satisfying the following equations for all $i, j, k, l < n$:

1. $c_i0 = 0$, $x \leq c_i x$, $c_i(x \odot c_i y) = c_i x \odot c_i y$, $c_i(x \odot x) = c_i x \odot c_i x$, $c_i c_i x = c_i c_i x$, and $c_i(x \rightarrow y) \leq c_i x \rightarrow c_i y$, where $a \rightarrow b$ is $\neg a \oplus b$.
2. $c_i \vec{r} = \vec{r}$.
3. $s^j_i x = x$.
4. $s^j_i$ is an $MV$ endomorphism,
5. $s^j_i c_i x = c_i x$.
6. $c_i s^j_i x = s^j_i x$ whenever $i \neq j$.
7. $s^j_i c_k x = c_k s^j_i x$, whenever $k \notin \{i, j\}$,
8. $c_i s^j_i x = c_i s^j_i x$.
9. $s^j_i c_k x = s^j_i s^j_i x$, whenever $\{i, j, k\} = 4$.
10. $s^j_i s^j_i x = s^j_i s^j_i x$.

We consider the case when algebras are dimension complemented: $\mathfrak{A}$ is such if $\alpha \setminus \Delta \alpha$ is infinite for every $x \in \mathfrak{A}$. Given such an $\mathfrak{A}$, we show that every $\tau \in FT_\alpha$ defines a Boolean endomorphism $s_\tau$ on $\mathfrak{A}$. First a piece of notation: If $\tau \in FT_\alpha$, let $\text{dom}(\tau) = \{i \in \alpha : \tau(i) \neq i\}$. Then we denote $\tau$ by $[u_0|v_0, u_1|v_1, \ldots, u_{k-1}|v_{k-1}]$, where $\text{dom}(\tau) = \{u_0, \ldots, u_{k-1}\}$ such that $u_0 < u_1 \ldots < u_{k-1}$ and $v_i = \tau(u_i)$ for
Given \( \tau = [u_0|v_0, u_1|v_1, \ldots, u_{k-1}|v_{k-1}] \), and \( x \in A \), let \( \pi_0, \ldots, \pi_{k-1} \) in this order be the first \( k \) ordinals in \( \alpha \setminus (\Delta x \cup \text{rng}(v) \cup \text{rng}(u)) \). Then set:

\[
S_\tau . x = s_{\pi_0} \ldots s_{\pi_{k-1}}^u s_{\pi_0} \ldots s_{\pi_{k-1}}^u . x.
\]

Then it can be proved exactly like the classical case (Henkin et al., 1971, Theorem 1.11.11) that \( S_\tau \) is a unary operations that satisfies all the axioms.

The algebra \( \mathfrak{A} = \mathfrak{F}^\# \mathfrak{S} \mathfrak{A}_\alpha \) is defined as before (definition 16), but we take into consideration the constants when forming subalgebras, and we study rather the pair \( (\mathfrak{A}, \{ \overline{r} : r \in L \}) \) which we with a slight abuse of notation continue to denote by \( \mathfrak{A} \). For \( H \subseteq A \) and \( a \in A \), set

\[
[a]_H = \bigvee \{ r \in L : \overline{r} \rightarrow a \in H \}.
\]

Now we prove interpolation. The idea is exactly like the idea used in the proof of theorem 19 but taking into consideration the new constants in \( L \). Let \( \mathfrak{A} = \mathfrak{F}^\# \mathfrak{S} \mathfrak{A}_\alpha \) be the given free dimension restricted free algebra on \( \beta \) generators and dimension restricting function \( \rho \). Let \( a \in \mathfrak{S} \mathfrak{g}^\mathfrak{A} X_1, b \in \mathfrak{S} \mathfrak{g}^\mathfrak{A} X_2 \) such that \( a \leq b \), then by theorem 15, we can neatly embed \( \mathfrak{A} \) into \( \mathfrak{N} \mathfrak{t}_\kappa \mathfrak{C} \), where \( \mathfrak{C} \in \mathfrak{S} \mathfrak{A}_\alpha \), \( \kappa \) a regular cardinal \( > max(|a|, |A|) \), so that \( \mathfrak{A} = \mathfrak{N} \mathfrak{t}_\kappa \mathfrak{C} \) and for \( X \subseteq A \), one has \( \mathfrak{S} \mathfrak{g}^\mathfrak{A} X = \mathfrak{N} \mathfrak{t}_\kappa \mathfrak{S} \mathfrak{g}^\mathfrak{C} X \). One argues like before. If an interpolant exists in \( \mathfrak{C} \), then an interpolant exists in \( \mathfrak{A} \). This can be used as the base of the induction to construct the proper filter \( H \) in \( \mathfrak{S} \mathfrak{g}^\mathfrak{A} (X_1 \cap X_2) \). From \( H \) one constructs maximal filters \( F_1 \) and \( F_2 \) in \( \mathfrak{S} \mathfrak{g}^\mathfrak{E} X_1 \) and \( \mathfrak{S} \mathfrak{g}^\mathfrak{E} X_2 \), respectively, such that \( F_1 \cap \mathfrak{S} \mathfrak{g}^\mathfrak{E} (X_1 \cap X_2) = H^* = F_2 \cap \mathfrak{S} \mathfrak{g}^\mathfrak{E} (X_1 \cap X_2) \), where \( H^* \) is a maximal filter of \( \mathfrak{S} \mathfrak{g}^\mathfrak{E} (X_1 \cap X_2) \) containing \( H \).

To handle the constants, one defines the homomorphisms differently as follows:

For any \( x \in \mathfrak{F} \mathfrak{T}_\alpha \), \( \psi_1(b)(x) = [s_x p]_{F_1} \) for \( b \in \mathfrak{S} \mathfrak{g}^\mathfrak{B} X_1 \) and \( \psi_2 \) is defined analogously.

Call a maximal filter Henkin, if whenever \( c_k x \in F \), then \( s_j^k x \in F \) for some \( j \notin \Delta x \). All of \( H^* \), \( F_1 \) and \( F_2 \) are Henkin. To complete the proof, we need to verify two things. Recall that \( \mathfrak{F} \mathfrak{T}_\alpha \) denotes the set of finite transformations on \( \alpha \):

**Theorem 26**

1. Let \( \mathfrak{C} \in \mathfrak{S} \mathfrak{A}_\alpha \). Let \( P \) be a Henkin maximal filter. Then the map \( \psi : \mathfrak{C} \rightarrow \mathfrak{F}(\mathfrak{F} \mathfrak{T}_\alpha, [0, 1]) \) defined by \( \psi(p)(x) = [s_x p]_P \) is a homomorphism.
2. Let \( \mathfrak{A}, \mathfrak{B}, F_1, F_2 \) be as in the above (sketch of) proof. If \( a \in \mathfrak{S} \mathfrak{g}^\mathfrak{B} (X_1 \cap X_2) \), then \( [a]_{F_1} = [a]_{F_2} \). This means that

\[
\Psi_1 | \mathfrak{S} \mathfrak{g}^\mathfrak{B} (X_1 \cap X_2) = \Psi_2 | \mathfrak{S} \mathfrak{g}^\mathfrak{B} (X_1 \cap X_2).
\]

**Proof.** We closely follow (Dragulici & Georgescu, 2001).

1. For every \( a, b \in C \) the following equalities hold:
   1. \( [a]_P = \bigwedge \{ r \in L : a \rightarrow \overline{r} \in P \} \)
   2. \( [a \circ b]_P = [a]_P \oplus [b]_P, \ [a \odot b]_P = [a]_P \odot [b]_P \) and \( [\neg a]_P = -[a]_P. \)

Then it can be easily checked that

\[
\psi(p \odot q)(x) = \psi(p) \odot \psi(q)(x),
\]

\[
\psi(p \circ q)(x) = \psi(p) \circ \psi(q)(x),
\]

\[
\psi(\neg p)(x) = -\psi(p)(x).
\]
Also
\[ \psi(\bar{r})(x) = r. \]

We only check cylindrifiers. We have:
\[ \psi(c_i p)(x) = [s_x c_i p]_P, \]
\[ c_i \psi(p)(x) = \bigvee \{ \psi(p)(y) : y \equiv_i x \}. \]

We need to show that
\[ [s_x c_i p] = \bigvee \{ s_y(p) : y \equiv_i x \}. \]

Here \( y \equiv_i x \) means that \( x(j) = y(j) \) for all \( j \neq i \). We have \( p \leq c_i p \), so \( s_y p \leq s_x c_i p \), thus for every \( r \in L \), we have \( r \to s_y p \leq r \to s_x c_i p \). From this we get that for every \( r \in L \), \( r \to s_y p \in P \) implies \( r \to s_x c_i p \) and so \( [s_y p]_P \leq [s_x c_i p]_P \), hence \( \bigvee s_y p \leq s_x c_i p \).

Conversely, assume that for all \( y, y \equiv_i x \), we have \( s_y p < r < s_x c_i p \). Then \( \exists s > r \) such that \( s \to s_x c_i p \). But \( r \to s_x c_i p \geq s \to s_x c_i p \), it follows that
\[ r \to s_x c_i p \in P. \]

This implies that \( r \to s_x c_i p \in P \). This is a contradiction and we are done.

(2) We proceed as follows. Let \( a \in \mathcal{G} g^{\mathfrak{B}}(X_1 \cap X_2) \). Then
\[ a \to \bar{r} \in F_1 \]
\[ \iff a \to \bar{r} \in F_1 \cap \mathcal{G} g^{\mathfrak{B}}(X_1 \cap X_2) \]
\[ \iff a \to \bar{r} \in F_2 \cap \mathcal{G} g^{\mathfrak{B}}(X_1 \cap X_2) \]
\[ \iff a \to \bar{r} \in F_2 \]

Then we can deduce that \([a]_{F_1} = [a]_{F_2}\) and we are done. \( \square \)

**Corollary 27**

1. If \( \beta \) is a non-zero cardinal, and \( \rho : \beta \to \psi(\alpha) \) is such that \( \mathfrak{A}_\beta^{\mathfrak{B}} \mathfrak{A}_\alpha \) is dimension complemented, then \( \mathfrak{A} \) has the interpolation property.
2. Every dimension complemented algebra is representable.

### 4. Representability and Interpolation for \( \mathfrak{MA} \alpha \)s

In this section, we drop the condition of *dimension complementedness* in free algebras. We deal with \( \mathfrak{A}_\beta^{\mathfrak{B}} \mathfrak{MA}_\alpha \) (\( \delta \) a non-zero cardinal and \( \alpha \) an infinite ordinal) when \( \rho = \delta \times \{ \alpha \} \). More succinctly, we deal with the (usual) free algebra \( \mathfrak{F}_\delta \mathfrak{MA}_\alpha \). We will get a weaker interpolation result. Furthermore, we show that this is the best possible result. In our positive result we use theorem 19. In our negative result, marking the boundaries of our weaker interpolation theorem, we use the counterexample given in (Sayed Ahmed, 2011a) in the classical case for \( \mathfrak{QA}_\alpha \)s.
Recall that RMA$_{\alpha}$ denotes the class of representable MA$_{\alpha}$s. Before embarking on interpolation and amalgamation theorems, we characterize the class of representable MA$_{\alpha}$s (which turns out to be a variety), via neat embeddings proving a many valued version of Henkin’s celebrated neat embedding theorem for cylindric algebras (Henkin et al., 1985, Theorem 3.2.10). The latter theorem is an algebraic version of the completeness of first order logic. So our proven theorem (first item of theorem 28) will be an algebraic version of the completeness of predicate many valued logic. Also, we prove several representability results analogous to (Henkin et al., 1985, Theorem 3.2.11) which implies completeness of several infinitary extension of predicate many valued logic via Hilbert style axiomatizations. The last implication will not be explicitly proved, but it follows from fairly standard known so-called bridge theorems in abstract algebraic logic, see e.g (Sayed Ahmed, 2005; Sain, 2000). We remind the reader that ordinals considered, unless otherwise indicated, are still infinite.

4.1 Representability

For any pair of ordinals $\alpha < \beta$ and $K \subseteq$ MA$_{\beta}$, recall that NR$_{\alpha}$K denotes the class $\{\text{NR}_{\alpha}\mathcal{B} : \mathcal{B} \in K\}$. Let Lf$_{\alpha}$ denote the class of locally finite MA$_{\alpha}$s; $\mathfrak{A} \in$ Lf$_{\alpha} \iff |\Delta x| < \omega$ for all $x \in \mathfrak{A}$. Clearly Lf$_{\alpha} \subseteq$ DC$_{\alpha}$.

Let WSet$_{\alpha}$ = $\{\text{SP}^{\alpha}X(p), [0, 1]) : p \in \mathcal{A}X$, $X$ is a non-empty set $\}$ and let Set$_{\alpha} = \{\text{SP}^{\alpha}X([0, 1]) : p \in \mathcal{A}X$, $X$ is a non-empty set $\}$. (Here $\mathcal{A}X(p)$ is the weak space consisting of all sequences in $\mathcal{A}X$ that agree co-finitely with $p$.) We use the standard notation Pu for the operation of forming ultraproducts.

**Theorem 28** Let $\alpha$ be an ordinal $\geq \omega$. Then

1. RMA$_{\alpha} = \text{SNR}_{\alpha} \text{MA}_{\alpha + \omega}$.
2. For any pair of ordinals $\alpha < \beta$, SNR$_{\alpha}$MA$_{\beta}$ is a variety. In particular, RMA$_{\alpha}$ is a variety.
3. For any class $K$, such that Lf$_{\alpha} \subseteq K \subseteq$ RMA$_{\alpha}$, SP$_{\alpha}K = \text{RMA}_{\alpha}$. In particular, SP$_{\alpha}$DC$_{\alpha} = \text{RMA}_{\alpha}$.
4. RMA$_{\alpha} = \text{HSP}_{\text{Set}_{\alpha}}$.

**Proof** (1) To show that RMA$_{\alpha} \subseteq$ SNR$_{\alpha}$MA$_{\alpha + \omega}$, it suffices to consider algebras in WSet$_{\alpha}$, because by definition 13 RMA$_{\alpha} = \text{SPWSet}_{\alpha}$ and as we will show in a minute, the class SNR$_{\alpha}$MA$_{\alpha + \omega}$ is a variety, hence it is closed under SP. Let $\alpha < \beta$ be infinite ordinals, and let $\mathfrak{A} = \mathfrak{F}(\alpha U[1, 0], [0, 1]) \in$ WSet$_{\alpha}$. Let $\mathcal{B} = \mathfrak{F}(\beta U[1, 0], [0, 1])$, then the map $\Psi : \mathfrak{A} \rightarrow \mathcal{B}$ defined by

$$\Psi(f)(s) = f(s \upharpoonright \alpha),$$

is the required neat embedding.

The converse inclusion (which is basically an algebraic version of a completeness theorem) is slightly more difficult, but the difficult part is already done with in theorem 21. Assume that $\mathfrak{A} \subseteq \text{NR}_{\alpha}\mathcal{B}$. Let $\mathcal{B}' = \mathcal{S}g^{\mathcal{B}}A$. Then $\mathcal{B}' \in$ MA$_{\alpha + \omega}$ and furthermore $|\beta \setminus \Delta b| \geq \omega$ for all $b \in \mathcal{B}'$. Indeed recall that $\beta = \alpha + \omega$. Then $|\beta \setminus \alpha| \geq \omega$, and because $\Delta a \subseteq \alpha$ for all $a \in \mathcal{A}$, it follows by a simple inductive argument that for all $b \in \mathcal{S}g^{\mathcal{B}}A$, we have $|\beta \setminus \Delta b| \geq \omega$. Hence, by the representability theorem, corollary 21 $\mathcal{B}'$ is representable, and so is $\mathfrak{A}$, for $\text{NR}_{\alpha}\mathcal{B}'$ is representable and a subalgebra of a representable algebra is representable. The neat reduct $\text{NR}_{\alpha}\mathcal{B}'$ is representable because of the following reasoning. Given a non-zero $a \in \text{NR}_{\alpha}\mathcal{B}'$, let $f : \mathcal{B}' \rightarrow \mathfrak{F}(\beta X(p), [0, 1])$ be a homomorphism such that
Then clearly
\[ (s \in \alpha X^{p(\alpha)}). \]

(2) We now show that for infinite ordinals \( \alpha < \beta \), \( \text{SN}_{\alpha} \text{MA}_\beta \) is a variety. It is closed under forming subalgebras by definition. It is closed under products, since for any system of algebras \( \{ \mathfrak{B}_i : i \in I \} \) in \( \text{MA}_\beta \), we have \( \prod_{i \in I} \mathfrak{B}_i = \mathfrak{B}_0 \prod_{i \in I} \mathfrak{B}_i \), via the map \( (a_i : i \in I) \mapsto (a_i : i \in I) \) which is evidently injective, surjective and a homomorphism. We show that the \( \text{SN}_{\alpha} \text{MA}_\beta \) is also closed under homomorphic images. Let \( \mathfrak{A} \subseteq \mathfrak{B} \), and \( R \) be a congruence on \( \mathfrak{A} \). We will show that the homomorphic image \( \mathfrak{A}/R \) of \( \mathfrak{A} \) is in \( \text{SN}_{\alpha} \text{MA}_\beta \). Let \( T \) be the congruence generated by \( R \) in \( \mathfrak{B} \). That is \( T = \bigcap\{ S : S \in \text{Co}\mathfrak{B} : R \subseteq S \} \). Then \( R \subseteq T \subseteq \mathfrak{B} \times \mathfrak{B} \) and \( T \cap (\mathfrak{A} \times \mathfrak{A}) = R \). Define \( f : \mathfrak{A}/R \to \mathfrak{B} \) by \( f(x/R) = x/T \), then \( f \) is a well-defined embedding, and we are done.

(3) (Henkin et al., 1971, Theorem 2.6.52).

(4) One proves that \( \text{Set}_\alpha \subseteq \text{SN}_{\alpha} \text{MA}_{\alpha + \omega} = \text{RMA}_\alpha \) exactly like for weak set algebras. Let \( \mathfrak{A} = \mathfrak{B}^\alpha \{0, 1\} \in \text{Set}_\alpha \). Let \( \beta = \alpha + \omega \), and let \( \mathfrak{B} = \mathfrak{B}^\beta \{0, 1\} \). Then the map \( \Psi : \mathfrak{A} \to \mathfrak{B} \) defined by \( \Psi(f)(s) = f(s \upharpoonright \alpha) \) is the required neat embedding. Hence \( \text{HSPSet}_\alpha \subseteq \text{RMA}_\alpha \) because \( \text{HSPRMA}_\alpha = \text{RMA}_\alpha \). For the other inclusion it suffices to show that \( \text{Lf}_\alpha \subseteq \text{SPSet}_\alpha \), since then we get from the previous item that \( \text{RMA}_\alpha = \text{HSPSet}_\alpha \subseteq \text{HSPSet}_\alpha \). The proof is like the dimension complemented case, but here we use all substitution in \( \alpha \omega \) which are definable exactly like in the cylindric case (Henkin et al., 1971, Theorem 1.11.14), hence the unit obtained is ‘square’. In more detail, let \( \mathfrak{A} \in \text{Lf}_\alpha \), and let \( a \in A \) be non-zero. Then by theorem 15, we get that \( \mathfrak{A} = \mathfrak{B}^\alpha \mathfrak{B}', \mathfrak{B}' \in \text{De}_\kappa \) and \( \kappa \) a regular cardinal \( \geq \max\{|\alpha|, |A|\} \).

Our next two theorems address finite-schema axiomatizability. The variety \( \text{RMA}_\alpha \), and for that matter \( \text{MA}_\alpha \), are not finitely axiomatizable for the simple reason that their signature contains infinitely many operations, and any finite axiomatization will have to contain only finitely many. But we show that \( \text{MA}_\alpha \) can be axiomatized by a finite schema, in a sense to be made precise.

If \( \rho : \omega \to \alpha \) is an injection, then \( \rho \) extends recursively to a function \( \rho^+ \) from \( \text{MA}_\omega \) terms to \( \text{MA}_\alpha \) terms. On variables \( \rho^+(v_k) = v_k \), and for compound terms like \( \mathfrak{C}_k \tau \), where \( \tau \) is a \( \text{MA}_\omega \) term, and \( k < \omega \), one sets \( \rho^+(\mathfrak{C}_k \tau) = \mathfrak{C}_{\rho(k)} \rho^+(\tau) \). For an equation \( e \) of the form \( \sigma = \tau \) in the language of \( \text{MA}_\omega \), \( \rho^+(e) \) is the equation \( \rho^+(\sigma) = \rho^+(\tau) \) in the language of \( \text{MA}_\alpha \). This last equation, namely, \( \rho^+(e) \) is called an \( \alpha \) instance of \( e \) obtained by applying the injection \( \rho \). Now as easily distilled from the different equivalent axiomatization given next for \( \text{MA}_\alpha \) (in a different signature), there is a strictly finite set of equations \( E \) in the language of \( \text{MA}_\alpha \) such all the \( \alpha \) instances of equations in \( E \) axiomatize \( \text{MA}_\alpha \).

Can the class \( \text{RMA}_\alpha \) be axiomatized by a finite schema, as is the case with \( \text{MA}_\alpha \)? In our next theorem 30, we show that the answer is negative by bouncing it back to the classical case. In the theorem following 30 we give a sharper non-finite axiomatizability result for infinitely many varieties that approximate \( \text{RMA}_\alpha \). For a start we give a simpler (equivalent) axiomatization of \( \text{MA}_\alpha \), bringing it in the form of a system of varieties definable by a schema of equations (Henkin et al., 1985, Definition 5.6.11-12).
**Definition 29** A quasi-polyadic MV-algebra of dimension $\alpha$, briefly a $\text{QMA}_\alpha$, is an algebra

$$\mathfrak{A} = (A, \oplus, \odot, \neg, c_i, s_{[i,j]} : i, j < \alpha)$$

where the reduct obtained by dropping the operations $s_{[i,j]}$, $i < j < \alpha$, satisfies all of \((1)-(10)\) in definition 25 except for (2), and additionally it satisfies the following equations for all $i, j, k < \alpha$:

1. $s_{[i,j]}(x) = s_{[i,j]}(x) = x$, and $s_{[i,j]} = s_{[j,i]}$,
2. $s_{[i,j]}s_{[i,k]} = s_{[j,k]}s_{[i,j]}$ if $|\{i,j,k\}| = 3$,
3. $s_{[i,j]}s_{[i,j]} = s_{[i,j]}x$.

The above axiomatization of $\text{MA}_\alpha$ is finitary in a two sorted sense. One for the indices $< \alpha$ and the other for the first order situation. It can be proved exactly like in the classical case (Sain & Thompson, 1990) that $\text{MA}_\alpha$ is term definitionally equivalent to $\text{QMA}_\alpha$. In particular, $\text{MA}_\alpha$ is definable by a finite schema of equations. It is known that in the classical case $\text{RQA}_\alpha$ cannot be axiomatized by a finite schema of equations (Sain & Thompson, 1990). We use this fact to show:

**Theorem 30** The variety $\text{RMA}_\alpha$ cannot be axiomatized by a finite schema. In particular, $\text{RMA}_\alpha \subseteq \text{MA}_\alpha$.

**Proof** Assume, for contradiction, that $\Gamma$ is a finite schema such that $\text{Mod}(\Gamma) = \text{RMA}_\alpha$. Let $\Sigma$ be the schema obtained from $\Gamma$ by adding the equations stipulating idempotency, namely, $x \odot x = x$ and $x \odot x = x$. Then $\Sigma$, of course, is still a finite schema with two additional equations and it is sound, meaning that if $\mathfrak{A} \in \text{RQA}_\alpha$ then $\mathfrak{A} \models \Sigma$. Now we show that it is also complete. Assume that $\mathfrak{A} \models \Sigma$. Then $\mathfrak{A} \in \text{RMA}_\alpha$ and every $a \in A$ is idempotent. Thus $\mathfrak{A} \in \text{RQA}_\alpha$, so $\mathfrak{A}$ is representable classically. But we know that this cannot happen because $\text{RQA}_\alpha$ cannot be axiomatized by a finite schema of equations.

The next theorem tells us that there are infinitely many varieties between $\text{RMA}_\alpha$ and $\text{MA}_\alpha$. For any positive $k$, $V_{k} = \text{SN}_{r,k}\text{MA}_{\alpha+k}$ is strictly between $\text{RA}_{\alpha}$ and $\text{MA}_{\alpha}$, and for $0 < k_1 < k_2$, $V_{k_1} \subseteq V_{k_2}$. Even more, the former cannot be axiomatized by a finite schema over the latter. In more detail:

**Theorem 31** For each finite $k \geq 1$, for each $r \in \omega$, there is $\mathfrak{B}^r \in \text{SN}_{r,k}\text{MA}_{\alpha+k}$ such that $\mathfrak{B}^r \notin \text{SN}_{r,k}\text{MA}_{\alpha+k+1}$ and $\Pi_r/\mathfrak{B}^r \in \text{SN}_{r,\alpha}\text{MA}_{\alpha+k+1}$. Hence, for any ordinal $\alpha \geq \omega$ and finite $k \geq 1$, the variety $\text{SN}_{r,k}\text{MA}_{\alpha+k+1}$ is not axiomatizable by a finite schema over $\text{SN}_{r,k}\text{MA}_{\alpha+k}$.

**Proof** It is shown in (Hirsch & Sayed Ahmed, 2014, Theorem 1.1) that for $3 \leq m < n < \omega$, and $r \in \omega$, there are $m$ dimensional algebras $\mathfrak{C}(m,n,r) \in \text{QA}_m$ (quasi-polyadic dimension $m$) such that:

1. $\mathfrak{C}(m,n,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$,
2. $\Pi_r/\mathfrak{C}(m,n,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$,
3. For each $k \in \omega$, $\mathfrak{C}(m,m+k,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$,
4. $\Pi_r/\mathfrak{C}(m,n,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$,
5. For $3 \leq m < n$ and $k \geq 1$, there exists $x_n \in \mathfrak{C}(n,n+k,r)$ such that $\mathfrak{C}(m,m+k,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$ and $\mathfrak{C}(m,n,k,r) \not\in \text{SN}_{r,m}\text{QA}_{n+1}$.

To lift the result to the transfinite, we proceed like in (Hirsch & Sayed Ahmed,
2014) using an ingenious lifting argument (Henkin et al., 1985, Theorem 3.2.87) due to Monk. Let \( I = \{ \Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega \} \). For each \( \Gamma \in I \), let \( M_{\Gamma} = \{ \Delta \in I : \Gamma \subseteq \Delta \} \), and let \( F \) be an ultrafilter on \( I \) such that \( \forall \Gamma \in I, M_{\Gamma} \in F \). For each \( \Gamma \in I \), let \( \rho_{\Gamma} \) be an injective function from \( |\Gamma| \) onto \( F \). Let \( C_{\Gamma} \) be an algebra similar to \( QA_\alpha \) such that \( \mathfrak{A}^{\rho_{\Gamma}}_{\Gamma} = \mathfrak{C}_{\Gamma} \). Then \( \mathfrak{B}^r = \Pi_{\Gamma/F} C_{\Gamma} \). Then we have \( \mathfrak{B}^r \in N_{\rho_{\Gamma}} QA_{\alpha+k} \) and \( \mathfrak{B}^r \not\in SN_{\rho_{\Gamma}} QA_{\alpha+k+1} \). These can be proved exactly like the proof of the first two items in (Hirsch & Sayed Ahmed, 2014, Theorem 3.1). Now assume for contradiction that \( \mathfrak{B}^r \in SN_{\rho_{\Gamma}} MA_{\alpha+k+1} \). Then \( \mathfrak{B}^r \subseteq \mathfrak{M} \) for some \( \mathfrak{M} \in MA_{\alpha+k+1} \). Let \( \mathfrak{B}' \) be the set of idempotent elements in \( \mathfrak{M} \). Then we claim that \( \mathfrak{B}' \) has a Boolean reduct, and is closed under cylindrifiers and substitutions. Clearly \( \mathfrak{B}' \) is closed under \( \oplus, \odot \) and \( \neg \) and in \( \mathfrak{B}' \) we have \( \mathfrak{B'} = \mathfrak{B}^r \). Furthermore, for \( x \in \mathfrak{B} \) and \( i < \alpha \), we have by item (5) and (6) of theorem 7 that \( c_i x = c_i (x \odot x) = c_i x \odot c_i x \). Substitution operations clearly preserve idempotency then we are done. Hence \( \mathfrak{B}' \) is closed under the operations, and is clearly the universe of a \( QA_\alpha \). Furthermore, \( \mathfrak{B}' \subseteq \mathfrak{M} \) because all elements of \( \mathfrak{B}^r \) are idempotent, which is a contradiction. Let \( \lambda_{\Gamma} : |\Gamma| + k + 1 \to n + k + 1 \) extend \( \rho_{\Gamma} : |\Gamma| \to \Gamma (\subseteq \alpha) \) and satisfy \( \lambda_{\Gamma}(|\Gamma| + i) = \alpha + i \) for \( i < k + 1 \). Then \( \mathfrak{A}'_{\Gamma} \) is a \( QA_{\alpha+k+1} \) type algebra such that \( \mathfrak{A}^{\lambda_{\Gamma}}_{\Gamma} = \mathfrak{A}'_{\Gamma} \). Then \( \Pi_{\Gamma/F} \mathfrak{A}'_{\Gamma} \in QA_{\alpha+k+1} \), and we have proceeding like in the proof of item 3 in (Hirsch & Sayed Ahmed, 2014, Theorem 3.1):

\[
\Pi_{\Gamma/F} \mathfrak{B}^r = \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathfrak{C}_{\Gamma} \\
\cong \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathfrak{C}_{\Gamma} \\
\subseteq \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathfrak{C}_{\Gamma} \\
= \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathfrak{C}_{\Gamma} \\
= \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathfrak{C}_{\Gamma}.
\]

Then \( \mathfrak{B} = \Pi_{\Gamma/F} \mathfrak{B}^r \subseteq SN_{\rho_{\Gamma}} QA_{\alpha+k+1} \) because \( \mathfrak{B} = \Pi_{\Gamma/F} \mathfrak{A}'_{\Gamma} \in QA_{\alpha+k+1} \) and \( \mathfrak{B} \subseteq \mathfrak{M} \). The last part follows using a standard L"os argument. \( \square \)

We know from theorem 19, that \( DC_{\alpha} \subseteq RMA_{\alpha} \). Next we define (also abstractly) a class larger than \( DC_{\alpha} \), but still consists solely of representable algebras. We use the first item of theorem 28 referred to henceforth as the neat embedding theorem.

**Definition 32** \( \mathfrak{A} \in MA_{\alpha} \) is called a substitution algebra of dimension \( \alpha \), if for all non-zero \( x \) in \( A \), for all finite \( \Gamma \subseteq \alpha \), there exist distinct \( i, j \in \alpha \setminus \Gamma \), such that \( s_i^j x \neq 0 \). Let \( SC_{\alpha} \) denote the class of substitution algebras.

Now we prove that \( SC_{\alpha} \subseteq RMA_{\alpha} \), using the neat embedding theorem. But first a definition:

**Definition 33** Let \( n < \beta \) be (infinite) ordinals. Let

\[
\mathfrak{A} = (A, \oplus, \odot, \neg, c_i, s_{[i,j]}, s^j_i)_{i,j < \beta} \subseteq MA_{\beta},
\]

and assume that \( \rho : n \to \beta \) is an injection, then \( \mathfrak{A}(\rho) \), or simply \( \mathfrak{A}(\rho) \), is defined to be the algebra

\[
(\mathfrak{A}(\rho), c_{\rho(i)}, s_{[\rho(i),\rho(j)]}, s^j_{\rho(i)} s^j_{\rho(i)})_{i,j < n}.
\]

It is easy to show that \( \mathfrak{A}(\rho) \subseteq MA_{\beta} \). Notice that when \( \rho \) is the inclusion map
then $\mathfrak{M}_0^n\mathfrak{A}$ denoted before by $\mathfrak{M}_n\mathfrak{A}$ is the usual reduct obtained by discarding operations with indices in $\beta \setminus n$.

**Theorem 34** $\mathfrak{S}_\alpha \subseteq \mathfrak{RMA}_\alpha$.

**Proof** It suffices by the neat embedding theorem to show that for $\mathfrak{A} \in \mathfrak{S}_\alpha$, there exists $\mathfrak{B} \in \mathfrak{MA}_{\alpha+\omega}$ such that $\mathfrak{A} \subseteq \mathfrak{M}_\alpha \mathfrak{B}$. So let $\mathfrak{A} \in \mathfrak{S}_\alpha$. To construct the required $\omega$-dilation $\mathfrak{B}$ we proceed as follows using ultraproducts. Let $R$ be the set of all quadruples $(\rho, n, k, l)$ such that: $\rho \in ^m\alpha$ is injective for some $m \in \omega$, $n \in \omega$, $k, l$ are injective (finite) sequences with $k, l \in ^n(\alpha \setminus \operatorname{rng} \rho)$ and $\operatorname{rng} k \cap \operatorname{rng} l = \emptyset$. For $\rho \in ^n\alpha$ ($n \in \omega$) injective, put $X_{\rho,n} = \{ (\sigma, m, k, l) \in R : \rho \subseteq \sigma$ and $n \leq m \}$. It is straightforward to check that the set consisting of all the $X_{\rho,n}$'s is closed under finite intersections. Accordingly, let $M$ be the proper filter of $\wp(R)$ generated by the $X_{\rho,n}$'s so that $M = \{ Y \subseteq R : X_{\rho,n} \subseteq Y$ for some $\rho$ and $n \in \omega \}$. For each $(\rho, n, k, l) \in R$, choose an injection $t((\rho, n, k, l)]$ from $\alpha + \omega$ onto $\alpha$ such that $t((\rho, n, k, l)] | \operatorname{rng} \rho \subseteq \operatorname{Id}$, and $t((\rho, n, k, l)](\alpha + j) = k_j$, for each $j < n$.

Let $\mathfrak{B} = \prod_{\rho \in R} \mathfrak{R}_d^{t(\phi)} \mathfrak{A}/M$. Here $\mathfrak{R}_d^{t(\phi)} \mathfrak{A}$ - the $t(\phi)$ reduct of $\mathfrak{A}$ - is an $\mathfrak{MA}_{\alpha+\omega}$, and so $\mathfrak{B}$ - an ultraproduct of these - is also a $\mathfrak{MA}_{\alpha+\omega}$. Note too, that for each $\phi \in R$, the algebra $\mathfrak{R}_d^{t(\phi)} \mathfrak{A}$ has universe $\mathfrak{A}$. Let $f$ be the function from $\mathfrak{A}$ into $\mathfrak{B}$ defined by $f(x) = (s_0^k \circ \ldots \circ s_{m-1}^k : (\rho, n, k, l) \in R)/M$.

Consider any $i < \alpha$. Then for each $(\rho, m, k, l) \in R$ such that $i \in \operatorname{rng} \rho$ we have

$$s_{i0}^k \ldots s_{im-1}^k c_i x = c_i s_{i0}^k \ldots s_{im-1}^k x$$

to every $x \in A$. Since $\{ (\rho, m, k, l) \in R : i \in \operatorname{rng} \rho) \} \subseteq M$, we have $f(c_i x) = c_i f(x)$ for all $x \in A$ and $i < \alpha$. Substitutions can be dealt with analogously. Now consider any $i \in (\alpha + \omega) \setminus \alpha$. Then for each $(\rho, m, k, l) \in R$ such that $i < \alpha + m$ we have

$$c_{i(t(\rho, m, k, l))} s_{i0}^k \ldots s_{im-1}^k x = s_{i0}^k \ldots s_{im-1}^k x,$$

since $t(\rho, m, k, l)(i) \in \operatorname{rng} k \setminus \operatorname{rng} l$. Hence $f(\mathfrak{A}) \subseteq \mathfrak{M}_\alpha \mathfrak{B}$.

Consider now any $x \in A$ such that $x \neq 0$. Then for any given finite $\Gamma \subseteq \alpha$, and any $m < \omega$, there exist by assumption, using a simple induction, injective sequences $k, l \in ^m(\alpha \setminus \Gamma)$ such that $\operatorname{rng}(k) \cap \operatorname{rng}(l) = \emptyset$ and

$$s_{i0}^k \ldots s_{im-1}^k x \neq 0,$$

thus $f$ is injective. Then $j$ is an embedding form $\mathfrak{A}$ into $\mathfrak{M}_\alpha \mathfrak{B}$. By the neat embedding theorem 28 we are done. \hfill \Box

It is easy to show that $\mathfrak{D}_\alpha \subseteq \mathfrak{S}_\alpha$. Indeed, let $\mathfrak{A} \in \mathfrak{D}_\alpha$. Let $x \in \mathfrak{A}$ be non-zero, and let $\Gamma$ be a finite subset of $\alpha$. Choose distinct $k, l$ outside $\Delta x \cup \Gamma$. This is possible since $\alpha \setminus \Delta$ is infinite and $\Gamma$ is finite. Then $s_{i0}^k x \neq 0$, and we are done. From this inclusion, together with theorem 28, we get that $\mathfrak{S} \mathfrak{P} \mathfrak{U} \mathfrak{S}_\alpha = \mathfrak{RMA}_\alpha$.

### 4.2 Interpolation and Amalgamation

Now we address interpolation properties in free algebras of certain varieties and the corresponding amalgamation properties in such varieties. We use theorem 19 to prove several strong amalgamation properties for classes of algebras that are not necessarily varieties using so-called bridge theorems in (abstract) algebraic logic, cf. (Andréka et al., 2000; Blok & Pigozzi, 1989).

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To prove a weak form of interpolation for $\exists ! \text{MA}_\alpha$, $\delta$ a non–zero cardinal, we need a technical lemma. For a term $t$ in the language of MA$_\beta$, $\beta$ an infinite ordinal, we write $\text{ind}(t)$ for the set of indices in $t$ and $\text{var}(t)$ for the set of variables in $t$. For example if $t = c(x_0 \odot x_1)$ ($i \in \beta$), then $\text{ind}(t) = \{i\}$ and $\text{var}(t) = \{x_0, x_1\}$.

**Lemma 35** Let $\alpha \geq \omega$. Let $K$ be a class of algebras such that $\text{RMA}_\alpha \subseteq K \subseteq \text{MA}_\alpha$. Then for any terms $\sigma, \tau$ of the language of MA$_\alpha$, $\sigma \subseteq \tau$, and $\var \subseteq \var$, there exists $j \in \mathbb{N}$, a term $\pi$ with $\var(\pi) \subseteq \var(\sigma) \cap \var(\tau)$ and a finite $\Delta \subseteq \alpha$ such that

$$K \models q(\Delta)\sigma^j \leq \pi \leq c(\Delta)\tau.$$  

**Proof** Assume that $K$ is in the hypothesis; $K \models \sigma \subseteq \tau$. We want to find an interpolant, i.e. a $\pi$ as in the conclusion. Let $L_\alpha$ be the language of MA$_\alpha$ and for $n \leq \omega$, let $L_n$ or simply $L^n$ be the language of MA$_{\alpha + n}$. We write $L$ for $L^{(0)}$.

For an assignment $s : \omega \rightarrow \mathcal{B}$, $\mathcal{B} \in \text{MA}_\alpha$, we write $\bar{s}$ for its extension, done recursively the usual way, to all terms of $L_\alpha$. Now since $\forall \text{RMA}_\alpha \subseteq K$, then for every $\mathcal{B} \in \text{MA}_{\alpha + \omega}$, for every $s : \omega \rightarrow \mathcal{B}$ such that $\text{rng}s \subseteq \forall \text{MA}_\alpha$, we have $\mathcal{B} \models (\sigma \subseteq \tau)[\bar{s}]$.

Hence, by theorem 19, there is a term $\pi$ of $L^{(\omega)}$ which contains only occurrences of variables which occur in both $\sigma$ and $\tau$, and $j \in \mathbb{N}$, which satisfy that for all $\mathcal{B} \in \text{MA}_{\alpha + \omega}$, for every $s : \omega \rightarrow \mathcal{B}$, such that $\text{rng}s \subseteq \forall \text{MA}_\alpha$, $\mathcal{B} \models (\sigma^j \subseteq \pi)[\bar{s}]$ and $(\pi \leq j\tau)[\bar{s}]$.

Expand each language $L^{(n)}$, $n < \omega$, by adjoining to its signature a fixed $\omega$–termed sequence $a = \{a_0, a_1, a_2, \ldots\}$ of distinct individual constants symbols. Let $\sigma', \tau'$ and $\pi'$ be the terms of the language extending $L^{(\omega)}$ that are obtained from $\sigma, \tau$ and $\pi$ by replacing each variable $v_k$ in all of its occurrences by the constant symbol $a_k$. For each $k < \omega$ let $\Pi^{(k)}$ be the set of all identities of the form $c_\mu a_\nu = a_\nu$, where $\alpha \leq \mu < \alpha + k$ and $v < \omega$. Now we have

$$\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)} \models (\sigma'^j \leq \pi') \land (\pi' \leq j\tau').$$

where $\Sigma^{(\omega)} = \text{Eq}(\text{MA}_{\alpha + \omega})$. Therefore, by the compactness theorem there is a finite subset $\theta$ of $\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)}$ such that

$$\theta \models (\sigma'^j \leq \pi') \land (\pi' \leq j\tau').$$

Then there is a finite ordinal, $\delta$, say such that $\theta \subseteq \Sigma^{(\delta)} \cup \Pi^{(\delta)}$. Now $\text{ind}(\pi) \subseteq \alpha + \delta$. Choose two distinct sets $\Gamma, \Delta \subseteq \alpha$ such that $|\Gamma| = |\Delta| = \delta$ and such that neither $\Gamma$ nor $\Delta$ contains any index occurring in $\sigma, \tau,$ or $\pi$.

Let $\mu, v$ be two sequences of length $\delta$ which enumerate the elements of $\Delta$, respectively (and hence are necessarily injective). Let $\bar{\sigma}$ and $\bar{\tau}$ be the terms of $L$ that are obtained from $\sigma$ and $\tau$, respectively, by replacing each variable $v_k$ in all of its occurrences by the term $s_{\delta_{0}}^{\mu_{0}} s_{\delta_{1}}^{\mu_{1}} \ldots s_{\delta_{\delta-1}}^{\mu_{\delta-1}} v_{k}$. Let $\bar{\pi}$ be obtained from $\pi$ by making these same replacements and also by replacing every index $\lambda$ in $\pi$ such that $\lambda \in (\alpha + \delta) \setminus \alpha$ to an ordinal in $\Gamma$; so that different
We can now readily conclude that $\Delta \subseteq \sigma, \tau$ contains an index occurring in $\alpha$–variables in $\bar{\alpha}$ such that $\rho(k + k) = \mu_k$ for every $k < \delta$.

Here we are using that $\alpha$ is infinite.

Then for every $\mathfrak{C} \in \mathit{MA}_\alpha$ we have, $\mathfrak{M}^{(\rho)} \mathfrak{C} \in \mathit{MA}_{\alpha + \delta}$ and by hypothesis, we get

$$s^{\mu_0 \sigma_{v_1}} \ldots s^{\mu_{i-1} \sigma} x \in \mathfrak{M}_\alpha \mathfrak{M}^{(\rho)} \mathfrak{C} \quad \text{for every} \quad x \in C.$$ 

We can now readily conclude that

$$\mathit{MA}_\alpha \models \bar{\sigma}^j \leq \bar{\pi} \quad \text{and} \quad \mathit{MA}_\alpha \models \bar{\pi} \leq j \bar{\tau}.$$ 

Now neither the $\Gamma$ nor $\Delta$ contains an index occurring in $\sigma, \tau$, or $\pi$ then

$$\mathit{MA}_\alpha \models \bar{\sigma}^j = s^{\mu_0 \sigma_{v_1}} \ldots s^{\mu_{i-1} \sigma} \text{ and } \mathit{MA}_\alpha \models j \bar{\tau} = s^{\mu_0 \sigma_{v_1}} \ldots s^{\mu_{i-1} \sigma} j \bar{\tau}.$$ 

Combining these results we get that

$$\mathit{MA}_\alpha \models s^{\mu_0 \sigma_{v_1}} \ldots s^{\mu_{i-1} \sigma} \leq \bar{\pi} \leq s^{\mu_0 \sigma_{v_1}} \ldots s^{\mu_{i-1} \sigma} j \bar{\tau};$$

in particular, in every member of $K$; the same is true of

$$q_{\mu_0} q_{\mu_1} \ldots q_{\mu_{i-1}} \sigma^j \leq \bar{\pi} \leq c_{\mu_0} c_{\mu_1} \ldots c_{\mu_{i-1}} j \bar{\tau}.$$ 

Therefore, since $\bar{\pi}$ is a term of $L$ and it contains like $\pi$ only occurrences of variables which occur at the same time in both $\sigma$ and $\tau$ we have shown that the inclusion $\sigma \leq \tau$ can indeed be interpolated relative to $K$. 

Terms in the signature of $\mathit{MA}_\alpha$ can be identified with formulas in an infinitary language with countably many relation symbols each of arity $\alpha$. For example the term $c_{0 \bar{x}}$, say, can be identified with the formula $\exists x_0 R(\bar{x})$, where $R$ is an $\alpha$–ary relation symbol and the $\alpha$–variables in $\bar{x}$ occur in their natural order $x_0, x_1, \ldots, x_i, \ldots: i < \alpha$ (Henkin et al., 1985, Definition 4.3.55).

So using standard methodology of abstract algebraic logic (Blok & Pigozzi, 1989), identifying free algebras in a certain signature with algebras of formulas, and identifying formulas with terms in this signature, we get using the previous lemma the following interpolation theorem (expressed algebraically for such infinitary languages):

**Theorem 36** Let $V$ be a variety, such that $\mathit{RMA}_\alpha \subseteq V \subseteq \mathit{MA}_\alpha$. Let $\beta$ be any cardinal $> 0$ and let $\mathfrak{A} = \mathfrak{A}_\beta V$. Then for any $X_1, X_2 \subseteq \mathfrak{A}$, if $a \in \mathfrak{A} g^{\beta} X_1$ and $b \in \mathfrak{A} g^{\beta} X_2$ are such that $a \leq b$, then there exists $j \in N$, $c \in \mathfrak{A} g^{\beta} (X_1 \cap X_2)$ and $\Delta \subseteq \omega \alpha$, such that $q(\Delta) a^j \leq c \leq q(\Delta) j b$.

According to definition 20, for the (classical) VIP studied in (Sayed Ahmed, 2012b), given an implication $\theta = \phi \to \psi$, one takes $f_\theta(x) = \exists x_{n+1} \ldots x_n x$, and $g_\theta(x) = \forall x_{n+1} \ldots x_n x$, for some $n$ depending on $\theta$ to find an interpolant. In the above theorem, which is its many valued version, one takes $f_\theta(x) = \exists x_{n+1} \ldots x_n x^n$.
and \( g_\theta(x) = \forall_{x_0,\ldots,x_n} nx \) for some \( n \) depending on \( \theta \). Again in case of idempotency the \( MV \) case proved in the previous theorem 36 reduces to the classical case proved in (Sayed Ahmed, 2012b) since \( x^n = x \) and \( nx = x \).

We refer to the interpolation property expressed in the previous theorem as weak varying interpolation property, \( WVIP \) for short. Now we show that for any variety \( \mathcal{V} \), such that \( RMA_\alpha \subseteq \mathcal{V} \subseteq MA_\alpha \), \( \mathcal{F}_\alpha \mathcal{V} \), though having \( WVIP \), it does not have \( VIP \). In other words, we cannot get rid of the ‘quantified variables’ in \( \Delta \) as specified in the previous proof and in the statement of theorem 36. Our proof is indirect. It depends on the following implication: \( \mathcal{F}_\alpha \mathcal{K} \) has \( VIP \) \( \implies \) \( \mathcal{K} \) has the amalgamation property (to be defined shortly) which we prove in a while when we work inside \( MA_\alpha \).

We need some preparation starting with the notion of filters in \( MA_\alpha \). A filter \( F \) of \( B \in MA_\alpha \) is a filter of \( \mathcal{MF}B \), such that if \( a \in F \), and \( k < \alpha \), then \( q_k a \in F \). It can be checked that filters defined this way correspond to congruences in the following sense. Let \( \mathfrak{A} \in MA_\alpha \) and let \( \text{Filt}(\mathfrak{A}) \) be the lattice of filters (with inclusion) on \( \mathfrak{A} \), and \( \text{Co}(\mathfrak{A}) \) be the lattice of congruences on \( \mathfrak{A} \). Then \( \Theta : \text{Filt}(\mathfrak{A}) \rightarrow \text{Co}(\mathfrak{A}) \) defined by \( \equiv \rightarrow \{ x \in \mathfrak{A} : x \equiv 1 \} \) is an isomorphism of lattices with inverse \( \Theta^{-1} : \text{Filt}(\mathfrak{A}) \rightarrow \text{Co}(\mathfrak{A}) \) defined by \( F \mapsto R = \{(a,b) \in A \times A : a \leftrightarrow b \in F \} \). Here \( a \leftrightarrow b \) is defined by \( -a \oplus b \cap -b \oplus a \), where \( \cap \) is the weak conjunction.

Furthermore, \( \Theta \) restricted to maximal filters is an isomorphism onto the set of maximal congruences (in both cases maximality is taken with respect to \( \subseteq \)). If \( \mathfrak{B} \in MA_\alpha \) and \( F \subseteq \mathfrak{B} \) is a filter, then \( \mathfrak{B}/F \) denotes the quotient algebra (in \( MA_\alpha \)) and its elements are denoted by \( b/F (b \in \mathfrak{B}) \). For an MV algebra \( \mathfrak{A} \), and \( X \subseteq \mathfrak{A} \) (recall that we denoted the MV filter as defined in 4 generated by a subset \( X \) of \( A \), by \( f^{\mathfrak{MV}_X}X \). For \( \mathfrak{A} \in MA_\alpha \) and \( X \subseteq \mathfrak{A} \), the \( MA_\alpha \) filter generated by \( X \) when we count in closure under universal quantifiers, will be denoted by \( \mathfrak{F}^{\mathfrak{MV}}X \). We need one more lemma:

**Lemma 37** Let \( \mathfrak{A} \in MA_\alpha \), \( \emptyset \neq X \subseteq \mathfrak{A} \) and \( M,N \) be filters of \( \mathfrak{A} \). Then the following hold:

1. \( M \) is proper \( \iff \emptyset \notin M \).
2. \( \mathfrak{F}^{\mathfrak{MV}}X = \{a \in \mathfrak{A} : \exists x_0,\ldots,x_{l-1} \in X, m_0,\ldots,m_{l-1} \in N, l \in \omega, \Delta \subseteq_{\omega} \alpha : q_\alpha (x_0^{m_0} \circ \cdots \circ x_{l-1}^{m_{l-1}}) \leq a \} \).
3. \( \mathfrak{F}^{\mathfrak{MV}}(M \cup N) = \{a \in \mathfrak{A} : \exists u \in M, v \in N, u \circ v \leq a \} \).

**Proof** Follows directly from the definition.

We shall violate a weaker version of the amalgamation property defined in the first item of the next definition:

**Definition 38** Let \( L \subseteq K \) be classes of algebras.

1. \( L \) has the amalgamation property with respect to \( K \) if for all \( \mathfrak{A}_1, \mathfrak{A}_2 \in L \) and injective homomorphisms \( i_1 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_1 \) \( i_2 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_2 \) there exist \( \mathfrak{D} \in K \) and injective homomorphisms \( m_1 : \mathfrak{A}_1 \rightarrow \mathfrak{D} \) and \( m_2 : \mathfrak{A}_2 \rightarrow \mathfrak{D} \) such that \( m_1 \circ i_1 = m_2 \circ i_2 \). \( \mathfrak{D} \) is called an amalgam of \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) over \( \mathfrak{A}_0 \), or simply an amalgam.

2. Let everything be as in the previous item. If, in addition, we have \( m_1 \circ i_1 (\mathfrak{A}_0) = m_1 (\mathfrak{A}_1) \cap m_2 (\mathfrak{A}_2) \), then \( \mathfrak{D} \) is called a strong amalgam and \( L \) is said to have the strong amalgamation property with respect to \( K \).

3. Let everything be as in the first item. If in addition \( (\forall x \in \mathfrak{A}_1)(\forall y \in \mathfrak{A}_2)(m_1(x) \leq m_2(y) \implies (\exists z \in \mathfrak{A}_0)(\exists n \in N)(x^n \leq i_j(z) \text{ and } i_k(z) \leq ny)) \) where \( \{j,k\} = \{1,2\} \), then \( \mathfrak{D} \) is called a generalized super amalgam, and \( L \) is said to have the generalized super amalgamation property with respect to \( K \). We say that \( \mathfrak{D} \) is a generalized super amalgam, or simply a super amalgam, of \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) over \( \mathfrak{A}_0 \).
The definition of the strong, generalized super, and amalgamation property, is the case when \( L = K \). That is, we say that \( L \) has the amalgamation property, if \( L \) has the amalgamation property with respect to itself, and similarly for strong and generalized super amalgamation properties. We write \( AP \) short hand for ‘amalgamation property’, \( SAP \) for ‘strong amalgamation’ and \( GSUPAP \) for ‘generalized super amalgamation’. The super amalgamation property, \( SUPAP \) for short, is the special case of \( GSUPSP \) when the exponent \( n = 1 \). Observe that \( GSUPAP \) implies \( SAP \) by writing the extra condition for \( SAP \) as follows:

\[
(\forall x \in \mathfrak{A}_1)(\forall y \in \mathfrak{A}_2)[m_1(x) = m_2(y) \implies (\exists z \in \mathfrak{A}_0)(\exists n \in N)(x^n = i_1(z) \text{ and } y^n = i_2(z))].
\]

Now we prove:

**Theorem 39** Let \( K \) be a subvariety of \( MA_\alpha \). Then the \( K \) free algebras has \( VIP \implies K \ has \ AP \).

**Proof** Let \( K \subseteq MA_\alpha \) be a variety for which the free algebras have \( VIP \). Let \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K \) be such that \( \mathfrak{C} \subseteq \mathfrak{A} \) and \( \mathfrak{C} \subseteq \mathfrak{B} \). We want an amalgam. That is, we want to find an algebra \( \mathfrak{D} \in K \) and injective homomorphisms \( f : \mathfrak{A} \to \mathfrak{D} \) and \( g : \mathfrak{B} \to \mathfrak{D} \), such that \( f \restriction \mathfrak{C} = g \restriction \mathfrak{C} \).

Let \( \mathfrak{F} \) be the free algebra on \( \beta \) generators, where \( \beta \) is a cardinal \( \geq \max|B|, |A| \).

Take \( \beta_1, \beta_2 \subseteq \beta \), such that \( \beta = \beta_1 \cup \beta_2 \), and such that there are surjective homomorphisms \( h : \mathfrak{F} \to \mathfrak{C} \), \( h_0 : \mathfrak{F} \to \mathfrak{A} \) and \( h_2 : \mathfrak{F} \to \mathfrak{B} \), with \( h \restriction \mathfrak{F} \) a bijection from \( \beta_1 \) to \( \mathfrak{A} \), \( h_2 \restriction \mathfrak{F} \) a bijection from \( \beta_2 \) to \( \mathfrak{B} \) and \( h \restriction \mathfrak{F} \) a bijection from \( \beta_1 \cap \beta_2 \) to \( \mathfrak{C} \).

Such homomorphisms exist by the freeness (and cardinality) of \( \mathfrak{F} \). Let \( \mathfrak{F}_1 = \mathfrak{F} \restriction \beta_1 \) and \( \mathfrak{F}_2 = \mathfrak{F} \restriction \beta_2 \). Let \( M = \{d \in \mathfrak{F}_1 : h_1(d) = 1\} \) and \( N = \{d \in \mathfrak{F}_2 : h_2(d) = 1\} \). Then \( M \) and \( N \) are filters. Let \( h_1 : \mathfrak{F}_1 \cap M \to \mathfrak{A} \), \( h_2 : \mathfrak{F}_2 \cap N \to \mathfrak{B} \) be the isomorphisms induced by \( h_1 \) and \( h_2 \) (by the first isomorphism theorem), that is, \( h_1(d/M) = h_1(d) \) and similarly for \( h_2 \).

Let \( \mathfrak{F}_0 = \mathfrak{F} \restriction (\beta_1 \cap \beta_2) \). Let \( l_1 : \mathfrak{F}_0 \cap M \to \mathfrak{A} \) be defined via \( d/\mathfrak{F}_0 \cap M \to h_1(d) \), and \( l_2 : \mathfrak{F}_0 \cap N \to \mathfrak{B} \) be defined via \( d/\mathfrak{F}_0 \cap N \to h_2(d) \). Then \( l_1 \) and \( l_2 \) are well defined, so \( \mathfrak{F}_0 \cap M = \mathfrak{F}_0 \cap M \).

We show that \( \mathfrak{F}_0 \cap (M \cup N) \) (the filter generated by \( M \cup N \) in \( \mathfrak{F} \)) is a proper filter of \( \mathfrak{F} \) and that \( \mathfrak{F}_0 \cap P \) is the required amalgam. Let \( x \in \mathfrak{F}_0 \cap (M \cup N) \cap \mathfrak{F}_1 \).

Then by the last item of lemma 37, there exist \( b \in M \) and \( c \in N \) such that \( b \cup c \leq x \). Thus \( c \leq x \cup -b \). But \( x \cup -b \in \mathfrak{F}_1 \) and \( c \in \mathfrak{F}_2 \), it follows from the assumption that \( \mathfrak{F} \) has \( VIP \), that there exist \( x \in \mathfrak{F}_0 \), \( j \in N \), such that \( jx \leq jx \cup [jx \cup -b] \). Observe that \( c \in N \), hence \( jx \leq N \), so \( d \in M \), because \( N \cap \mathfrak{F}_0 = M \cap \mathfrak{F}_0 \). We have \( d \leq jx \cup [jx \cup -b] \), hence

\[(*) \quad d \cup jx \leq jx .\]

Now \( d \) and \( b \) are in \( M \) and \( b \leq b \cup \ldots b \) (j times) \( \leq jb \), so \( jb \) is in \( M \). But

\[d \leq d \cup jb \leq d \cup jb \in M \text{ and so } d \cup jb \in M.\]

But then \( d \cup jb \neq 0 \), so that \( jx \neq 0 \), hence by \((*) \) \( x \neq 0 \), too. Thus \( 0 \notin \mathfrak{F}_0 \cap (M \cup N) \) and so by the first item of lemma 37 \( P = \mathfrak{F}_0 \cap (M \cup N) \) is a proper \( MV \) filter of \( \mathfrak{F} \). The algebra \( \mathfrak{F}_0 \cap P \) is the required amalgam via the maps \( k \) and \( h \) defined from \( \mathfrak{F}_1 \cap M \) and \( \mathfrak{F}_2 \cap N \) to \( \mathfrak{F} \) by \( d/\mathfrak{F} \to d/P \) \( (d \in \mathfrak{F}_1 \cap M) \) and \( d/\mathfrak{F} \to d/P \) \( (d \in \mathfrak{F}_2 \cap N) \), respectively. Recall that \( \mathfrak{F}_1 \cap M \cong \mathfrak{A} \), \( \mathfrak{F}_2 \cap N \cong \mathfrak{B} \) and \( \mathfrak{F}_0 \cap M \cap \mathfrak{F}_0 = \mathfrak{F}_0 \cap N \cap P \cong \mathfrak{C} \).

Our next result, like theorems 30 and 31, is obtained by bouncing it back to the classical case. Recall that \( QA_\alpha \) denotes the class of quasi-polycadic algebras of dimension \( \alpha \) and \( RQA_\alpha \) is the class of representable \( QA_\alpha \)s as defined in (Sain & Thompson, 1990). Then \( \mathfrak{A} \in (R)QA_\alpha \iff \mathfrak{A} \in (R)MA_\alpha \) and \( R_{MV} \mathfrak{A} \) is a Boolean
algebra.

**Theorem 40** $\text{RMA}_\alpha$ does not have the amalgamation property with respect to $\text{MA}_\alpha$.

**Proof** We use the fact that $\text{RQA}_\alpha$ does not have the amalgamation property (Sayed Ahmed, 2011a). Assume to the contrary that $\text{RMA}_\alpha$ has the amalgamation property with respect to $\text{MA}_\alpha$. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \text{RQA}_\alpha$, $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{A} \to \mathfrak{C}$ be injective homomorphisms for which there is no amalgam in $\text{QA}_\alpha$. Such algebras exist (Sayed Ahmed, 2011a). Then $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \text{RMA}_\alpha$. By assumption there is an amalgam $\mathfrak{D}' \in \text{MA}_\alpha$, that is, there are $\mathfrak{D}' \in \text{MA}_\alpha$, injective homomorphisms $m : \mathfrak{B} \to \mathfrak{D}'$, $n : \mathfrak{C} \to \mathfrak{D}'$ such that $m \circ f = n \circ g$. Let $\mathfrak{D} = \{a \in \mathfrak{D}' : a$ is idempotent\}.

Then like in the proof of theorem 31, $\mathfrak{D} \in \text{QA}_\alpha$. Also $m(\mathfrak{B})$ and $n(\mathfrak{C})$ are contained in $\mathfrak{D}$, because the elements of $\mathfrak{B}$ and $\mathfrak{C}$ are idempotent, since they have a Boolean reduct, and homomorphisms preserve idempotency. Thus $\mathfrak{D}$ is an amalgam of $\mathfrak{B}$ and $\mathfrak{C}$ over $\mathfrak{A}$ which is a contradiction. \hfill $\square$

The finite dimensional version of the above theorem for classical logic is proved by Comer (Comer, 1969). Theorems 30, 31 taken together with theorems 36, 40 give necessarily infinitely many varieties whose free algebras on $\omega$ generators have $\text{WVIP}$ but not VIP.

**Corollary 41** For $\alpha \geq \omega$, for any $k \in \omega$, $\exists \alpha \text{SNr}_\alpha \text{MA}_{\alpha+k}$ does not have VIP but has WVIP.

In theorem 72 below, we show that the WVIP is equivalent to that the the class of simple algebras (algebras having no proper MV filters) in $\text{SNr}_\alpha \text{MA}_{\alpha+k}$ for each finite $k$ has the amalgamation property. On the other hand, by theorem 40 such varieties do not have the amalgamation property.

From theorem 19 alone, one can obtain all (positive) results analogous to the positive results proved for cylindric algebras (Pigozzi, 1971; Madarasz & Sayed Ahmed, 2012; Madarasz & Sayed Ahmed, 2009).

We explain how after proving the following theorem which depends essentially on theorem 19 and then we prove two positive such results as a representative sample, namely, that $\text{Dc}_\alpha$ has the generalized super amalgamation property with respect to $\text{RMA}_\alpha$; in fact, we prove something stronger, theorem 45, and that $\text{Sc}_\alpha$ has the amalgamation property, theorem 46. But first:

**Theorem 42** The class $\textbf{K} = \{\mathfrak{A} \in \text{MA}_{\alpha+\omega} : \exists g^\mathfrak{A} \text{Nr}_\alpha \mathfrak{A} = \mathfrak{A}\} \subseteq \text{Dc}_{\alpha+\omega}$ has the generalized super amalgamation property.

**Proof** We first prove amalgamation. Then we show that the amalgam obtained is in fact a super amalgam. The first part of the proof is analogous to the proof of theorem 39. In fact the only difference is that we use the dimension restricted free algebras as in theorem 19 instead of (usual) free algebras. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \textbf{K}$ be such that $\mathfrak{C} \subseteq \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{B}$. We want an amalgam as in the proof of theorem 39 that is, in addition, a super amalgam. Let $\mathfrak{F}$ be the dimension restricted free algebra on $\beta$ generators, with dimension restricting function $\rho : \beta \to \phi(\alpha + \omega)$, such that $\rho(i) = \alpha$ for all $i < \beta$ where like in the proof of op.cit $\beta$ is chosen to satisfy the following conditions:

There are $\beta_1, \beta_2 \subseteq \beta$, such that $\beta = \beta_1 \cup \beta_2$, and furthermore there are surjective homomorphisms $h : \mathfrak{F} \to \mathfrak{C}$, $h_1 : \mathfrak{F} \to \mathfrak{A}$ and $h_2 : \mathfrak{F} \to \mathfrak{B}$, such that $h_1 \upharpoonright \beta_1$ is a bijection from $\beta_1$ to $\mathfrak{A}$, $h_2 \upharpoonright \beta_2$ is a bijection from $\beta_2$ to $\mathfrak{B}$ and $h \upharpoonright \beta_1 \cap \beta_2$ is a bijection from $\beta_1 \cap \beta_2$ to $\mathfrak{C}$. Since $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \textbf{K}$ each has a set of generators whose dimension sets are contained in $\alpha$. So such homomorphisms exist by the universal
property of $\mathfrak{F}$ formulated in theorem 17.

The rest of the proof is exactly like the proof of theorem 39 using now theorem 19, namely, that the dimension restricted free algebra $\mathfrak{F}$ has WVIP. One defines $\mathfrak{F}_1 = \mathfrak{F}^g\beta_1$, $\mathfrak{F}_2 = \mathfrak{F}^g\beta_2$, and the filters $M \subseteq \mathfrak{F}_1$ and $N \subseteq \mathfrak{F}_2$ like in the proof of op.cit. Then $P = \mathfrak{F}^W(M \cup N)$ is a proper MV filter by theorem 19, so that $\mathfrak{F}/P$ is the required amalgam via the maps $k : \mathfrak{F}_1/M \rightarrow \mathfrak{F}/P$ and $h : \mathfrak{F}_1/N \rightarrow \mathfrak{F}/P$ defined, like in the proof of theorem 39, via $d/M \rightarrow d/P$ ($d \in \mathfrak{F}_1$) and $d/N \rightarrow d/P$ ($d \in \mathfrak{F}_2$), respectively. But we can go further.

We now prove that $\mathfrak{F}/P$ is actually a super amalgam implying that $\mathbf{K}$ has the generalized super amalgamation property. Assume that $a \in \mathfrak{F}_1/M$ and $b \in \mathfrak{F}_2/N$ and that $k(a) \leq h(b)$. By symmetry it suffices to show that there is a $t \in \mathbb{C}$ and $j \in \mathbb{N}$ such that $t \geq a^j$ and $t \leq jb$. There exists $x \in \mathfrak{F}_1$ such that $x/P = k(a)$ and there exists $z \in \mathfrak{F}_2$ such that $z/P = h(b)$. Since $x/P \leq z/P$, we get $-x \oplus z \in P$. Therefore, by the last item of lemma 37, there are $r \in \mathbb{R}$, $s \in \mathbb{N}$, such that $r \oplus s \leq -x \oplus z$. Hence $r \oplus x \leq z \oplus s$.

Now $r \oplus x \in \mathfrak{F}_1$ and $z \oplus s \in \mathfrak{F}_2$, it follows using theorem 19 (again) that there are $u \in \mathfrak{F}_0$, and $j \in \mathbb{N}$, such that $[r \oplus x]^j \leq u \leq [j(z \oplus s)]$. Thus $r^j \oplus f \oplus x^j \leq u$, for some $f \in \mathfrak{F}$. Hence $-x^j \oplus u \geq r^j \oplus f$, but the latter is in $P$ because $r \in \mathbb{R} \subseteq P$, so $r^j \in P$. Also $r^j \oplus t \in P$ because $r^j \leq r^j \cup t \leq r^j \oplus t$. Therefore $-[x^j] \oplus u \in P$, so we conclude that (*) $u/P \geq x^j/P$. Similarly, $-u \oplus jz \in P$, hence we have (**): $u/P \leq jz/P$. Since $\mathfrak{F}_0 \cap M = \mathfrak{F}_0 \cap N$ and $\mathbb{C} \cong \mathfrak{F}_0/M \cap \mathfrak{F}_0$ and $\mathbb{C} \cong \mathfrak{F}_0/N \cap \mathfrak{F}_0$, as proved in theorem 39, then there is a $t \in \mathbb{C}$ such that $t = u/M$ and $t = u/N$. Recall that $x/P = k(a)$. So by (*) we have

$$k(t) = k(u/M) = u/P \geq x^j/P = (x/P)^j = k(a)^j.$$  

Thus $k(t) \geq k(a)^j$. Since $k$ is injective, we get that $t \geq a^j$. Similarly, from (**), we get that $h(t) \leq jh(b)$, hence again by injectivity we get that $t \geq jb$, and we are done.

Having theorem 42 at hand, the theme for all positive results proved henceforth is basically the same. In fact, the strong amalgamation property suffices (these two notions are surprisingly distinct as proved by Sagi and Shelah (Sagi & Shelah, 2006).)

Until and including theorem 45, $\mathbf{K}$ denotes the class $\{\mathfrak{A} \in \mathbf{MA}_\alpha + \omega : \mathfrak{F}^\mathfrak{A} \mathfrak{R} \alpha \mathfrak{A} = \mathfrak{A}\}$ proved to have GSUPAP in theorem 42.

We want to prove that a certain class $\mathbf{L} \subseteq \mathbf{RMA}_\alpha$ has the amalgamation property. The neat embedding theorem can be used, to ‘lift’ algebras required to be amalgamated in $\mathbf{L}$ to $\mathbf{K}$. One finds an amalgam $\mathfrak{D} \in \mathbf{K}$ using op.cit, and then returns to ‘the original diagram’ using the neat reduct operator truncating $\omega$ dimensions forming the required amalgam $\mathfrak{R} \alpha \mathfrak{D}$. This ‘dilating then compressing dimension’ technique works when we want to prove only amalgamation with the amalgam in $\mathbf{RMA}_\alpha$.

Proving stronger forms of amalgamation matters can be more delicate and subtle. We may lose the additional generalized super amalgamation property, when we apply the neat reduct operator $\mathfrak{R} \alpha : \mathbf{K} \rightarrow \mathbf{RMA}_\alpha$. It can happen that the two successive operations of ‘dilating then compressing’ does not bring us back where we started (Sayed Ahmed, 2011b).

A necessary and sufficient condition that $\mathfrak{R} \alpha \mathfrak{D}$ is a super amalgam is that $\mathbf{L}$ has the so-called NS and UNEP as defined in (Sayed Ahmed, 2012a, Definitions 5.2.1, 5.2.2). We recall the definitions of NS and UNEP reformulated for $\mathbf{MA}_\alpha$s.
Definition 43 (Sayed Ahmed, 2012a, Definition 5.2.1) Let $\mathfrak{A} \in \text{RMA}_\alpha$. Then $\mathfrak{A}$ has the UNEP (short for Unique Neat Embedding Property) if for all $\mathfrak{A}' \in \text{MA}_\alpha$, $\mathfrak{B}, \mathfrak{B}' \in \text{MA}_{\alpha+\omega}$, isomorphism $i : \mathfrak{A} \rightarrow \mathfrak{A}'$, embeddings $e_A : \mathfrak{A} \rightarrow \mathfrak{R}_\alpha \mathfrak{B}$ and $e_{A'} : \mathfrak{A}' \rightarrow \mathfrak{R}_\alpha \mathfrak{B}'$ such that $\mathfrak{S}^{\alpha}e_A(A) = \mathfrak{B}$ and $\mathfrak{S}^{\alpha}e_{A'}(A') = \mathfrak{B}'$, there exists an isomorphism $i : \mathfrak{B} \rightarrow \mathfrak{B}'$ such that $i \circ e_A = e_{A'} \circ i$.

Definition 44 (Sayed Ahmed, 2012a, Definition 5.2.2) Let $\mathfrak{A} \in \text{RMA}_\alpha$. Then $\mathfrak{A}$ has the NS property (short for Neat reduces commuting with Subalgebras) if for all $\mathfrak{B} \in \text{MA}_{\alpha+\omega}$ if $\mathfrak{A} \subseteq \mathfrak{R}_\alpha \mathfrak{B}$, then for all $\mathfrak{X} \subseteq \mathfrak{A}$, $\mathfrak{S}^{\alpha} \mathfrak{X} = \mathfrak{R}_\alpha \mathfrak{S}^{\alpha} \mathfrak{X}$.

It is not easy to construct representable algebras that do not have the UNEP. Cylindric algebras that do not have UNEP, were explicitly constructed in (Sayed Ahmed, 2011b), confirming a conjecture of Tarski’s on cylindric algebras, see also (Sayed Ahmed, 2012a, Theorem 5.4.2) and the comments following it for an indirect proof of the existence of cylindric algebras that do not have UNEP.

We say that a class has the UNEP if each of its members have the UNEP. One can prove exactly like the classical case that the class $\text{Dc}_\alpha$ has NS and UNEP (Henkin et al., 1971, Theorems 2.6.69, 2.6.71, 2.6.73). We use these two properties to show:

Theorem 45 If $\mathfrak{C} \in \text{Dc}_\alpha$, and $\mathfrak{A}, \mathfrak{B} \in \text{RMA}_\alpha$ are such that $\mathfrak{C} \subseteq \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{B}$, then there is a super amalgam $\mathfrak{D} \in \text{RMA}_\alpha$ of $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathfrak{C}$. In particular, $\text{Dc}_\alpha$ has the generalized super amalgamation property with respect to $\text{RMA}_\alpha$.

Proof Let $\mathfrak{C} \in \text{Dc}_\alpha$, let $\mathfrak{A}, \mathfrak{B} \in \text{RMA}_\alpha$, and let $f : \mathfrak{C} \rightarrow \mathfrak{A}$ and $g : \mathfrak{C} \rightarrow \mathfrak{B}$ be injective homomorphisms. Here we consider that $\mathfrak{C}$ embeds in $\mathfrak{A}$ via the injective homomorphism $f$, and similarly for $\mathfrak{B}$ and $g$. We want to find a super amalgam. The proof is divided into three parts:

(1) We perform the dilating operation by ‘lifting’ the algebras required to be amalgamated to the class $\mathfrak{K}$ using the neat embedding theorem. Here UNEP is used:

Let $\beta = \alpha + \omega$. Then by the neat embedding theorem there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+ \in \text{MA}_\beta$, $e_A : \mathfrak{A} \rightarrow \mathfrak{R}_\alpha \mathfrak{A}^+$, $e_B : \mathfrak{B} \rightarrow \mathfrak{R}_\alpha \mathfrak{B}^+$ and $e_C : \mathfrak{C} \rightarrow \mathfrak{R}_\alpha \mathfrak{C}^+$. We can assume, without loss, that $\mathfrak{S}^{\alpha}e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. Let $f(C)^+ = \mathfrak{S}^{\alpha}e_A(f(C))$ and $g(C)^+ = \mathfrak{S}^{\alpha}e_B(g(C))$, so that $\mathfrak{A}^+, \mathfrak{B}^+$ and $\mathfrak{C}^+$ are in $\mathfrak{K}$. Then by the UNEP, there exist $\tilde{f} : \mathfrak{C}^+ \rightarrow f(C)^+$ and $\tilde{g} : \mathfrak{C}^+ \rightarrow g(C)^+$ such that $(e_A \mid f(C)) \circ \tilde{f} = f \circ e_C$ and $(e_B \mid g(C)) \circ \tilde{g} = g \circ e_C$. Both $\tilde{f}$ and $\tilde{g}$ are injective homomorphisms and we know by theorem 42 that $\mathfrak{K}$ has the generalized super amalgamation property, hence there is a $\mathfrak{D}^+ \in \mathfrak{K}$ and $k : \mathfrak{A}^+ \rightarrow \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \rightarrow \mathfrak{D}^+$ such that $k \circ f = h \circ g$.

(2) We do the (inverse) operation of compressing, implemented by forming the $\alpha$–neat reduct of the super amalgam $\mathfrak{D}^+$, obtaining an amalgam in $\text{RMA}_\alpha$:

We know from (1) that $k$ and $h$ are injective homomorphisms. Also $k \circ e_A : \mathfrak{A} \rightarrow \mathfrak{R}_\alpha \mathfrak{D}^+$ and $h \circ e_B : \mathfrak{B} \rightarrow \mathfrak{R}_\alpha \mathfrak{D}^+$ are injective homomorphisms and $k \circ e_A \circ f = h \circ e_B \circ g$. Let $\mathfrak{D} = \mathfrak{R}_\alpha \mathfrak{D}^+$. Then we obtain $\mathfrak{D} \in \text{RMA}_\alpha$ and $m : \mathfrak{A} \rightarrow \mathfrak{D}$, $n : \mathfrak{B} \rightarrow \mathfrak{D}$ such that $m \circ f = n \circ g$, where $m = k \circ e_A$ and $n = h \circ e_B$. We have proved that $\mathfrak{D} \in \text{RMA}_\alpha$ is an amalgam.

(3) Finally, we use NS property to show that $\mathfrak{D}$ is a super amalgam of $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathfrak{C}$: To prove that $\mathfrak{D}$ is the required super amalgam, it suffices to show, by symmetry, that if $m(a) \leq n(b)$, for $a \in A$ and $b \in B$, then there exists $t \in \mathfrak{C}$ and $j \in \mathbb{N}$, such that $a^j \leq f(t)$ and $jg(t) \leq b$. So let $a$ and $b$ be as indicated. We have $(k \circ e_A)(a) \leq (h \circ e_B)(b)$, so $k(e_A(a)) \leq h(e_B(b))$. 

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Since $K$ has the generalized super amalgamation property, there exist $z \in C^+$ and $j \in N$ such that $[e_A(a)]^2 \leq f(z)$ and $\overline{g}(z) \leq j[e_B(b)]$. Let $\Gamma = \Delta z \setminus \alpha$ and $z' = e_{\Gamma}(z)$. (Note that $\Gamma$ is finite.) So, we obtain that $e_A(e_{\Gamma}(a)) \leq \overline{f}(e_{\Gamma}(z))$ and $\overline{g}(e_{\Gamma}(z)) \leq e_B(e_{\Gamma}(jb))$. It follows that $e_A(a') \leq \overline{f}(z')$ and $\overline{g}(z') \leq e_B(jb)$. Now by $NS$, we have $z' \in \alpha e_{\Gamma} = Sg^{\alpha e_{\Gamma}}(e_C(C)) = e_C(C)$. So, there exists $t \in C$ with $z' = e_C(t)$. Then we get $e_A(a') \leq \overline{f}(e_C(t))$ and $\overline{g}(e_C(t)) \leq e_B(jb)$. It follows that $e_A(a') \leq (e_A \circ f)(t)$ and $(e_B \circ g)(t) \leq e_B(jb)$. Hence, $a' \leq f(t)$ and $g(t) \leq jb$ and we are done. \hfill \Box

Now we show using the same technique of ‘dilating then compressing’ that $S_{\alpha}$ has the amalgamation property. Here we cannot prove strong amalgamation, a fortiori generalized super amalgamation, because it can be proved, like the classical case, that $S_{\alpha}$ does not have $NS$ (Sayed Ahmed, 2013).

**Theorem 46**

(1) Let $A_0$ and $A_1 \in S_{\alpha}$ Then there exist $B_0$, $B_1 \in K \subseteq \text{RMA}_{\alpha + \omega} j_0 : A_0 \rightarrow \alpha B_0$ and $j_1 : A_1 \rightarrow \alpha B_1$ such that for every injective homomorphism $f : A_0 \rightarrow A_1$ there exists an injective homomorphism $g : B_0 \rightarrow B_1$ such that $g \circ j_0 = j_1 \circ f$.

(2) $S_{\alpha}$ has the amalgamation property.

**Proof** Let $A_0$ and $A_1$ be as in the hypothesis. Fix $i \in \{0, 1\}$. Let $B_i'$ be the dilations constructed in theorem 34, of $A_i$, so that, using the notation in op.cit, $B_i' = \Pi_{R \subseteq \text{RMA}_{\alpha}^{(\sigma)}\overrightarrow{A}/M}$. Let $j_i$ be the neat embedding functions from $A_i$ into $B_i'$, defined, as in op.cit by

$$f_i x = (s_{l_0}^{k_0} \circ \ldots \circ s_{l_{n-1}}^{k_{n-1}} x : (\rho, n, k, l) \in R)/M.$$ 

($f_i$ is an embedding form $A_i$ to $\alpha B_i'$). Now let $h$ be the function from $B_0'$ into $B_1'$ defined by:

$$h(\langle x_\phi : \phi \in R \rangle/M) = \langle f x_\phi : \phi \in R \rangle/M.$$ 

Then it is easy to check that $h$ is a well defined homomorphism such that $h \circ j_0 = j_1$. Take $B_i = Sg^{\alpha e_{\Gamma}} j_i(A_i)$ for $i \in \{0, 1\}$ and define $g = h \upharpoonright B_0$. Then it is not hard to check that $g$ is the desired “lifting” function.

For the second part one uses a similar (but not identical) argument, to that used in the first (lifting) and second (compressing) part of the proof of the previous theorem 45 to obtain an amalgam in $D \in \text{RMA}_{\alpha}$. Here the UNEP is not needed, because we have already lifted the algebras required to be amalgamated to $K$ using ultraproducts. So far, we obtained a representable amalgam. But $S_{\alpha} \subseteq \text{RMA}_{\alpha}$, so we want to obtain an amalgam in (the smaller) class $S_{\alpha}$.

We this amalgam as a homomorphic image of $D$ as follows: Define a transfinite sequence of filters $M_i$ where $i$ ranges over arbitrary ordinals, $M_0 = \{0\}$ and for each $i > 0$ $M_i$ is the set of all elements $x \in D$ such that for some finite $\Gamma \subseteq \alpha$ and all distinct $k, l < \alpha \setminus \Gamma$, $s_k^l x$ is not in $\bigcup_{j < i} M_j$. Then $M_i$ is a filter in $D$. Since $M_i \subseteq M_j$ whenever $i < j$ it follows that $M_i = M_{i+1}$ for sufficiently large $i$, if $\beta$ is the least such ordinal with this property, then $D/M_\beta$ is in $S_{\alpha}$ and is the required amalgam. \hfill \Box

The following easy lemma on extension of filters to super algebras will be used in our next theorem.

**Lemma 47** Let $A, B \in \text{MA}_{\alpha}$ with $B \subseteq A$. Let $M$ be a filter of $B$. Then for every filter $N$ of $A$ such that $N \cap B \subseteq M$, there is a filter $N'$ in $A$ such that $N \subseteq N'$ and
$N' \cap B = M$. Furthermore, if $M$ is a maximal filter of $\mathcal{B}$, then $N'$ can be taken to be a maximal filter of $\mathcal{A}$.

**Proof** The special case when $N = \{1\}$ is straightforward. The general case follows from this one, by considering $\mathcal{A}/N$, $\mathcal{B}/(N \cap \mathcal{B})$ and $M/(N \cap \mathcal{B})$, in place of $\mathcal{A}$, $\mathcal{B}$ and $M$, respectively. \hfill \Box

We understand a *semi-simple* algebra to be a subdirect product of simple algebras (this is the usual algebraic universal definition).

**Theorem 48** The class of semi-simple MA$_\alpha$s has the amalgamation property.

**Proof** Let us first consider simple algebras. Given simple algebras $\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \subseteq \mathcal{A}_2$, we have by theorem 46, since the class of simple algebras is contained in Sc$_\alpha$ an amalgam $\mathcal{C} \in$ RMA$_\alpha$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$. The following reasoning ensures that $\mathcal{C}$ can be chosen to be simple. Let $\mathcal{A} \subseteq \mathcal{C}$. Suppose that $\mathcal{A}$ is simple, and let $I$ be a filter of $\mathcal{C}$. Then $\mathcal{A}$ can be embedded in $\mathcal{C}/I$. Indeed, $I \cap A$ is a filter of $\mathcal{A}$, and is proper, by lemma 37, since $0 \notin I$. Hence $I \cap A = \{1\}$ because $\mathcal{A}$ is simple. But then $\mathcal{A} \cong \mathcal{A}/(I \cap A)$ and the latter embeds in $\mathcal{C}/I$. By taking a maximal filter $I$ we are done.

Now we show that the class of semi-simple algebras has AP. Assume that $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{B}$ are semi-simple and let $h : \mathcal{C} \to \mathcal{A}$ and $g : \mathcal{C} \to \mathcal{B}$ be injective homomorphisms. We want to find a semi-simple amalgam. By semisimplicity, there exist a system $(\mathcal{C}_l : l \in I)$ of simple algebras and $k : \mathcal{A} \to \prod_{l \in I} \mathcal{C}_l$ such that $\pi_l \circ k(\mathcal{A}) = \mathcal{C}_l$ for each $l$ where $\pi_l$ is the projection map.

Fix $l \in I$. Let $t_l = \pi_l \circ k \circ h \circ g^{-1}$. Then $t_l : g(\mathcal{C}) \to t_l(g(\mathcal{C}))$ is a surjective homomorphism and $i_{g(l)} : g(\mathcal{C}) \to \mathcal{B}$. We claim that there exist simple $\mathcal{D}_l$, surjective $j_l : \mathcal{B} \to \mathcal{D}_l$ and injective $i_l : t_l(g(\mathcal{C})) \to \mathcal{D}_l$, such that $j_l \circ i_{g(l)} = i_l \circ t_l$. To see why, let $M$ be a maximal filter of $g(\mathcal{C})$ such that $g(\mathcal{C})/M \cong t_l(g(\mathcal{C}))$ via $t_l(a) = t_l(a)/M$. Then $i_{g(l)}(M)$ is a filter in $\mathcal{B}$ by lemma 47, let $N$ be a maximal filter of $\mathcal{B}$ such that $N \cap i_{g(l)}(g(\mathcal{C})) = i_{g(l)}(M)$. Let $\mathcal{D}_l = \mathcal{B}/N$ and define $j_l(b) = b/N$. Then $j_l : \mathcal{B} \to \mathcal{D}_l$. Let $u = t_l(g(\mathcal{C}))$. Choose $x \in t_l^{-1}(a)$ and define $i_l(a) = j_l \circ i_{g(l)}(x)$. Then $i_l$ is well defined, injective, and as required. Now $\mathcal{C}_l$ and $\mathcal{D}_l$ amalgamate over $t_l(g(\mathcal{C}))$, hence there exist $\mathcal{F}_l$ simple and $m_l, n_l$ such that $n_l \circ i_l = m_l$. Define $m(a) = \langle m_l \circ \pi_l \circ k(a) : l \in I \rangle$ and $n(b) = \langle n_l \circ j_l(b) : l \in I \rangle$ Then for $l \in I$, we have $(m(h(c)))_l = m_l \circ \pi_l \circ k \circ h(c) = n_l \circ j_l \circ g(c) = m_l \circ t_l \circ c = n_l \circ j_l \circ g(c) = n_l \circ j_l \circ g(c) = \langle n(b), n(c) \rangle$. Therefore $m \circ h = n \circ g$, and we are done. \hfill \Box

We define another restricted varying interpolation property.

**Definition 49** Let $\mathcal{A} \in$ MA$_\alpha$. $\mathcal{A}$ has the restricted varying interpolation property, briefly RVIP, if whenever $x \leq z$, $x \in \mathcal{G}^Y$, and $z \in \mathcal{G}^Z$, with $Y \cap Z = \emptyset$, then there exists $n \in N$, such that either $x^n = 0$ or $ny = 1$.

**Theorem 50** Let $\beta$ be any cardinal $> 0$ and $\alpha$ be an infinite ordinal. Then $\mathfrak{R}_{\beta}\text{RMA}_\alpha$ has the restricted varying interpolation property.

**Proof** Let $\mathcal{A} = \mathfrak{R}_{\beta}\text{RMA}_\alpha$ and let $X_1, X_2 \subseteq \beta$ be disjoint sets. We can assume without loss of generality that $X_1 \cup X_2 = \beta$. Assume that $a \in \mathcal{A}_1 = \mathcal{G}^aX_1$ and $b \in \mathcal{A}_2 = \mathcal{G}^bX_2$ such that $a \leq b$. Since $X_1 \cap X_2 = \emptyset$, we have $\mathcal{A}_0 = \mathcal{G}^a(X_1 \cap X_2)$ is embeds in $\mathcal{G}^aX_1$ and $\mathcal{G}^bX_2$, respectively via the inclusion maps $i_0$ and $i_1$. From theorem 45, since $\mathcal{A}_0 \in$ Sc$_\alpha$ there are a $\mathcal{D} \in$ RMA$_\alpha$ and injective homomorphisms $m_1 : \mathcal{A}_1 \to \mathcal{D}$ and $m_2 : \mathcal{A}_2 \to \mathcal{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and $(\forall x \in A_1)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(\exists n \in N)(x^n \leq i_j(z) \text{ and } i_k(z) \leq ny))$ where $\{j, k\} = \{1, 2\}$. Now since $\mathcal{A}$ is free, there exists a
homomorphism $f : \mathfrak{A} \to \mathfrak{D}$ such that $f \upharpoonright \mathfrak{A}_1 = m_1$ and $f \upharpoonright \mathfrak{A}_2 = m_2$. Since $f(a) \leq f(b)$ it follows that $m_1(a) \leq m_2(b)$. Hence there exists $z \in \mathfrak{A}_0$ such that $a^n \leq z \leq nb$. But $z \in \{0, 1\}$, hence the required. \hfill $\Box$

Before stating our next result, we need:

**Definition 51** $K$ has the (strong) embedding property if it has the (strong) amalgamation property when the base algebra is the two element algebra.

Using exactly the methods in the proof of theorem 45 by observing that the two element algebra is in $\mathfrak{D}_\alpha$, together with theorem 50, we get:

**Corollary 52** Let $\alpha \geq \omega$. Then $\text{RMA}_\alpha$ has the strong embedding property.

On the other hand, most negative results in (Madárasz & Sayed Ahmed, 2009; Madárasz & Sayed Ahmed, 2012) can be obtained by bouncing them back to the classical case, as we already did once before in theorem 40. As a sample we give another negative property, namely, that epimorphisms (in the categorial sense) in the category $\text{RMA}_\alpha$ are not surjective. This implies that the corresponding many valued logic does not have the Beth definability property (Madárasz & Sayed Ahmed, 2012). We say that a class $K$ of algebras has $ES$ if epimorphisms are surjective in $K$. Else, we say that $ES$ fails in $K$.

**Theorem 53** In any class $K$, such that $\text{MA}_\alpha \subseteq K \subseteq \text{RMA}_\alpha$, $ES$ fails.

**Proof** In (Madárasz & Sayed Ahmed, 2009) two weak set algebras in the signature of $\text{QA}_\alpha$, $\mathfrak{A}$ and $\mathfrak{B}$ are given such that $\mathfrak{A} \subseteq \mathfrak{B}$ is a non-surjective epimorphism in $\text{QA}_\alpha$.

Take $U_0, U_1, \ldots, U_n, n < \omega$ a system of mutually disjoint finite sets such that $|U_0| = 3$ and $|U_{i+1}| = 2$ for $i \in \omega$. Let $U = \bigcup \{U_i : i \in \omega\}$, and let $T^+ = U_0 \times U_1 \times U_2 \times \ldots$. Then $T^+ \subseteq \omega U$. Now fix $q \in T^+$, and let $V$ be the weak space $V = \omega U^{(q)}$. Let $T = V \cap T^+$, and let $C$ be the full weak set algebra with unit $V$. Then $T \subseteq C$. $T$ is split into two relations $R$ and $T \setminus R$ as follows: Suppose that $U_0 = \{a, b, c\}$ with $a, b, c$ pairwise distinct. Define $X = \{s \in T : s_0 = a$ and $|\{i \in \omega : s_i \neq q_i\}|$ is even $\}$. Define $Y = \{s \in T : s_0 \in \{b, c\}$ and $|\{i \in \omega : s_i \neq q_i\}|$ is odd $\}$.

Now define $R = X \cup Y$ and $R^- = T \setminus R$. Let $\mathfrak{B} = \mathfrak{C}g^\mathfrak{C}\{R\}$ and $\mathfrak{A} = \mathfrak{C}g^\mathfrak{C}\{R, X\}$. Then it is shown in $\text{op.cit}$ that $X \notin \mathfrak{B}$, and that the inclusion $\mathfrak{B} \subseteq \mathfrak{A}$ is an epimorphism in $\text{QA}_\alpha$. Assume for contradiction, that $\mathfrak{B} \subseteq \mathfrak{A}$ is not an epimorphism in $\text{MA}_\alpha$. Then there exists $\mathfrak{D}' \in \text{MA}_\alpha$ that witnesses that this inclusion is not an epimorphism, meaning that there are epimorphisms $f, g : \mathfrak{D} \to \mathfrak{D}'$ such that $f(X) \neq g(X)$. Like before, take $\mathfrak{D}$ to be the set of idempotent elements in $\mathfrak{D}'$, then $\mathfrak{D} \subseteq \text{QA}_\alpha$ and $\mathfrak{D}$ witnesses that $\mathfrak{B} \subseteq \mathfrak{A}$ is not an epimorphism in $\text{QA}_\alpha$. Indeed, we have $f(\mathfrak{B}) \cup g(\mathfrak{B}) \subseteq \mathfrak{D}$, as $f$ and $g$ preserve idempotency because they are homomorphisms. This is a contradiction and we are done. \hfill $\Box$

**Example 54** (1) We show that the inclusions $\mathfrak{D}_\alpha \subseteq \mathfrak{S}_\alpha \subseteq \text{RMA}_\alpha$ are all proper. The first inclusion can be witnessed on Boolean polyadic algebras. As we work in the classical case, we find it more appropriate to deal with ideals the dual of filters. In this connection if $\mathfrak{A}$ is a $\text{QA}_\alpha$ then $I \subseteq A$ is a $\rightarrow$ $I$ is a Boolean ideal, and furthermore if $k < \alpha$, then $\mathfrak{C}k\mathfrak{A} \in \mathfrak{A}$. For the first inclusion. Let $m \geq 2$ be a finite ordinal. Take $\mathfrak{A} = \mathfrak{C}(^\omega m)$, with the Boolean operations of intersection and (usual) complementation, and cylindrifiers and substitution operations defined like in example 11. Then $\mathfrak{A} \in \text{RMA}_\alpha$, of course, in fact it is easy to see that $\mathfrak{A} \in \mathfrak{S}_\alpha$. However, $\mathfrak{A}$ is not in $\mathfrak{D}_\alpha$ because for every $s \in ^\omega m$, we have $\Delta(s) = \{i \in \alpha : c_i\{s\} \neq \{s\}\} = \alpha$.

For the second inclusion, let $\mathfrak{A} = \mathfrak{C}(^\alpha \alpha, [0, 1])$ as defined in example 11 taken in the
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type \((FT_\alpha,\psi_\omega(\alpha))\). Then \(\mathfrak{A} \in \text{RMA}_\alpha\). We show that \(\mathfrak{A} \notin \text{Sc}_\alpha\). Let \(\Theta\) be a bijection from \(\alpha\) to \(\alpha\). Let \(f : \alpha \alpha \to [0,1]\) be defined by \(f(\Theta) = 1\) and \(f(x) = 0\), otherwise. Then for distinct \(i,j \in \alpha\) and \(x \in \alpha\), \(\delta^i_j f(x) = f(x \circ [i,j]) = 0\), because for any \(x \in \alpha\), the map \(x \circ [i,j]\) satisfies \(x(i) = x(j)\), hence is not a bijection.

(2) Now we show that \(\text{Dc}_\alpha\) does not have the amalgamation property (recall that it has the super amalgamation property but with respect to the larger class \(\text{RMA}_\alpha\), theorem 45) and that \(\text{Sc}_\alpha\) does not have the strong amalgamation property (recall that \(\text{Sc}_\alpha\) has the amalgamation property, theorem 46). For the first required, let \(\mathfrak{A}, \mathfrak{B} \in \text{Dc}_\alpha\), for which there exist \(x \in \mathfrak{A}\) and \(y \in \mathfrak{B}\), such that \(\Delta x \cup \Delta y = \alpha\). Clearly such algebras cannot be amalgamated by a \(\text{Dc}_\alpha\) over the common minimal 2 element subalgebra of \(\mathfrak{A}\) and \(\mathfrak{B}\).

For the second required we use the well known fact (*): An algebra \(\mathfrak{A}\) (in the universal algebraic sense) is semisimple \(\iff\) the intersection of all maximal proper congruences on \(\mathfrak{A}\) is the identity (congruence) relation, see e.g. (Henkin et al., 1971, Theorem 0.3.49). In (Madárasz & Sayed Ahmed, 2009), it is shown that the algebras \(\mathfrak{A}\) and \(\mathfrak{B}\) in theorem 53 can be modified to be semisimple as follows:

Let \(U_0, \ldots, U_i, \ldots, i < \omega\) be as in the proof of theorem 53. That is \((U_i : i < \omega)\) is a system of mutually disjoint finite sets such that \(|U_0| = 3\) and \(|U_{i+1}| = 2\) for \(i \in \omega\). Let \(n \in \omega\). Define

\[
U_n = \bigcup_{i<n} U_i
\]

\[
T_n = \{s \in \omega U_n : (\forall i < n) s_i \in U_i\}
\]

\[
X_n = \{s \in T_n : s_0 = a \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is even}\}
\]

\[
Y_n = \{s \in T_n : s_0 \in \{b, c\} \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is odd}\}
\]

\[
R_n = X_n \cup Y_n
\]

\(\mathfrak{C}_n\) is the full quasi-polyadic set algebra with base \(U_n\). Now let

\[
\mathfrak{C} = \prod \langle \mathfrak{C}_{n+2} : n \in \omega \rangle
\]

\[
\mathfrak{R} = \langle R_{n+2} : n \in \omega \rangle
\]

\[
\mathfrak{X} = \langle X_{n+2} : n \in \omega \rangle
\]

\[
\mathfrak{B} = \mathfrak{C}^\mathfrak{C} \{\mathfrak{R}\}
\]

\[
\mathfrak{A} = \mathfrak{C}^\mathfrak{C} \{\mathfrak{R}, \mathfrak{X}\}.
\]

Then it is shown in (Madárasz & Sayed Ahmed, 2009) that \(\mathfrak{B}\), \(\mathfrak{A}\) are semisimple quasi-polyadic equality algebras and that the inclusion \(\mathfrak{B} \subseteq \mathfrak{A}\) is an epimorphism that is not surjective in \(\text{QA}_\alpha\), because \(\mathfrak{X} \notin \mathfrak{B}\).

Arguing like in the last part of the proof of the previous theorem 53, we get that \(\mathfrak{B} \subseteq \mathfrak{A}\) is an epimorphism in \(\text{MA}_\alpha\). Also \(\mathfrak{B}\) and \(\mathfrak{A}\) are semisimple as \(\text{MA}_\alpha\) algebras, because in this case \(\text{MA}_\alpha\) filters coincide with \(\text{QA}_\alpha\) filters. So in \(\mathfrak{A}\) we have \(\bigcap \{F : F \text{ a maximal } \text{MA}_\alpha \text{ filter in } \mathfrak{A}\} \cap \{F : F \text{ a maximal } \text{QA}_\alpha \text{ filter in } \mathfrak{A}\} = \{0\}\), and an entirely analogous situation holds in \(\mathfrak{B}\). By (*) both \(\mathfrak{A}\) and \(\mathfrak{B}\) are semisimple as \(\text{MA}_\alpha\) algebras. We have shown that \(\text{Sc}_\alpha\) does not have ES. It is known (Madárasz & Sayed Ahmed, 2009) that for a class \(K\) or which \(\text{SK} = K\), the amalgamation property and \(\text{ES}\) holds for \(K\) \(\iff\) \(K\) has the strong amalgamation property. Since \(\text{SSc}_\alpha = \text{Sc}_\alpha\), and as just shown \(\text{ES}\) fails for \(\text{Sc}_\alpha\), we readily conclude that \(\text{Sc}_\alpha\) does

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not have the strong amalgamation property.

In what follows, we allow finite reducts (that is algebras with finitely many indices) defined by injections $\rho : n \to \alpha$, with finite domain $n$ as in definition 33. The following class is the MV analogue of the class of cylindric algebras introduced in (Henkin et al., 1971, Item (iii) of Theorem 2.6.50). Using the neat embedding theorem 28 yet again, together with ultraproducts, it can be shown exactly like in (Henkin et al., 1971) that such a class consists only of representable $\text{MA}_\alpha$’s.

**Definition 55** $\mathfrak{A} \in \text{MA}_\alpha$ is called a weak substitution algebra, a $\text{WSC}_\alpha$ for short, if for every finite injective map $\rho$ into $\alpha$, and for every $x, y \in \mathfrak{A}$, $x \neq y$, there is a function $h$ and $k < \alpha$ such that $h$ is an endomorphism of $\mathfrak{A}_k = \mathfrak{A}$, $k \in \alpha \setminus \text{rng}(\rho)$, $c_k \circ h = h$ and $h(x) \neq h(y)$.

**Theorem 56** For any ordinal $\alpha \geq \omega$, $\text{WSC}_\alpha \subseteq \text{RMA}_\alpha$.

It is clear that $\text{SC}_\alpha \subseteq \text{WSC}_\alpha$. We build on the example in 54 to show that $\text{SC}_\alpha$ is properly contained in $\text{WSC}_\alpha$.

**Example 57** Let $\mathfrak{A} = \mathfrak{F}(\omega, [0, 1])$. We have seen in example 54 that $\mathfrak{A} \notin \text{SC}_\alpha$. We now show that $\mathfrak{A} \in \text{WSC}_\alpha$. Let $x \in \mathfrak{A}$ be non-zero. Let $\rho$ be a one to one finite function with $\text{rng}(\rho) \subseteq \alpha$. We want to find $H$ as in the conclusion of the definition of a $\text{WSC}_\alpha$. Let $\tau \in \omega \alpha$ such that $k \notin \text{rng} \tau$, $\tau \upharpoonright \text{rng} \rho \subseteq 1d$ and $\tau$ is one to one. Let $H : \mathfrak{A} \to \mathfrak{A}$ by $H(Y) = \{ \phi \in \mathfrak{F}(\omega, [0, 1]) : \phi \circ \tau \in Y \}$. Then $H$ is as required.

**Theorem 58** The following conditions are equivalent:

1. $\text{WSC}_\alpha$ is elementary.
2. $\text{WSC}_\alpha$ is closed under ultraproducts.
3. $\text{WSC}_\alpha = \text{RMA}_\alpha$.
4. $\text{WSC}_\alpha$ is a variety.

**Proof**

(1) $\to$ (2) is trivial.

(2) $\to$ (3). If $\text{WSC}_\alpha$ is closed under ultraproducts, then because it is closed under forming subalgebras, we would have $\text{RMA}_\alpha \supseteq \text{WSC}_\alpha = \text{SPuWSC}_\alpha \supseteq \text{SPuSC}_\alpha = \text{RMA}_\alpha$.

(3) $\to$ (4). Since $\text{RMA}_\alpha$ is a variety.

(4) $\to$ (1). Trivial.

We do not know whether any of the above statements is true or not implying that $\text{WSC}_\alpha = \text{RMA}_\alpha$. If equality holds then by theorem 40, $\text{WSC}_\alpha$ will not have the amalgamation property. If $\text{WSC}_\alpha \subseteq \text{RMA}_\alpha$, we do not know whether it has the amalgamation property or not.

We refer to (Sayed Ahmed, 2011b) for undefined notation and terminology used in the next definition and theorem from category theory.

**Definition 59** The neat reduct functor, $\text{Nr}$ for short, is defined from $K = \{ \mathfrak{A} \in \text{MA}_{\alpha + \omega} : \mathfrak{A} = \mathfrak{A}_\alpha \}$ to $\text{RMA}_\alpha$ by sending every object $\mathfrak{A} \in K$, to $\text{Nr}(\mathfrak{A})$, and sending injective homomorphisms to their restrictions, that is for $\mathfrak{A}, \mathfrak{B} \in K$ and $f : \mathfrak{A} \to \mathfrak{B}$, an injective homomorphism, $\text{Nr}(f) = f \upharpoonright \text{Nr}(\mathfrak{A})$.

It is clear that $f(\text{Nr}(\mathfrak{A})) \subseteq \text{Nr}(\mathfrak{B})$, hence this functor is well defined. Here we are restricting morphisms to only injective homomorphisms.

**Theorem 60** Let $\alpha$ be an infinite ordinal. Then the following hold:

1. $\text{RMA}_\alpha$ does not have UNEP.
(2) There is an $\mathfrak{A} \in \text{RMA}_\alpha$ that does not have a universal map with respect to the functor $\text{Nr}$. Hence $\text{Nr}$ does not have a right adjoint.

Proof

(1) Let $\mathfrak{A} \in \text{RQA}_\alpha$ be an algebra and $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{A} \to \mathfrak{C}$ be injective homomorphisms, for which there are no $\mathfrak{D} \in \text{RQA}_\alpha$ and no injective homomorphisms $m : \mathfrak{B} \to \mathfrak{D}$, $n : \mathfrak{C} \to \mathfrak{D}$, such that $m \circ f = n \circ g$. Such $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $f$ and $g$ exist (Sayed Ahmed, 2011a). Assume for contradiction that $\mathfrak{A}$ has the UNEP. Then exactly like in the proof of theorem 45, $f$ and $g$ lift to injective homomorphisms on their unique dilations, and an amalgam $\mathfrak{D}$ can be found in $\mathfrak{K}$, giving an amalgam $\text{Nr}_\alpha \mathfrak{D}$ in $\text{RMA}_\alpha$. Taking the subalgebra consisting of idempotent elements, like in the proof of theorem 40, we get an amalgam in $\text{RQA}_\alpha$, which is a contradiction.

(2) This is the same argument used in (Sayed Ahmed, 2011b) for the classical case. Assume that $\mathfrak{A} \in \text{RMA}_\alpha$ does not have UNEP. Then by negating definition 43 $\mathfrak{A}$ generates two non-isomorphic $\omega$–dilations in $\text{RMA}_{\alpha+\omega}$. The existence of a universal map for $\mathfrak{A}$ will force that these two algebras are actually isomorphic, fixing $\mathfrak{A}$ pointwise, and this cannot happen.

In more detail, assume that $\mathfrak{A}$ neatly embeds into $\mathfrak{B}$ via $e_B$ and into $\mathfrak{B}'$ via $e_{B'}$. Let $(e, \mathfrak{C})$ be a universal map for $\mathfrak{A}$, so that $\mathfrak{A}$ neatly embeds into $\mathfrak{C}$ via $e$. (See the above diagram). By universality, there exists isomorphisms $f : \mathfrak{C} \to \mathfrak{B}$ and $k : \mathfrak{C} \to \mathfrak{B}'$ such that $f \circ e = e_B$ and $k \circ e = e_{B'}$. The maps are injective by definition, they are surjective, because $\mathfrak{A}$ is contained in $\mathfrak{C}$ and it generates both $\mathfrak{B}$ and $\mathfrak{B}'$.

We infer that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic, but we want more. We want to exclude special isomorphisms (in principal, isomorphisms can exist as long as they do not fix $\mathfrak{A}$ pointwise). We have $h = k \circ f^{-1} : \mathfrak{B} \to \mathfrak{B}'$ is an isomorphism such that $h \circ e_B = e_{B'}$, and this isomorphism is as required, leading to a contradiction. For the last part, it is known (Herrlich & Strecker, 1973, Theorem 27.3 pp.196), using the notation in op.cit that if $G : K \to L$ is a functor such that each $\mathfrak{B} \in \text{Ob}(K)$ has a $G$ universal map $(\mu_B, \mathfrak{A}_B)$, then there exists a unique adjoint situation $(\mu, \epsilon) : F \to G$ such that $\mu = (\mu_B)$ and for each $\mathfrak{B} \in \text{Ob}(L)$, $F(\mathfrak{B}) = \mathfrak{A}_B$. Conversely, if we have an adjoint situation $(\mu, \epsilon) : F \to G$ then for each $\mathfrak{B} \in \text{Ob}(L)$ $(\mu_B, F(\mathfrak{B}))$ have a $G$ universal map.
5. Interpolation in Various Reducts of $PMA_\alpha$

We start with $PMA_\alpha$ itself. That is, we address the case when $G = \circ \alpha$ and $T = \varphi(\alpha)$. The corresponding logic is the many valued version of the logic studied by Keisler (Keisler, 1963) algebraically reflected by Halmos’ polyadic algebras (Halmos, 1962). We recover Keisler’s completeness theorem (and more) in the many valued context. The results in (Daigneault, 1963; Daigneault & Monk, 1963; Keisler, 1963) become special cases when we have idempotency.

5.1 Interpolation for Free $PMA_\alpha$s

We need the notion of neat reducts of polyadic MV algebras when the type is $(\circ \alpha, \varphi(\alpha))$.

**Definition 61** (Henkin et al., 1985, Definition 5.4.16). Let $\beta$ be an infinite ordinal. Let $\mathfrak{A} = (A, \oplus, \odot, \ominus, 0, 1, c(\Gamma), s_\tau)_{\Gamma \subseteq \beta, \tau \in \alpha} \in PMA_\beta$. Let $\mathfrak{N}_J \mathfrak{B} = \{a \in A : c_{[\beta \setminus J]}a = a\}$. Then $\mathfrak{N}_J \mathfrak{B} = (\mathfrak{N}_J \mathfrak{B}, \oplus, \odot, \ominus, c(\Gamma), s_\tau')_{\Gamma \subseteq \beta, \tau \in J}$ where for $\tau \in J$, $s_\tau' = s_{\tau \cup \{\iota \setminus J\}}$. The structure $\mathfrak{N}_J \mathfrak{B}$ is an algebra, called the $J$ compression of $\mathfrak{B}$. When $J = \alpha$, $\alpha$ an infinite ordinal, then $\mathfrak{N}_\alpha \mathfrak{B} \in PMA_\alpha$ and it is called the neat $\alpha$ reduct of $\mathfrak{B}$. The elements of $\mathfrak{N}_\alpha \mathfrak{B}$ are said to be $\alpha$-dimensional.

The next definition is taken from (Daigneault & Monk, 1963). Formulated for polyadic (Boolean) algebras of dimension $\alpha$ in op.cit, they apply equally well to $PMA_\alpha$s.

**Definition 62** Let $\mathfrak{A} \in PMA_\alpha$.

(1) If $J \subseteq \alpha$, an element $a \in A$ is independent of $J$ if $c_{[\beta \setminus J]}a = a$; $J$ supports $a$ if $a$ is independent of $\alpha \setminus J$. The effective degree of $\mathfrak{A}$ is the smallest cardinal $\epsilon$ such that each element of $\mathfrak{A}$ admits a support whose cardinality does not exceed $\epsilon$.

(2) The local degree of $\mathfrak{A}$ is the smallest cardinal $m$ such that each element of $\mathfrak{A}$ has support of cardinality $< m$. The effective cardinality of $\mathfrak{A}$ is $\epsilon = |\mathfrak{N}_J \mathfrak{A}|$ where $|J| = \epsilon$. (This is independent of $J$).

**Theorem 63** Let $\beta$ be a non–zero cardinal and $\alpha$ be an infinite ordinal. Then $\mathfrak{N}_\beta PMA_\alpha$ has the VIP.

**Proof** Here we use the techniques in (Sayed Ahmed, 2010a) together with those in the above proof for $\mathfrak{C}$. Assume that $a \leq b$ is the given inequality required to be interpolated. Like in the proof of theorem 19, we divide the proof into (the analogous) three parts. The second part is here more involved, than its counterpart in the proof of theorem 19, and requires non–trivial manipulation of cardinals. In this (second) part, we use the axiom of choice. Proving that the ‘representability’ maps defined during the proof are homomorphisms, is tedious but not too difficult, once we have built the (three) maximal filters, denoted below by $H^*$, $F_1$ and $F_2$, corresponding to the three maps proving representability. In fact, the arguments used for this aforementioned part of the proof is very similar, but not identical, to the arguments used in the corresponding parts in theorem 19. To draw further analogy with the proof of theorem 19, we use the same notation concerning the three maximal filters built in both proofs from which we construct the (required) representability functions and, using the terminology in the outline preceding theorem 19, the corresponding models, ending up with a model of $a – b$, contradicting $a \leq b$. 

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(1) **Forming a suitable dilation:** Let \( m \) be the local degree of \( \mathfrak{A} = \mathfrak{Tr}_p \mathbb{PMA}_\alpha \), \( \epsilon \) its effective cardinality and \( n \) be any cardinal such that \( n \geq \epsilon \) and \( \sum_{s < m} n^s = n \). Let \( X_1, X_2 \subseteq A \), \( a \in \mathfrak{S}_g^X X_1 \) and \( b \in \mathfrak{S}_g^X X_2 \) such that \( a \leq b \). We want to find an interpolant. Then there exists \( \mathfrak{B} \in \mathbb{PMA}_\alpha \) such that \( \mathfrak{A} \subseteq \mathfrak{Tr}_p \mathfrak{B} \) and \( \mathfrak{A} \) generates \( \mathfrak{B} \). Being a minimal dilation of \( \mathfrak{A} \), the local degree of \( \mathfrak{B} \) is the same as that of \( \mathfrak{A} \), in particular each \( x \in \mathfrak{B} \) admits a support of cardinality \( < \mathrm{m} \). Also for all \( X \subseteq A \), \( \mathfrak{S}_g^X X = \mathfrak{Tr}_p \mathfrak{S}_g^X X \), this can be proved exactly like the polyadic case (Daigneault & Monk, 1963, Theorems 3.1, 3.2 and pp.166).

(2) **Constructing the two maximal Henkin filters eliminating quantifiers:** Assume for contradiction such an interpolant does not exist. Then like in the proof of theorem 19, there exists no interpolant in \( \mathfrak{S}_g^X (X_1 \cap X_2) \). Indeed, let \( c \) be an interpolant in \( \mathfrak{S}_g^X (X_1 \cap X_2) \). Let \( \Gamma = (n \setminus \alpha) \cap \Delta c \). Then \( a \leq c' \). Also \( b = c_{(\Gamma)} b \), so that \( c' \leq b \). Hence \( a \leq c' \leq b \). But

\[
c' \in \mathfrak{Tr}_p \mathfrak{S}_g^X (X_1 \cap X_2) = \mathfrak{S}_g^X (X_1 \cap X_2) = \mathfrak{S}_g^X (X_1 \cap X_2).
\]

Let \( \mathfrak{A}_1 = \mathfrak{S}_g^X X_1 \) and \( \mathfrak{A}_2 = \mathfrak{S}_g^X X_2 \). Let \( Z_1 = \{(J, p) : J \subseteq n, |J| < m, p \in A_1\} \) and define \( Z_2 \) similarly with \( A_2 \) replacing \( \mathfrak{A}_1 \). Then \( |Z_1| = |Z_2| \leq n \). To show that \( |Z_1| \leq n \), let \( K \) be a subset of \( n \) of cardinality \( \epsilon \), the effective degree of \( \mathfrak{A}_1 \). Then every element \( p \) of \( \mathfrak{A}_1 \) is of the form \( s_{\sigma q} \) with \( q \in A_{1K} \) and \( \sigma \in \mathfrak{N} n \). The number of subsets \( J \) of \( n \) such that \( |J| < m \) is at most \( \sum_{s < m} n^s = n \). Let \( q \in A_{1K} \) have a support of cardinality \( s < m \). Then the number of distinct elements \( s_{\sigma q} \) with \( \sigma \in \mathfrak{N} n \) is at most \( n^s \leq n \). Hence there is at most \( n \cdot \epsilon \) elements \( s_{\sigma q} \) with \( \sigma \in \mathfrak{N} n \) and \( q \in A_{1K} \). Hence \( |Z_1| \leq n \cdot n \cdot \epsilon = n \). Let

\[
\langle (k_i, x_i) : i \in n \rangle \quad \text{and} \quad \langle (l_i, y_i) : i \in n \rangle
\]

be enumerations of \( Z_1 \) and \( Z_2 \) respectively, possibly with repetitions. Now there are two functions \( u \) and \( v \) such that for each \( i < n \) \( u_i, v_i \) are elements of \( \mathfrak{N} n \) with

- \( u_i \mid n \setminus k_i = Id \),
- \( u_i \mid k_i \) is injective,
- \( x_j \) and \( y_j \) and \( a \) and \( b \) are independent of \( u_i(k_i) \) for all \( j \leq i \),
- \( s_{u_i} \) \( x_j \) is independent of \( u_i(k_i) \) for all \( j < i \),
- The same conditions replacing \( u_i \) by \( v_i \) and \( k_i \) by \( l_i \) hold,
- \( v_i(l_i) \cap u_i(k_j) = \emptyset \) for all \( j \leq i \).

The existence of such \( u \) and \( v \) can be proved by by transfinite recursion (Sayed Ahmed, 2010a). As before, let

\[
Y_1 = \{a\} \cup \{-c_{(k_i)} x_i \oplus s_{u_i} x_i : i \in n\},
\]

\[
Y_2 = \{-b\} \cup \{-c_{(l_i)} y_i \oplus s_{v_i} y_i : i \in n\},
\]

\[
H_1 = \mathfrak{f}^{\mathfrak{Tr}_p, \mathfrak{S}_g^X (X_1)} Y_1, \quad H_2 = \mathfrak{f}^{\mathfrak{Tr}_p, \mathfrak{S}_g^X (X_2)} Y_2,
\]

\[
H = \mathfrak{f}^{\mathfrak{Tr}_p, \mathfrak{S}_g^X (X_1 \cap X_2)} [(H_1 \cap \mathfrak{S}_g^X (X_1 \cap X_2) \cup (H_2 \cap \mathfrak{S}_g^X (X_1 \cap X_2))]
\]

Then we claim that again \( H \) is a proper filter of \( \mathfrak{S}_g^X (X_1 \cap X_2) \). To prove this it is sufficient to consider any pair of finite, strictly increasing sequences of ordinals \( \eta(0) < \eta(1) \cdots < \eta(n - 1) < n \) and \( \xi(0) < \xi(1) < \cdots < \xi(m - 1) < n \),

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and to prove that the following condition holds:

For any \( b_1, b_2 \in \mathfrak{S} \mathfrak{g}^\mathfrak{B}(X_1 \cap X_2) \) such that

\[
d^l \odot \prod_{i<m} (-c_{(k_\eta(i))} x_\eta(i) \oplus s_{u_\eta(i)} x_\eta(i))^k_i \leq b_1
\]

and

\[
(-b)^k \odot \prod_{i<n} (-c_{(l_\xi(i))} y_\xi(i) \oplus s_{v_\xi(i)} y_\xi(i))^k_i \leq b_2
\]

we have

\[
b_1 \odot b_2 \neq 0.
\]

Fixing \( k, l \in \omega \) this can be proved by induction on \( n + m \). The base of the induction is exactly like in the proof of theorem 19. Now assume that \( n + m > 0 \). For the time being suppose that \( \eta(n - 1) > \xi(m - 1) \). Let \( i < n \). Then \( u_i \in [n, n] \), and \( u_i(k_i) \subseteq n \). Let \( k'_{\eta(n-1)} = u_{\eta(n-1)}(k_{\eta(n-1)}) \). Apply \( c_{k'_{\eta(n-1)}} \) to both sides of the first inclusion of (1). By \( a \) is independent of \( k'_{\eta(n-1)} \) we have \( c_{k'_{\eta(n-1)}} a = a \), and by noting that \( c_{(\Gamma)}(c_{(\Gamma)} x \oplus y) = c_{(\Gamma)} x \odot c_{(\Gamma)} y \), we get

\[
d^l \odot c_{k'_{(\eta(n-1))}} \prod_{i<n} (-c_{(k_\eta(i))} x_\eta(i) \oplus s_{u_\eta(i)} x_\eta(i)) \leq c_{k'_{(\eta(n-1))}} b_1,
\]

hence,

\[
d^l \odot \prod_{i<n} c_{k'_{(\eta(n-1))}} (-c_{(k_\eta(i))} x_\eta(i) \oplus s_{u_\eta(i)} x_\eta(i)) \leq c_{k'_{(\eta(n-1))}} b_1.
\]

Let \( c^\partial_{(\Gamma)}(x) = -c_{(\Gamma)}(x) \). \( c^\partial_{(\Gamma)} \) is the algebraic counterpart of the universal quantifier. Now apply \( c^\partial_{k'_{(\eta(n-1))}} \) to the second inclusion of (1). By noting that \( c^\partial_{(\Gamma)} \), the dual of \( c_{(\Gamma)} \), distributes over the Boolean meet and by \( b \) independent of \( k'_{\eta(n-1)} \) we get

\[
(-b)^k \prod_{j<m} c^\partial_{k'_{(\eta(n-1))}} (-c_{(l_\xi(j))} y_\xi(j) \oplus s_{v_\xi(j)} y_\xi(j))^k_j \leq c^\partial_{k'_{(\eta(n-1))}} b_2.
\]

Before proceeding, like we did in the dimension complemented case, we formulate (and prove) a subclaim that will enable us to eliminate the generalized quantifier \( c_{k'_{(\eta(n-1))}} \) (and its dual) from (1) (and (4)) above.

For the sake of brevity set for each \( i < n \) and each \( j < m \):

\[
z_i = -c_{(k_\eta(i))} x_\eta(i) \oplus s_{u_\eta(i)} x_\eta(i)
\]

and

\[
t_j = -c_{(l_\xi(j))} y_\xi(j) \oplus s_{v_\xi(j)} y_\xi(j).
\]

Then (5) and (6) below hold:
\[ c(k_{\eta(n-1)}')z_i = z_i \text{ for } i < n - 1 \text{ and } c(k_{\eta(n-1)}')z_{n-1} = 1 \]  

(5)

\[ c^0(k_{\eta(n-1)}')t_j = t_j \text{ for all } j < m. \]  

(6)

**Proof of** \( c(k_{\eta(n-1)}')z_i = z_i \text{ for } i < n - 1. \)

Both \( c(k_{\eta(n-1)})x_{\eta(i)} \) and \( k_{\eta(n-1)}x_{\eta(i)} \) are independent of \( k'_{\eta(n-1)} \), it thus follows that

\[ c(k_{\eta(n-1)})(-c(k_{\eta(n-1)})x_{\eta(i)}) = -c(k_{\eta(n-1)})x_{\eta(i)} \]

and

\[ c(k_{\eta(n-1)}')s_{u_{\eta(n-1)}}x_{\eta(i)} = s_{u_{\eta(n-1)}}x_{\eta(i)}. \]

Finally, by \( c(k_{\eta(n-1)}) \) distributing over the Boolean join, we get

\[ c(k_{\eta(n-1)}')z_i = z_i \text{ for } i < n - 1. \]

**Proof of** \( c(k_{\eta(n-1)}')z_{n-1} = 1. \)

Computing we get, by \( x_{\xi(n-1)} \) independent of \( k'_{\eta(n-1)} = u_{\eta(n-1)}(k_{\eta(n-1)}) \), the following:

\[ d = c(k_{\eta(n-1)}')(-c(k_{\eta(n-1)})x_{\eta(n-1)} \oplus s_{u_{\eta(n-1)}}x_{\eta(n-1)}) \]

\[ = c(k_{\eta(n-1)}') - c(k_{\eta(n-1)})x_{\eta(n-1)} \oplus c(k_{\eta(n-1)})s_{u_{\eta(n-1)}}x_{\eta(n-1)} \]

\[ = -c(k_{\eta(n-1)})x_{\eta(n-1)} \oplus c(k_{\eta(n-1)})s_{u_{\eta(n-1)}}x_{\eta(n-1)} \]

To carry on with this computation, we let for the sake of brevity and better readability \( X = k_{\eta(n-1)} \), \( u = u_{\eta(n-1)} \), \( Y = u(X) = k'_{\eta(n-1)} \), and \( x = x_{\eta(n-1)}. \) Choose \( t \in \mathbb{n} \) with \( t \upharpoonright (n \setminus Y) \cup X = u \upharpoonright (n \setminus Y) \cup X \) and \( Y \cap (Y \setminus X) = \emptyset. \) Then \( t^{-1}(Y) = X \) and \( t \upharpoonright X \) is one to one. Now

\[ c(Y)s_{u}x = c(Y)s_{u}c(Y)x \]

\[ = c(Y)s_{u}c(Y)x \]

\[ = s_{u}c(X)c(Y)x \]

\[ = s_{u}c(X)c(Y)x \]

\[ = c(X)c(Y)x. \]

It follows that

\[ d = -c_{k_{\eta(n-1)}}x_{\eta(n-1)} \oplus c_{k_{\eta(n-1)}}x_{\eta(n-1)} = 1. \]
With this the proof of (5) in our subclaim is complete. Now we prove (6). Let $j < m$. Then we have
\[
\mathcal{C}_{k'_{n(n-1)}}(-c(l_{\xi(j)})y_{\xi(j)}) = -c(l_{\xi(j)})y_{\xi(j)}
\]
and
\[
\mathcal{C}_{k'_{n(n-1)}}(s_{v_{\xi(j)}}y_{\xi(j)}) = s_{v_{\xi(j)}}y_{\xi(j)}.
\]
Indeed, computing we get
\[
\mathcal{C}_{k'_{n(n-1)}}(-c(l_{\xi(j)})y_{\xi(j)}) = -c(k'_{n-1}) - (-c(l_{\xi(j)})y_{\xi(j)})
= -c(k'_{n-1})c(l_{\xi(j)})y_{\xi(j)}
= -c(l_{\xi(j)})y_{\xi(j)}.
\]
Similarly, we have
\[
\mathcal{C}_{k'_{n(n-1)}}(s_{v_{\xi(j)}}y_{\xi(j)}) = -c(k'_{n-1}) - (s_{v_{\xi(j)}}y_{\xi(j)})
= -c(k'_{n-1})(s_{v_{\xi(j)}} - y_{\xi(j)})
= -s_{v_{\xi(j)}} - y_{\xi(j)}
= s_{v_{\xi(j)}}y_{\xi(j)}.
\]
By \(\mathcal{C}_{k_{n_1}}(\mathcal{C}_{k}x \oplus y) = \mathcal{C}_{k_{n_1}}x \oplus \mathcal{C}_{k_{n_1}}y\) we get from the above that
\[
\mathcal{C}_{k'_{n(n-1)}}(t_j) = \mathcal{C}_{k'_{n(n-1)}}(-c(l_{\xi(j)})y_{\xi(j)} \oplus s_{v_{\xi(j)}}y_{\xi(j)})
= \mathcal{C}_{k'_{n(n-1)}}(-c(l_{\xi(j)})y_{\xi(j)} + \mathcal{C}_{k'_{n(n-1)}}s_{v_{\xi(j)}}y_{\xi(j)}
= -c(l_{\xi(j)})y_{\xi(j)} \oplus s_{v_{\xi(j)}}y_{\xi(j)}
= t_j.
\]
We have proved (6). By the above proven subclaim, i.e. by (5), (6), we have
\[
\mathcal{C}_{k'_{n(n-1)}} \prod_{i<n} z_i^{l_i} = \mathcal{C}_{k'_{n(n-1)}} [\prod_{i<n-1} z_i^{l_i} \circ z_{n-1}^{n-1}]
= \mathcal{C}_{k'_{n(n-1)}} \prod_{i<n-1} z_i^{l_i} \circ \mathcal{C}_{k'_{n(n-1)}} z_{n-1}^{n-1}
= \prod_{i<n-1} \mathcal{C}_{k'_{n(n-1)}} z_i^{l_i} \circ \mathcal{C}_{k'_{n(n-1)}} z_{n-1}^{n-1}
= \prod_{i<n-1} z_i^{l_i} \circ \mathcal{C}_{k'_{n(n-1)}} z_{n-1}^{n-1}
= \prod_{i<n-1} z_i^{l_i} \circ [\mathcal{C}_{k'_{n(n-1)}} z_{n-1}^{n-1}]^{n-1}
= \prod_{i<n-1} z_i^{l_i}.
\]

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Combined with (3) we obtain
\[ a^t \odot \prod_{i<n-1} (c_{(k_{\eta(i)})}x_{\eta(i)} \oplus s_{\eta(i)}x_{\eta(i)}) \leq c_{(k'_{\eta(n-1)})}b_1. \]

On the other hand, from (5), (6) and (4), it follows that
\[ (-b)^k \odot \prod_{j<m} (-c_{(l_{\xi(j)})}y_{\xi(j)} \oplus s_{\xi(j)}y_{\xi(j)}) \leq c^\partial_{(k'_{\eta(n-1)})}b_2. \]

Now making use of the induction hypothesis we get:
\[ c_{(k'_{\eta(n-1)})}b_1 \odot c^\partial_{(k'_{\eta(n-1)})}b_2 \neq 0; \]
and hence that
\[ b_1 \odot c^\partial_{(k'_{\eta(n-1)})}b_2 \neq 0. \]

From
\[ b_1 \odot c^\partial_{(k'_{\eta(n-1)})}b_2 \leq b_1 \odot b_2 \]

we reach the desired conclusion, i.e. that
\[ b_1 \odot b_2 \neq 0. \]

The other case, when \( \eta(n - 1) \leq \xi(m - 1) \) can be treated in a similar manner and is therefore left to the reader. We have proved that \( H \) is a proper filter. We proceed exactly as in the proof of theorem 19. Proving that \( H \) is a proper filter of \( \mathfrak{B}g^\mathfrak{B}(X_1 \cap X_2) \), let \( H^* \) be a (proper MV) maximal filter of \( \mathfrak{B}g^\mathfrak{B}(X_1 \cap X_2) \) containing \( H \). We obtain maximal filters \( F_1 \) and \( F_2 \) of \( \mathfrak{B}g^\mathfrak{B}(X_1) \) and \( \mathfrak{B}g^\mathfrak{B}(X_2) \), respectively, such that
\[ H^* \subseteq F_1, \quad H^* \subseteq F_2 \]
and (*)
\[ F_1 \cap \mathfrak{B}g^\mathfrak{B}(X_1 \cap X_2) = H^* = F_2 \cap \mathfrak{B}g^\mathfrak{B}(X_1 \cap X_2). \]

Now for all \( x \in \mathfrak{B}g^\mathfrak{B}(X_1 \cap X_2) \) we have
\[ x \in F_1 \text{ if and only if } x \in F_2. \]

Let \( i \in \{1, 2\} \). Then \( F_i \) by construction satisfies the following quantifier elimination condition:

For each \( p \in \mathfrak{B} \) and each subset \( J \subseteq n \) with \( |J| < m \), there exists \( \rho \in \mathfrak{n}n \) such that
\[ \rho \upharpoonright n \setminus J = Id_{n-J} \]
and

\[-c_{(j)}p \oplus s_\rho p \in F_i.\]

Since

\[s_\rho p \leq c_{(j)}p,\]

we have (**)

\[c_{(j)}p \in F_i \iff s_\rho p \in F_i.\]

Observe that this condition is more complex than the quantifier elimination condition encountered before in the proof of theorem 19. The reason is that in the present infinitary context, we are allowing infinitary cylindrification. However, if we restrict our attention to only finite cylindrification, then the two notions coincide.

(3) Building the required representation: Let \(D_i = \mathcal{S}g^\mathbb{N}X_i, \ i = 1, 2.\) Let

\[\psi_i : D_i \to \mathfrak{S}(^\alpha n, D_i/F_i)\]

be defined as follows:

\[\psi_i(a)(x) = \bar{s}_\beta a/F.\]

Note that \(\bar{\tau} = \tau \cup Id_{\beta \setminus \alpha}\) is in \(^\mathbb{n}n,\) so that substitutions are evaluated in the big algebra \(\mathcal{B}.\) Then, we claim that \(\psi_i\) is a homomorphism. Then using freeness we paste the two maps \(\psi_1 \psi_2,\) obtaining that \(\psi = \psi_1 \cup \psi_2\) is homomorphism from the free algebra to \(\mathfrak{S}(^\alpha n, [0, 1])\) such that \(\psi(a \odot -b) \neq 0\) which is a contradiction. As usual, we check cylindrifiers, and abusing notation for a while, we omit superscripts, in particular, we write \(\psi\) instead of \(\psi_1.\) Let \(x \in ^\alpha n, M \subseteq \alpha, p \in D.\) Then

\[\psi(c_{(M)}p)(x) = s_x c_{(M)}p/F.\]

Let \(K\) be a support of \(p\) such that \(|K| < m\) and let \(J = M \cap K.\) Then

\[c_{(J)}p = c_{(M)}p.\]

Let

\[\sigma \in ^\mathbb{n}n, \sigma \upharpoonright \mathbb{n} \setminus J = \bar{\tau} \upharpoonright \mathbb{n} \setminus J,\]

\[\sigma J \cap \tau(K \setminus J) = \emptyset\]

and

\[\sigma \upharpoonright J\] is one to one.
Then
\[ s_x c(J)p = c(\sigma J)s_\sigma p. \]

By (**), let \( \rho \) be such that
\[ \rho \upharpoonright \sigma J = Id_{n \setminus \sigma J} \]
and
\[ c(\sigma J)s_\sigma p \in F \iff s_\rho s_\sigma p \in F \iff s_{\rho \sigma} p \in F. \]

It follows that
\[ c(\sigma J)s_\sigma p / F = s_\rho s_\sigma p / F = s_{\rho \sigma} / F. \]

Let \( y \in ^n n \) such that
\[ y \upharpoonright M = \rho \circ \sigma \upharpoonright M \]
and (***)
\[ y \upharpoonright n \setminus M = \tau \upharpoonright n \setminus M. \]

If \( k \in K \setminus M, \rho \circ \sigma(k) = \rho \circ \tau(k) \) and since \( \tau k \notin \sigma J \) we have
\[ \rho \tau(k) = \tau(k) = y(k). \]

So
\[ y \upharpoonright K = \rho \circ \sigma \upharpoonright K. \]

Now we have using (**) and (***):
\[
\psi(c(M)p)x \\
= s_x c(M)p / F \\
= s_x c(J)p / F \\
= c(\sigma J)s_\sigma p / F \\
= s_\rho s_\sigma p / F \\
= s_{\rho \sigma} p / F \\
= s_y p / F \\
\leq c(M)\psi(p)x.
\]

The other inclusion is left to the reader. The proof is complete. \( \square \)

5.2 Interpolation for Reducts of PMA_\alpha s

Let CPMA_\alpha be the reduct obtained from PMA_\alpha by discarding infinitary cylindrifiers, so here we have \( G = ^\alpha \alpha \) and \( T = \wp(\alpha) \). Such algebras are called
cylindric-polyadic MV algebras. Algebras in CPMA, have a double facet. One is that they have only finite cylindrifiers; in this respect they are like cylindric algebras. The other facet is that they have all substitutions available in their signature like polyadic algebras. Classical versions of such algebras, introduced by Ferenczi (Ferenczi, 2012), are studied in (Sayed Ahmed, 2011b).

We need the following lemma:

**Lemma 64** Let \( \alpha \) be an infinite ordinal. Let \( \mathcal{D} \in \text{CPMA}_\alpha \). Then for any ordinal \( n > \alpha \), there exists \( \mathcal{B} \in \text{CPMA}_n \) such that \( \mathcal{D} = \mathfrak{R}_\alpha \mathcal{B} \).

**Proof** Assume that \( \mathcal{D} \in \text{CPMA}_\alpha \) and \( n > \alpha \). If \( |\alpha| = |n| \), then one fixes a bijection \( \rho : n \to \alpha \), and defines the \( n \)-dimensional dilation of \( \mathcal{D} \), having the same universe as \( \mathcal{D} \), by re-shuffling the operations of \( \mathcal{D} \) along \( \rho \) (Daigneault & Monk, 1963).

Now assume that \( |n| > |\alpha| \). Write \( F(A, B) \) for the set of all functions from \( A \) to \( B \).

Then the set \( F(\alpha, \mathcal{D}) \) can be endowed with a Boolean structure defined pointwise. For \( \tau \in \alpha \) and \( f \in F(\alpha, \mathcal{D}) \), put \( s_\tau f(x) = f(x \circ \tau) \).

The map \( H : \mathcal{D} \to F(\alpha, \mathcal{D}) \) defined by \( H(p)(x) = s_\rho p \) is easily checked to be an embedding preserving the Boolean operations and substitution operations where the \( s_\tau \)-s in \( \mathcal{D} \) are the cylindrifiers in \( \mathcal{D} \) and \( s_\tau \)-s in \( F(\alpha, \mathcal{D}) \) as just defined. Furthermore, \( K : F(\alpha, \mathcal{D}) \to F(\alpha, \mathcal{D}) \) defined by \( K(f)x = f(x | \alpha) \) is an embedding, preserving the Boolean operations and substitution operations, defined for both the domain and codomain of \( K \) like for \( F(\alpha, \mathcal{D}) \). Then \( K \circ H \) defines an embedding for \( \mathcal{D} \) to \( \mathcal{B}^{-c} = F(\alpha, \mathcal{D}) \), preserving Boolean operations and substitution operations. These facts are straightforward to establish, cf. (Daigneault & Monk, 1963, Theorems 3.1, 3.2).

One defines cylindrifiers on \( \mathcal{B}^{-c} \), for each \( i \in n \), via (Daigneault & Monk, 1963):

\[
\mathcal{c}_i s_\tau \mathcal{B} = s_\rho \mathcal{c}_i \mathcal{D}, \mathcal{c}_\tau \mathcal{D} = s_\rho \mathcal{c}_\tau \mathcal{B}.
\]

(p \( \in \mathcal{B} \)). Call the expanded structure \( \mathcal{B} \). Then it can be proved that \( \mathcal{B} \in \text{CPMA}_n \) and that \( K \circ H : \mathcal{D} \to \mathfrak{R}_\alpha \mathcal{B} \) is an isomorphism preserving all the operations (Daigneault & Monk, 1963, Theorem 3.10). That is \( K \circ H \) is an injective and surjective homomorphism.

The definition of rich semigroups to be formulated next is exactly like in (Sayed Ahmed, 2004, definition 1.4) to which we refer to for notation used.

**Definition 65** Let \( \alpha \) be a countable ordinal. Let \( T \subseteq (\alpha, \circ) \) be a semigroup. We say that \( T \) is rich if \( T \) satisfies the following conditions:

(1) \( (\forall i,j \in \alpha)(\forall \tau \in T)\tau[i,j] \in T \).

(2) There exist \( \sigma, \pi \in T \), called distinguished elements of \( T \), such that \( (\pi \circ \sigma = \text{Id}, \text{rng}\sigma \neq \alpha) \), satisfying

\[
(\forall \tau \in T)(\sigma \circ \tau \circ \pi)[(\alpha \setminus \text{rng}\sigma)]\text{Id} \in T.
\]

If \( G \) is a rich countable sub-semigroup of \( \omega \omega \), and \( T = \varphi_\omega(\omega) \), then we denote \( \text{MV}_{G,T} \) by \( \text{PMA}_G \). We give concrete examples of rich semigroups.

**Example 66** The semigroup \( T \) generated by \( \{[i,j], [i,j], i,j \in \omega, \text{suc, pred} \} \) is strongly rich. Here \( \text{suc} \) abbreviates the successor function on \( \omega \) and \( \text{pred} \) acts as its quasi-right inverse, the predecessor function, defined by \( \text{pred}(0) = 0 \) and for other \( n \in \omega \), \( \text{pred}(n) = n - 1 \). Here the transformation \( \text{suc} \) plays the role of \( \sigma \) while the transformation \( \text{pred} \) plays the role of \( \pi \), hence \( \text{suc} \) and \( \text{pred} \) are the distinguished elements of \( T \).
Theorem 67.  

(1) If $G$ is a rich countable semigroup on $\omega$, and $\beta$ is any countable non-zero cardinal, then $\mathfrak{F}_\beta \mathcal{PMA}_G$ has VIP.  

(2) If $\alpha \geq \omega$, and $\beta$ is any non-zero cardinal, then $\mathfrak{F}_\beta \mathcal{CPMA}_\alpha$ has VIP.  

Proof. (1) The proof is similar to the proof of theorem 19 by adapting the technique in (Sayed Ahmed, 2004) to the many valued present context. The idea is the same idea used before in theorem 19 and 63. We will be sketchy. Let $\mathfrak{A} = \mathfrak{F}_\beta \mathcal{PMA}_G$. Let $\alpha = \omega \cdot \omega$. By (Sayed Ahmed, 2004) there exists $\mathfrak{C} \in \mathcal{PMA}_G$ such that $\mathfrak{A} = \mathfrak{F}_\alpha \mathcal{C}$, where $\tilde{G}$ is the subsemigroup of $(^\langle \omega, \alpha, \circ \rangle, \circ)$ generated by $\{\tau \cup \text{Id}_{\alpha \setminus \omega}, [i, j], [i \cdot j], \tau \in G, i < j < \alpha\}$. Let $X_1, X_2 \subseteq A$ and assume that $a \in \mathcal{G}^X_\alpha X_1$ and $b \in \mathcal{G}^X_\alpha X_2$ are such that $a \leq b$. Assume that no interpolant exists in $\mathfrak{A}$. Then, as in the proof of theorem 19, no interpolant exists in $\mathfrak{C}$.  

Like before, we can define by recursion $\omega$-termed sequences of witnesses:

$$\langle u_i : i \in \omega \rangle \text{ and } \langle v_i : i \in \omega \rangle$$

such that for all $i \in \omega$ we have: $u_i \in \alpha \setminus (\Delta a \cup \Delta b) \cup \cup_{j < i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}$ and $v_i \in \alpha \setminus (\Delta a \cup \Delta b) \cup \cup_{j < i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}$.  

As before, in our proofs of the interpolation property, let

$$Y_1 = \{a\} \cup \{-c_{k} x_i \oplus s_{u_i}^{k} x_i : i \in \omega\},$$

$$Y_2 = \{-b\} \cup \{-c_{l} y_i \oplus s_{v_i}^{l} y_i : i \in \omega\},$$

$$H_1 = \mathfrak{f}^{\mathcal{G}^X_\alpha \mathcal{C}}(X_1) Y_1, \ H_2 = \mathfrak{f}^{\mathcal{G}^X_\alpha \mathcal{C}}(X_2) Y_2,$$

$$H = \mathfrak{f}^{\mathcal{G}^X_\alpha \mathcal{C}}(X_1 \cap X_2) \{H_1 \cap \mathcal{G}^X_\alpha (X_1 \cap X_2) \cup (H_2 \cap \mathcal{G}^X_\alpha (X_1 \cap X_2)).$$

Then $H$ is a proper filter of $\mathcal{G}^X_\alpha (X_1 \cap X_2)$; same reasoning as in the proof of theorem 19. Let $V = \bigcup_{x \in G} \omega x^{(\tau)}$. Then for $s \in V$, $s = s \cup \text{Id}_{\alpha \setminus \omega} \in \tilde{G}$. Define $x \in V$, $\psi_1(a) = s^X_\alpha / F_1$ and $\psi_2 = s^X_\beta / F_2$. Then these are homomorphisms that can be pasted together to give a homomorphism $\psi$ with domain $\mathfrak{A}$ such that $\psi(a \circ b) \neq 0$. This contradicts $a \leq b$.  

(2) When $G = \mathfrak{F}_\alpha$, the first part of the proof is almost identical to the proof of theorem 19. Let $\mathfrak{A} = \mathfrak{F}_\beta \mathcal{CPMA}_\alpha X_1, X_2 \subseteq \mathfrak{A}, a \in \mathcal{G}^X_\alpha X_1$ and $b \in \mathcal{G}^X_\alpha X_2$ such that $a \leq b$. Let $\kappa$ be a regular cardinal $> \text{max}(|A|, \alpha)$, and let $\mathfrak{B}$ be an algebra of dimension $\kappa$ such that $\mathfrak{A} = \mathfrak{F}_\kappa \mathcal{B}$. Such a $\mathfrak{B}$ by lemma 64.  

Then one proceeds as in the proof of theorem 19; assuming no interpolant exists in $\mathfrak{A}$, then no interpolant of $a$ and $b$ exists in $\mathfrak{B}$, and then one constructs the proper filter $H$ of $\mathcal{G}^X_\alpha (X_1 \cap X_2)$ as in the above proof. In defining the representability functions, instead of $V$ one takes $\kappa$, because all substitutions are at hand, and from thereon the proof is exactly like before.  

Dictated by the above interpolation theorems, representability of $\mathcal{PMA}_\alpha$ and $\mathcal{CPMA}_\alpha$ are by algebras of the form $\mathfrak{F}(^\langle \kappa U, [0, 1]\rangle)$ as in example 11 taken in the suitable type (signature) while that of $\mathcal{PMA}_G$ is via algebras of the form $\mathfrak{F}(V, [0, 1])$, also in the suitable signature, where $V$ is a union of weak spaces having the same base. Such a $V$ is called a compressed space (Henkin et al., 1985, Definition 3.1.5). Using the interpolation theorem proved together with an argument entirely analogous to that used in theorem 42, we get:  

Corollary 68. Every algebra in $\mathcal{PMA}_\alpha$ or $\mathcal{PMA}_G$ or $\mathcal{CPMA}_\alpha$ is representable. All
three classes PMA$_\alpha$, PMA$_G$ and CPMA$_\alpha$ have the generalized super amalgamation property.

Let $K \in \{\text{PMA, CPMA}\}$. Then like in (Sayed Ahmed, 2011b), one can show that the neat reduct operator $\text{Nr}_\alpha : K_{\alpha+\omega} \to K_\alpha (\alpha \geq \omega)$ as defined in 59 is invertible (it is an equivalence). Recall that in item(iii) of theorem 59 we showed that $\text{Nr}_\alpha$ for MA$_\alpha$ does not even have a right adjoint.

As opposed to negative non–finite axiomatizability results obtained for representable algebras (and their approximations as shown in the first row of the previous table) that abound in the literature of cylindric–like algebras, our next result is a positive finitizability result (such results are rare in algebraic logic, witness (Sayed Ahmed, 2005) for an overview of this phenomena). Before formulating such a result for MV polyadic algebras, circumventing the two negative non-finite axiomatizability results proved in theorems 30, 31, we need a definition.

**Definition 69** Let $G$ be a rich semigroup. Let $B \in \text{PMA}_G$. By $\text{R}_f B$ we understand the reduct of $B$ obtained by discarding infinitary substitutions, and $\text{R}_f K$, $K \subseteq \text{PMA}_G$, denotes $\{\text{R}_f B : B \in K\}$.

Let $G$ be a rich finitely presented subsemigroup of $(\omega \omega, \circ)$. Such semigroups exist (Sain, 2000). Let $K_G = \{S(V, [0, 1]) : V$ is a compressed space $\}$, where algebras in $K_G$ are taken in the type $(G, \wp(\omega))$ as in example 11. We let $I$ denote the operation of forming isomorphic copies.

**Theorem 70** Assume that $G$ is a rich finitely presented subsemigroup of $(\omega \omega, \circ)$ and $K_G$ as just described. Then the following hold:

(i) $\text{PMA}_G = IK_G$.
(ii) $S\text{R}_f \text{PMA}_G = \text{RMA}_\omega$. In other words, the class of representable MA$_\omega$s coincides with the class of subreducts of PMA$_G$.
(iii) $IK_G$ is term–definitionally equivalent to a finitely axiomatizable variety (in a finite signature) and $IK_G$ has the amalgamation property.

**Sketch of proof.** The gaps in the following sketch can be easily filled in using the analogous result for the classical case fully proved in (Sayed Ahmed, 2004; Sain, 2000). We know from corollary 68 that if $G$ is rich, then $\text{PMA}_G = IK_G$. If $G$ is rich and finitely presented, then using exactly the techniques in (Sain, 2000) one can truncate the axiomatization of PMA$_G$ given in definition 8, restricted to a rich finitely presented semigroups, to be strictly finite. This entails that the signature is also finite. It can be proved that $IK_G$ is term–definitionally equivalent to a variety $V_S$ in a finite signature containing only the operations $\oplus$, $\odot$ and $\neg$ together with $\{c_0, s_\tau : \tau \in S\}$, where $S$ is a finite set presenting $G$.

The rough idea here is that the successor like transformation $\sigma \in S$ (which is one of the distinguished elements of $G$) generates the rest of the operations. The variety $V_S$ is finitely axiomatized by the finite set of equations resulting from restricting the equations defining PMA$_G$ to the hitherto obtained finite signature.

The previous theorem is the many valued version of the solution to the so–called finitizability problem in algebraic logic, a central problem posed by Monk and Tarski in the seventies of the last century (Sain, 2000; Sayed Ahmed, 2005). The (infinitary) logic corresponding to PMA$_G$, when $G$ is rich and finitely presented, is an algebraizable in the Blok–Pigozzi sense (Blok & Pigozzi, 1989) extension of predicate many valued logic that admits a finite complete and sound Hilbert style axiomatization, and furthermore has VIP.
6. Summary of Results in Tabular Form

We will summarize our results in tabular form. We need one last theorem which we give with a fairly complete sketch of proof. By a simple algebra we understand an algebra having no proper congruences which is the usual universal algebraic definition. For MAVs this is equivalent to having no proper MV filters as defined before lemma 37. Recall that for an algebra B, CoB stands for the set of all congruence relations on B. We let MC0B stand for the set of maximal congruences on B (with respect to ⊆). To formulate and prove a part of the next theorem, we need:

**Definition 71** Let A be an algebra, and C ⊆ ∪B⊂Α CoB. A is said to have the coextension extension property relative to C if for any x1, x2 ⊆ A if R ∈ Co(Sa3X1) ∩ C and S ∈ Co(Sa3X2) ∩ C, such that R ∩ 2 Sa3(X1 ∩ X2) = S ∩ 2 Sa3(X1 ∩ X2) ⊆ C then there is a T ∈ C ∩ CoA such that T ∩ 2 Sa3X1 = R and T ∩ 2 Sa3X2 = S. We refer to T as the amalgamating congruence of R and S. If C = ∪B⊂Α CoB, then we say that A has CP, and when C = ∪B⊂Α MC0B, then we say that A has MCP.

For MAVs we identify (maximal) congruences with (maximal) filters. (Here maximality in both cases is defined the usual way relative to the order ⊆.) In the next theorem WVIP is the weak interpolation property addressed in lemma 35.

**Theorem 72** (1) (Madárasz & Sayed Ahmed, 2009) For a class K of algebras such that HK = SK, SAP ≡ ES.

(2) If K ⊆ MAV is a variety, then its free algebras have WVIP ≡⇒ the class of simple algebras in K has AP.

**Proof** We give a fairly complete sketch of (2). For a variety K ⊆ MAV, it can be shown that K has the amalgamation property ≡⇒ the free algebras have CP, and that simple algebras in K have the amalgamation property ≡⇒ the free algebras have the MCP. This is similar to the classical case proved in (Madárasz & Sayed Ahmed, 2009). Now we show that if the free algebra A has WVIP ≡⇒ it has the MCP by which we will be done. We work with filters in place of congruences. Assume A has MCP. Let x1, x2 ⊆ A, and x ∈ Sa3X1 and z ∈ Sa3X2, such that x ≤ z and assume for contradiction that there is no y, no n, m ∈ N, and no finite Θ ⊆ α, such that q(Θ)x ⊗ y ≤ c(Θ)mz. Then q(Θ)x ⊗ y > 0 or q(Θ)|−z|m ⊗ y > 0 for all n, m ∈ N, whenever y ∈ Sa3(X1 ∩ X2). Hence for any finite subsets Δ, θ of α, for any n, m ∈ N, we have u ⊗ w > 0 for all u, w ∈ Sa3(X1 ∩ X2) such that u ≥ q(Δ)xn and w ≥ q(θ)|−z|m. Let

\[ P = \exists l \cdot (q(l)x \cap z) \cup (q(l)y \cup z \cap q(l)x \cap z). \]

Then by lemma 37, P is proper, so let P′ be a maximal proper filter in Sa3(X1 ∩ X2) containing P.

By lemma 47, there are maximal filters M of Sa3X1 and N of Sa3X2 such that

\[ \exists l \cdot (q(l)x \cap z) \subseteq M \text{ and } \exists l \cdot (q(l)x \cap z) \subseteq N, \]

and M ∩ Sa3(X1 ∩ X2) = P′ = N ∩ Sa3(X1 ∩ X2). By MCP, since M and N are maximal filters in A, we have M′ = M ∪ N is proper, and so it is not the case that x ≤ z. For if x ≤ z, then x ∩ (−z) = 0 and so by (*) we get 0 ∈ M′ = M ∪ N and
so \( F^\exists(M \cup N) = \emptyset \), which is a contradiction, and we are done.

Conversely, assume that \( \mathfrak{A} \) has \( WVIP \). Let \( X_1, X_2 \subseteq A \) and \( M \) be a maximal filter of \( \mathfrak{G}^\exists X_1 \) and \( N \) be a maximal filter of \( \mathfrak{G}^\exists X_2 \) such that \( M \cap \mathfrak{G}^\exists(X_1 \cap X_2) = N \cap \mathfrak{G}^\exists(X_1 \cap X_2) \). It suffices to show that \( F = F^\exists(M \cup N) \) is a proper filter, for then it amalgamates the maximal congruences induced by \( M \) and \( N \).

Assume for contradiction that \( F \) is not proper. Then \( F = \mathfrak{A} \), so by the last item of lemma 4, there exist \( x \in M \), \( z \in N \) such that \( x \circ z = 0 \). By \( WVIP \) there are \( y \in \mathfrak{G}^\exists(X_1 \cap X_2) \), \( n \), \( m \in N \), and a finite \( \Gamma \subseteq \alpha \) such that \( q_\Gamma(x^n) \geq y \in c_\Gamma(mz) \).

By the second item of the same lemma, \( y \in \mathfrak{F}\mathfrak{g}^\exists\{x\} \subseteq M \) and \( -y \in \mathfrak{F}\mathfrak{g}^\exists\{z\} \subseteq N \). Thus \(-y \in N \cap \mathfrak{G}^\exists(X_1 \cap X_2) = M \cap \mathfrak{G}^\exists(X_1 \cap X_2) \subseteq M \), and \( y \in M \cap \mathfrak{G}^\exists(X_1 \cap X_2) \subseteq M \). But then, by the definition of \( MV \) filters, \( 0 = -y \circ y \in M \), which is impossible by first item of lemma 4, because \( M \) is proper. We conclude that \( \mathfrak{F}\mathfrak{g}^\exists(M \cup N) \) is a proper filter, and we are done. \( \Box \)

### Table 1. Summary of interpolation and amalgamation properties

<table>
<thead>
<tr>
<th>Free algebras</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists MA_0 )</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists RA_0 )</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists SN_0 MA_{\omega+k} )</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists MA_\alpha )</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists PMA_\alpha )</th>
<th>( \mathfrak{F}\mathfrak{g}^\exists CPMA_\alpha )</th>
<th>Citation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( VIP )</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>( WVIP )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( ES )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( SEP )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( AP )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( GSUPAP )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

In the Table 1 the left hand most column addresses different dimension restricted free algebras, and the upper most row addresses various interpolation and amalgamation properties. We write \((S)AP\) short for the (strong) amalgamation property and \( GSUPAP \) is short for the generalized super amalgamation property as defined in item \((3)\) of definition 38. \( ES \) abbreviates that epimorphisms (right cancellative maps) are surjective. \( VIP \) stands for varying interpolation property, \( WVIP \) stands for the weak varying interpolation property as formulated in theorem 36, and \( SEP \) stands for the strong embedding property as defined in 51.

The columns under \( AP \), \( GSUPAP(SAP) \) \( ES \) and \( SEP \) refers to the status of this property for the class \( K \) whose \( \omega \) dimensional restricted free algebra lies at the left hand most column. For example in the third row \( VIP, WVIP \) address the free algebra \( \mathfrak{F}\mathfrak{g}^\exists RA_0 \) while \( AP \), \( GSUPAP(SAP) \), \( ES \), \( SEP \) address the (corresponding) variety \( RA_0 \).

In the first column intersected with the fourth row \( k \) occurring in \( MA_{\omega+k} \) is a natural number, that is \( k \in \omega \). In the first row the ‘yes’ in the column under \( AP \) refers to that \( AP \) holds in \( K \), where \( K \) is the class in theorem 42. In the first and second rows, the function \( \rho: \omega \to \phi(\alpha) \) satisfies \( |\alpha \setminus \rho(i)| > \omega \) for all \( i \in \omega \) so that here we deal with dimension restricted (by \( \rho \)) free algebras. The ‘yes’ in the second row addresses the class defined similarly to \( K \) for \( SAT_\alpha \) as in definition 25 taking the constants \([0, 1] \cap Q \) into consideration.

By free algebras in the upper most column, we understand free algebras in the broader sense of dimension restricted free algebras as in definition 16, though this notion is used only in the first two rows.
In the remaining rows we deal with the (ordinary) notion of free algebras. ‘AP’ in the rows 3, 4, 5, abbreviate that: AP does not hold for the class in question, but it holds for the class of its simple members. In the table we write e.g 36 in place of Theorem 36 for typographic reasons. The same convention applies to corollaries.

References


